CS 70 Spring 2025 — Discrete Mathematics and Probability Theory — Rao — DIS 2A

Graph Theory I

Note 5

A graph $G = (V,E)$ consists of a set of vertices V and a set of pairs of vertices $(u,v) \in E$ with $u,v \in V$. In a directed graph, an edge $(u,v) \in E$ is directed from u to v. In an undirected graph the pair is unordered. Unless otherwise specified, graphs in this class are undirected and simple (no self-loops or multiple edges).

Degree: An edge (u,v) is incident to u and v. The degree of a vertex v is the number of edges incident to it, denoted $\deg(v)$.

相邻

Handshaking Lemma: $\sum_{v\in V} \deg(v) = 2|E|$. The total number of edge vertex incidences is the sum of the degrees by definition of degree, and also twice the number of edges as each edge is incident to 2 vertices. It's called handshaking since two people participate in a handshake just as two vertices are incident to an edge. 无向图

Connected: (u,v) are connected in $G = (V,E)$ if there is a path between u and v. Formally, there is a sequence of vertices $u = v_0, \ldots, v_k = v$ where successive vertices are in an edge, i.e., $(v_i, v_{i+1}) \in E$. A graph is connected if all pairs of vertices are connected.

Bipartite graph: A graph G with two groups of vertices such that all edges are incident to one vertex in each group. 二分图

Tree: A graph is a tree iff it satisfies any of the following:

① • connected and acyclic
② • connected and has $|V|-1$ edges
③ • connected, and removing any edge disconnects the graph
④ • acyclic, and adding any edge creates a cycle

① ⇒ ② : $n=1$ ✓

$n=k$ 时 ✓

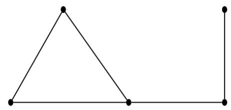
$n=k+1$

1 Degree Sequences

Note 5

The *degree sequence* of a graph is the sequence of the degrees of the vertices, arranged in descending order, with repetitions as needed. For example, the degree sequence of the following graph is $(3,2,2,2,1)$.

$\sum \deg = 2 \cdot |E|$.



(d): 2 nodes have edges to every node.

So couldn't has a node with deg 1.

For each of the parts below, determine if there exists a simple undirected graph G (i.e. a graph without self-loops and multiple-edges) having the given degree sequence. Justify your claim.

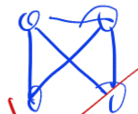
(a) $(3,3,2,2)$

(b) $(3,2,2,2,2,1,1)$

(c) $(6,2,2,2)$

(d) $(4,4,3,2,1)$

(a)



✓

(c) only 4 vertices, impossible to have 6 as deg ree

(b) $\sum \deg = 3 + 2 \times 4 + 2 = 13$ is odd $\neq 2|E|$

the num of odd degree is even

2 Build-Up Error?

Note 5

What is wrong with the following "proof"? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger of build-up error.

False Claim: If every vertex in an undirected graph with $|V| \geq 2$ has degree at least 1, then it is connected.

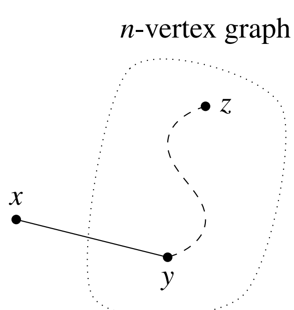
Proof? We use induction on the number of vertices $n \geq 2$.

Base case: The only valid graph has two vertices joined by an edge. This graph is connected, so the base case is true.

Inductive hypothesis: Assume the claim is true for some $n \geq 2$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.

$\boxed{\text{"shrink down grow back"}}$



wrong direction!

