Interest Rate Curves in DeFi Lending Markets: Total Borrows Matters More than Utilization

The RiskDAO

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1 Introduction

DeFi lending pools (e.g., Aave, Compound and Euler) decide on the borrower interest rate of an asset according to the asset utilization (the ratio between asset borrow and supply). The interest rate curve serves two purposes: (i) Mitigate high utilization, which makes it harder for suppliers to withdraw their deposits, and for liquidators to liquidate a position; and (ii) Optimize the protocol KPIs, e.g., revenues and/or liquidity. In this writeup, we focus on the second objective and argue that the current way interest rate curves are setup is sub-optimal. We assume the demand for supply and borrow can be approximated by linear functions and prove that under this assumption:

1. The optimal (borrow) interest rate is such that the borrow amount is exactly half of the borrow demand that exists when the interest rate is 0.
2. A protocol is more likely to reach the optimal interest rate if it set a constant interest rate function, rather than a function that linearly increases with utilization. Moreover, a protocol is even more likely to reach the optimal interest rate if the interest rate function linearly grows with the borrow size (independently from utilization size).

In the next section we prove these assertions. After that we give some additional intuition about our mathematical model, with the help of few examples. And in the last section we discuss the limitations of the model.

In this paper, we ignore the risks of high utilization, but we conjecture that similar conclusions would hold by introducing a kink utilization threshold at which interest rate does become dependent on utilization.

2 Mathematical model

A pooled lending market consists of *suppliers* who deposit funds into a pool, and *borrowers* who borrow funds from the pool (and possibly provide a collateral as a backing to their debt). In this setup, borrowers pay an interest rate, and the paid interest rate is distributed among all suppliers. Formally, we denote the total supplied amount by S, and the total borrowed amount by B. Let I_b be the interest rate that is paid by the borrowers, and let I_s be the interest rate that the suppliers receive, then it always holds that:

$$\frac{I_s}{I_b} = \frac{B}{S} \tag{1}$$

We assume that when the interest rate is 0, then the demand to supply is 0, and it linearly increases as the interest rate goes up. Formally:

$$S(I_s) = a \cdot I_s \tag{2}$$

We further assume that the demand for borrow linearly decrease as the interest rate goes up.

$$B(I_b) = c - b \cdot I_b \tag{3}$$

Where c is the borrow demand when the interest rate is 0. Recall that by definition

$$u = \frac{I_s}{I_b} = \frac{B(I_b)}{S(I_s)} \tag{4}$$

hence, we get

$$\frac{c - bI_b}{aI_b u} = u \tag{5}$$

and that

$$I_b = \frac{c}{au^2 + b} \tag{6}$$

As $I_s = I_b \cdot u$ we get

$$I_s = \frac{c \cdot u}{a \cdot u^2 + b} \tag{7}$$

Corollary 0.1. *The supply interest rate is maximal when* $u = \sqrt{b/a}$.

Proof. Calculate the derivative of I_s and compare to 0. $\square$

As the demand for supply is correlated with the supply interest rate, we get that the highest supply is obtained when $u = \sqrt{b/a}$. Further, the protocol revenues are in the form of certain % of the supply interest rate. Hence, maximizing the value of the interest rate and the size of the supply results in maximal revenues to the protocol.

Putting Corollary 0.1 and Equations 3 and 6 together we get:

Corollary 0.2. *At the optimal state,* $I_b = \frac{c}{2b}$, $B = c/2$.

When a DAO decides on an interest rate curve it chooses two parameters m, n such that the interest rate for utilization u is $I_b(u) = m \cdot u + n$. In order to reach the optimal interest rate it must hold that

$$m \cdot \sqrt{b/a} + n = \frac{c}{2b} \tag{8}$$

In practice however, it is very hard to asses the real values of a, b, c, and therefore we assume a value of $\sqrt{b/a} + \epsilon$ and $\frac{c}{2b} - \epsilon$ are estimated instead. Under this assumption we get that the selected m, n satisfy:

$$m \cdot (\sqrt{b/a} + \epsilon) + n = \frac{c}{2b} - \epsilon \tag{9}$$

From Equation 9 we get that the interest rate curve that was set by the DAO would miss the target interest rate by $-\epsilon(1 + m)$. Clearly this is minimal when $m = 0$, and the following conclusion holds:

