Induction Intro

Note 3

Natural numbers start at 0, and there is always a next one. For predicates on natural numbers the *principle of induction* is: $\forall n \in \mathbb{N}, P(n) \equiv P(0) \wedge \forall n, P(n) \implies P(n+1)$.

That is, to prove $P(n)$ for natural numbers one proves $P(0)$, the *base* case, and $\forall n, P(n) \implies P(n+1)$, the *induction step*. In the induction step, the assumption that $P(n)$ is true is called the *induction hypothesis* which is typically used to argue that $P(n+1)$ is true.

An example is the statement $P(n) = \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$. The base case, $P(0)$, is the observation that $\sum_{i=0}^{0} i = 0$. In the induction step, the induction hypothesis, $P(n)$, is $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$. The induction step proceeds as follows:

$$\sum_{i=0}^{n+1} i = \sum_{i=0}^{n} i + n + 1 = \frac{(n)(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}.$$

The first equality follows from the definition of the notation, $\sum$, the second substitutes the induction hypothesis and the last is algebra. And what is proven is $P(n+1)$, which is that $\sum_{i=0}^{n+1} i = \frac{(n+1)(n+2)}{2}$.

Another and equivalent view of the natural numbers are that there are the numbers 0 to n and then there is $n+1$. The *strong induction principle* is that

$$\forall n \in \mathbb{N}, P(n) \equiv P(0) \wedge \forall n, ((\forall k \leq n) P(k)) \implies P(n+1).$$

Here the induction hypothesis is that $P(k)$ is true for all values $k \leq n$. To prove that every natural number $n \geq 2$ can be written as a product of primes, we take the base case as $P(2)$ which can be written as 2, which is a product of a prime. And for any n, if it is prime, it can be written as itself, otherwise $n = ab$ and by the inductive hypotheses $P(a)$ and $P(b)$ is that each can be written as a product of primes. Thus, we can write n as the product of the primes in both a and b. Note here that the base case starts at 2, which illustrates that one chose the base case as is relevant to the statement being proven.

Strengthening the induction hypothesis is a technique proves a stronger theorem. The example, the notes consider the theorem *The sum of the first n odd numbers is a perfect square.* In fact, the notes inductively prove the stronger theorem *The sum of the first n odd numbers is* n^2. Here, the stronger inductive hypothesis allows the induction step to proceed easily.

1 Natural Induction on Inequality

Note 3

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1+x)^n \geq 1+nx$.

base case: $n=0$: $1 \geq 1$ ✓

induction hypothesis: $n=k$, $(1+x)^k \geq 1+kx$

induction steps: $(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x)$

$= 1 + x + kx + kx^2 \geq 1 + (k+1)x$

$\square$

2 Make It Stronger

Note 3

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

$a_1 \leq 3^{(2^1)} = 9$ ✓

$n=k$: $a_k \leq 3^{(2^k)}$

$n=k+1$: $a_{k+1} = 3a_k^2 \leq 3 \cdot (3^{2^k})^2 = 3 \cdot 3^{2^{k+1}}$, there's a additional 3!

(b) Try to instead prove the statement $a_n \leq 3^{(2^n-1)}$ using induction.

$a_1 = 1 \leq 3^{2^1-1} = 3$ ✓

$n=k$: $a_k \leq 3^{(2^k-1)}$

$n=k+1$: $a_{k+1} = 3 \cdot a_n^2 \leq 3 \cdot (3^{2^k-1})^2 = 3 \cdot 3^{2^{k+1}-2} = 3^{2^{k+1}-1}$ ✓

(c) Why does the hypothesis in part (b) imply the overall claim?

Because $3^{(2^n-1)} \leq 3^{(2^n)}$

(b) $\Rightarrow$ overall claim

3 Binary Numbers

Note 3

Prove that every positive integer n can be written in binary. In other words, prove that for any positive integer n, we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

for some $k \in \mathbb{N}$ and $c_i \in \{0, 1\}$ for all $i \leq k$.

$n = 1$ ✓

induction hypothesis. $n = 1 \cdots k$, n could be write as binary

Induction steps. $n = k+1$: ① if $2 \mid k+1$,

then $\frac{k+1}{2} = C_m 2^m + C_{m-1} 2^{m-1} + \cdots + C_0 2^0$. ($\frac{k+1}{2} \leq k$)

$k+1 = \frac{k+1}{2} \times 2 = C_{m+1} 2^{m+1} + C_m 2^m + \cdots C_1 2^1 + 0 \cdot 2^0$

② if $2 \nmid k+1$

then $\lfloor \frac{k+1}{2} \rfloor = C_m 2^m + \cdots + C_0 2^0$ ($\lfloor \frac{k+1}{2} \rfloor \leq k$)

$k+1 = \lfloor \frac{k+1}{2} \rfloor \times 2 + 1 = C_{m+1} 2^{m+1} + \cdots C_1 2^1 + 1 \cdot 2^0$

4 Fibonacci for Home

Note 3

Recall, the Fibonacci numbers, defined recursively as

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$.

Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

To prove F_{3n} is even ($n = 1, 2, 3 \cdots$)

$n = 1$: $F_3 = F_1 + F_2 = 2$ is even

$n = k$: F_{3k} is even

$n = k+1$: $F_{3(k+1)} = F_{3k+3} = F_{3k+2} + F_{3k+1}$

$= F_{3k+1} + F_{3k} + F_{3k+1}$

$= 2F_{3k+1} + F_{3k} = 2a + 2b = 2(a+b)$

$=$ even + even = even ✓