(hypothesis to step)

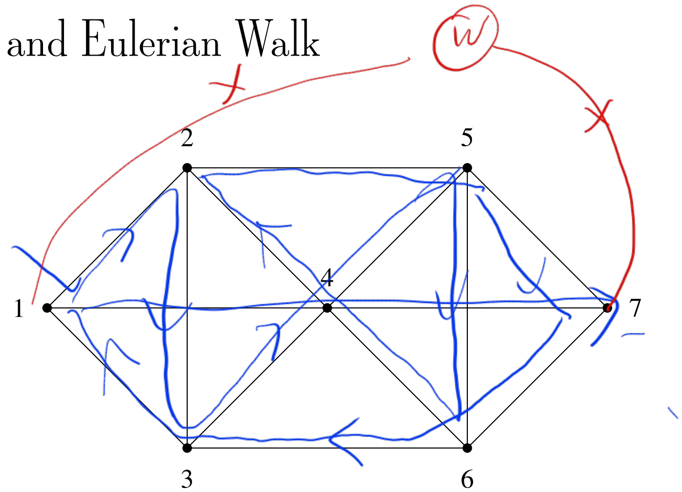
we should construct hypothesis from step

All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$. $\square$

every edge exactly once

3 Eulerian Tour and Eulerian Walk

Note 5



(a) Is there an Eulerian tour in the graph above? If no, give justification. If yes, provide an example.

No, 1 and 7 have odd degree

(b) Is there an Eulerian walk in the graph above? An Eulerian walk is a walk that uses each edge exactly once. If no, give justification. If yes, provide an example.

Yes $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 4 \to 2 \to 6 \to$
$3 \to 1 \to 4 \to 7$

(c) What is the condition that there is an Eulerian walk in an undirected graph? Briefly justify your answer.

connected and there are only 2 odd degree vertices, and the other are even

or all are even

证法:
① 补一个连接两个奇度数的节点
② 类似 note

iff: (except for isolated vertices)
(connected and at most 2 odd degree vertices, since there's no graph with one degree)

$|\text{Leaves}| = L$

4 Coloring Trees

$n = |V|$

(度为1的节点 ≥ 2个)

Note 5

(a) Prove that all trees with at least 2 vertices have at least two leaves. Recall that a leaf is defined as a node in a tree with degree exactly 1.

a tree has $n-1$ edges

$\sum \text{degree} = 2(n-1)$

Suppose there are only < 2 leaves

then degree $\geq 1 + (n-1) \times 2 = 2n-1$, contradict

or:

$$\sum \deg = 2(n-1) = \sum_{\text{leaves}} \deg + \sum_{\text{not leaves}} \deg \geq |L| + (n-|L|)\cdot 2 = 2n - |L|$$

$\Rightarrow |L| \geq 2$ ✓

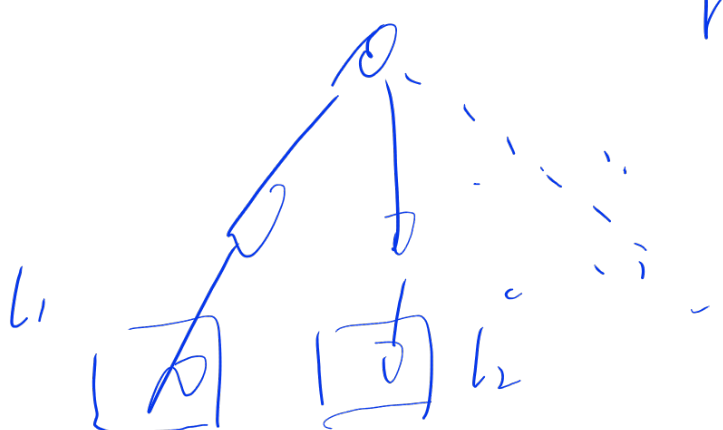
(b) Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]

$n=2$, $n=3$ ✓ (or proof by $k \Rightarrow k+1$) ✓

Suppose $n \leq k$ ✓

when $n = k+1$: choose 2 leaves delete them, color the rest, since the two leaves is not incident, it could be easily colored. ✓



if l_1 connect to $v \in L$ then $l_1 \in R$

if l_2 connect to $v \in R$ $l_2 \in L$