Corollary 0.3. *If some partial information about the supply and borrow demand exists, and the protocol interest rate function is of the form of $m \cdot u + n$, then a better approximation is obtained when $m = 0$.*

Next, we consider an interest function of the form of $I = m \cdot B$, where B is the total borrow size. Recall that at the optimal interest rate we have $I = c/2b$ and $B = c/2$. Hence, an optimal interest rate is obtained for $m = 1/b$. For a constant interest rate function one has to estimate the value of $c/2b$, while for $I = m \cdot B$ function one has to only approximate the value of $1/b$, we get

Corollary 0.4. *If some partial information about the supply and borrow demand exists, then an interest rate function of the form of $I = m \cdot B$ would better approximate the optimal interest rate w.r.t a constant function.*

Corollaries 0.1, 0.3 and 0.4 prove the main assertions of this paper. In the next section we give additional intuition about the mathematical model, and in the last section we discuss its limitations.

3 Example

In our example we have the following supply demand

$$S(I_s) = 6 \cdot I_s \tag{10}$$

and the following borrow curve

$$B(I_b) = 20 - 5 \cdot I_b \tag{11}$$

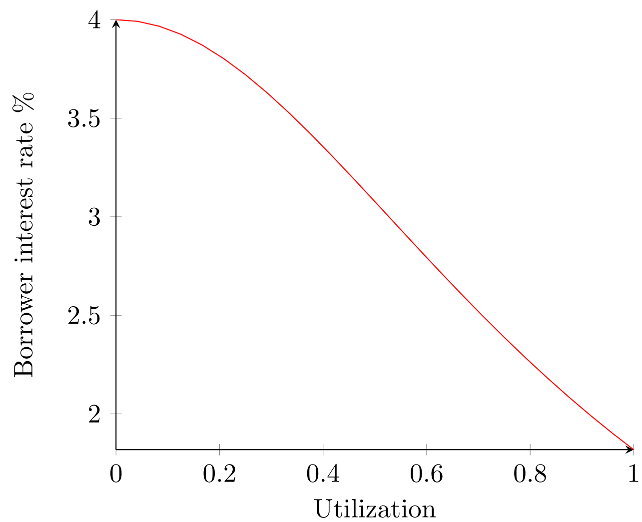
Recall that $u = \frac{I_s}{I_b} = \frac{B(I_b)}{S(I_s)}$, hence, we get

