CS 70	Discrete Mathematics and Probability Theory	
Spring 2025	Rao	HW 01

Due: Saturday, 2/1, 4:00 PM
Grace period until Saturday, 2/1, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Calculus Review

In the probability section of this course, you will be expected to compute derivatives, integrals, and double integrals. This question contains a couple examples of the kinds of calculus you will encounter.

(a) Compute the following integral:

$$\int_0^\infty \sin(t)e^{-t}\,\mathrm{d}t.$$

$= -\int_0^\infty \sin(t)\, d(e^{-t})$

$= -\left(\sin t\, e^{-t} - \int_0^\infty e^{-t} \cos t\, dt\right)$

$= -\sin t\, e^{-t} + \int_0^\infty d(e^{-t}) \cos t$

$= -\sin t\, e^{-t} - \left(e^{-t} \cos t + \int_0^\infty e^{-t} \sin t\, dt\right)$

$\Rightarrow \int_0^\infty \sin t\, e^{-t}\, dt = -\frac{e^{-t}(\sin t + \cos t)}{2} = \frac{1}{2}$

(b) Compute the double integral

$$\iint_R 2x + y\,\mathrm{d}A,$$

where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.

(b) $= \int_0^1 \left(\int_0^x (2x+y)\, dy\right) dx = \int_0^1 \frac{5}{2} x^2\, dx = \frac{5}{2} \cdot \frac{x^3}{3}\Big|_0^1 = \frac{5}{6}$

2 More Logical Equivalences

Note 1

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

$x \in \{0, 1, 2\}$ $P(x): x > 0$ $Q(x): x > 10$

Then RHS $\equiv$ T, LHS is false when $x = 1, 2$

F (a) $\forall x (P(x) \implies Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \implies \forall x\, Q(x)$

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T (b) $\neg(\exists x (P(x) \vee Q(x))) \stackrel{?}{\equiv} \forall x (\neg P(x) \wedge \neg Q(x))$ LHS $\equiv \forall(x)\left(\neg(P(x) \vee Q(x))\right) \equiv \forall(x)\left(\neg P(x) \vee \neg Q(x)\right)$

T (c) $\forall x ((P(x) \implies Q(x)) \wedge Q(x)) \stackrel{?}{\equiv} \forall x\, P(x)$

$Q(x) \equiv T$ and

$P(x) \equiv F$

3 Prove or Disprove

Note 2

For each of the following, either prove the statement, or disprove by finding a counterexample.

$p \Rightarrow q \equiv \neg p \vee q$

LHS $\equiv \forall x (\neg p \vee q)$ $\neq$ $\exists x(\neg P(x) \vee \neg Q(x))$

RHS $\equiv \neg(\forall x P(x)) \vee \neg(\forall x\, Q(x)) \equiv (\exists x\, \neg P(x)) \vee (\exists x\, \neg Q(x))$

(a) $n = 2k+1$, $k = 0, 1, 2 \ldots$
$n^2+4n = 4k^2+4k+1+8k+4$
$= 4k^2+12k+4+1$
$= 2(2k^2+6k+2)+1$ = odd.

T (a) $(\forall n \in \mathbb{N})$ if n is odd then n^2+4n is odd.

T (b) $(\forall a, b \in \mathbb{R})$ if $a+b \leq 15$ then $a \leq 11$ or $b \leq 4$. by contradiction. if $a \geq 11 \wedge b > 4 \Rightarrow a+b > 15$

T (c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational. by contradiction. $r = \frac{p}{q} \Rightarrow r^2 = \frac{p^2}{q^2}$, rational

F (d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers) $n = 10$: $5 \times 10^3 \not> 10! = 3628800$

T (e) The product of a non-zero rational number a and an irrational number b is irrational. if $a \cdot b$ is rational, $a = \frac{p}{q}$ then $a \cdot b = \frac{p'}{q'}$, $b = \frac{p'}{q'} \times \frac{q}{p}$ is ra...

T (f) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. (Recall that $A' \in \mathscr{P}(A)$ if and only if $A' \subseteq A$.)

$A \subseteq B \Rightarrow \forall A' \subseteq \mathscr{P}(A)!$ $A' \subseteq A \Rightarrow A' \subseteq B \Rightarrow A' \subseteq \mathscr{P}(B)$
$\Rightarrow \mathscr{P}(A) \subseteq \mathscr{P}(B)$

4 Twin Primes

Note 2 (a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

(3, 5) (5, 7)

(a) proof by contrapositive. if p is not the form of $3k+1$ or $3k-1$. $\Rightarrow p = 3k \Rightarrow p$ is composite

(b) proof by contriction: if $\exists\, p \geq 7$. $p-2, p, p+2$ is prime. $\begin{cases} p-2 = 3k+1 \Rightarrow p = 3k+3 = \text{composite} \\ p-2 = 3k-1 \Rightarrow p+2 = 3k+3 = \text{composite} \end{cases}$ Contradiction

5 Airport

Note 3 Suppose that there are $2n+1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

$n = 1$, $2n+1 = 3$, A B C let B C be closest, then A satisfy

$n = k$, satisfy

$n = k+1$, delete closest pair, then there's $2n+1$ nodes, assume M satisfies in the remaining $2n+1$ nodes, put back other nodes

6 Grid Induction

Note 3 Pacman is walking on an infinite 2D grid. He starts at some location $(i, j) \in \mathbb{N}^2$ in the first quadrant, and is constrained to stay in the first quadrant (say, by walls along the x and y axes).

Every second he does one of the following (if possible):

(i) Walk one step down, to $(i, j-1)$.

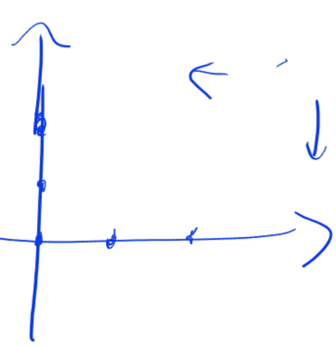
(ii) Walk one step left, to $(i-1, j)$.

For example, if he is at $(5,0)$, his only option is to walk left to $(4,0)$; if Pacman is instead at $(3,2)$, he could walk either to $(2,2)$ or $(3,1)$.

Prove by induction that no matter how he walks, he will always reach $(0,0)$ in finite time.

(*Hint*: Try starting Pacman at a few small points like $(2,1)$ and looking all the different paths he could take to reach $(0,0)$. Do you notice a pattern in the number of steps he takes? Try to use this to strengthen the inductive hypothesis.)

$i+j$ reduce 1 in each step



① $i+j = 0$, $i = 0$, $j = 0$, take 0 steps to reach $(0,0)$

② $i+j=k$ could reach $(0,0)$ in finite time, regardless of what path

③ $i+j=k+1$ suppose i to then reduce i for 1, $i+j=k$ ...