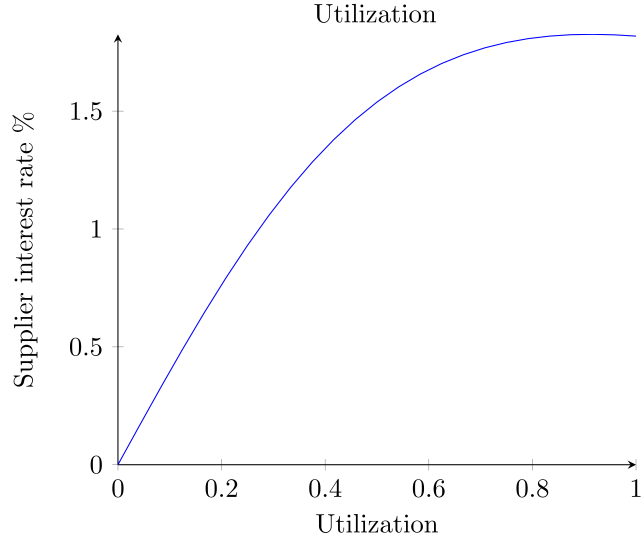
$$20 - 5I_b = 6I_b u^2 \tag{12}$$

and therefore

$$I_b = \frac{20}{5 + 6u^2} \tag{13}$$

$$I_s = \frac{20u}{5 + 6u^2} \tag{14}$$



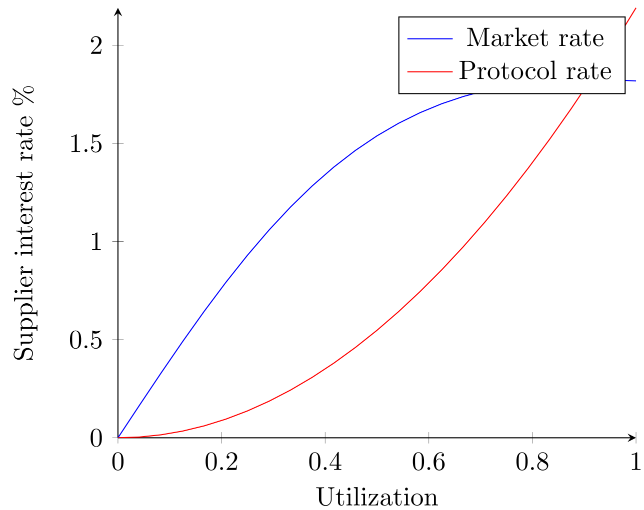


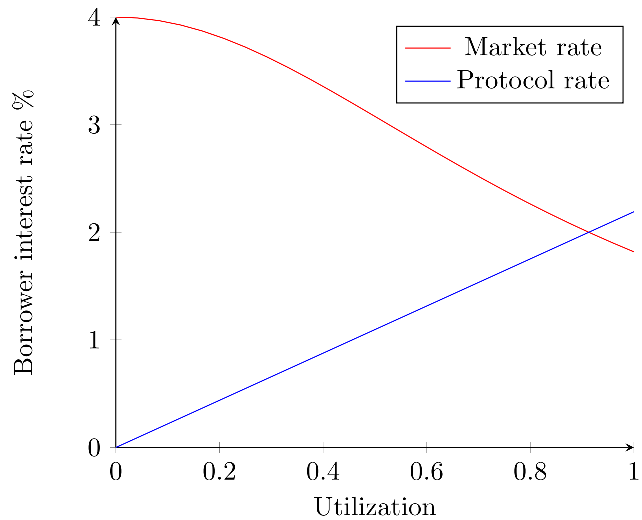
The supply interest rate get a maximal value at $u = \sqrt{5/6}$, and it is an easy observation that the supply amount is maximal when the supply interest rate is maximal. Further, as the protocol takes fees in the form of % over the supply interest rate, the protocol revenues are also maximized when $u = \sqrt{5/6}$. However, the utilization will reach this point only if the protocol interest rate for such utilization will be equal to

$$I_b(\sqrt{5/6}) = \frac{20}{5 + 6 * \sqrt{5/6}^2} = 2$$

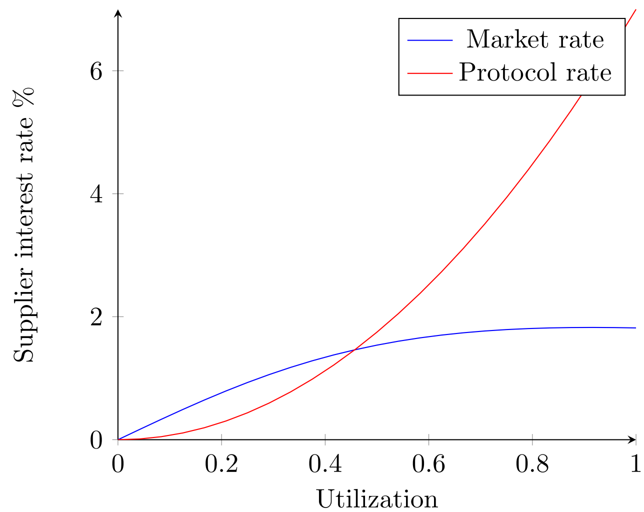
. If the protocol selects a linear interest function, it will have the following interest rate curve:

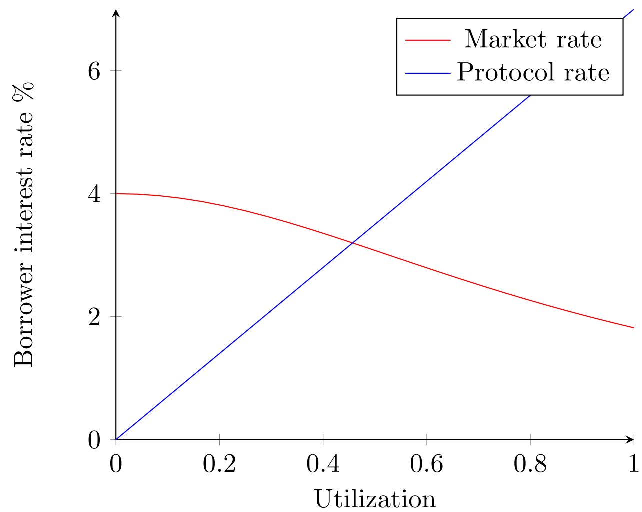
$$I_b(u) = 2 \cdot \sqrt{6/5} \cdot u \tag{15}$$



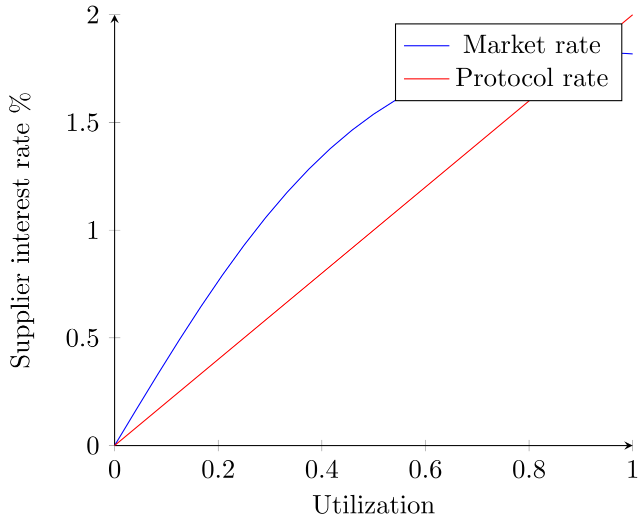


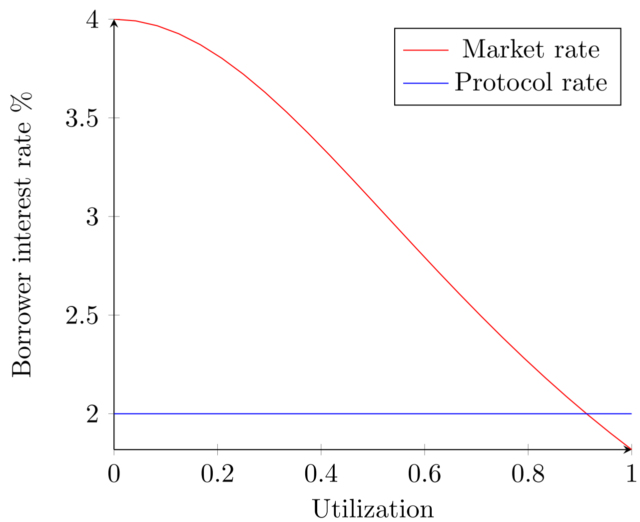
However, if the interest rate curve would have been $I_b(u) = 7 \cdot u$ we would get:



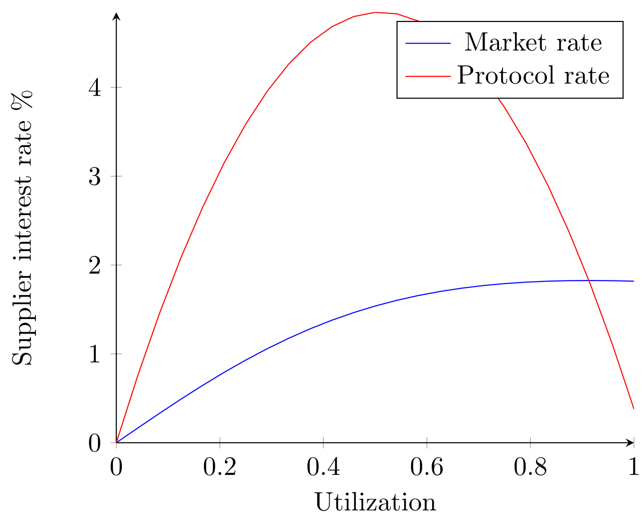


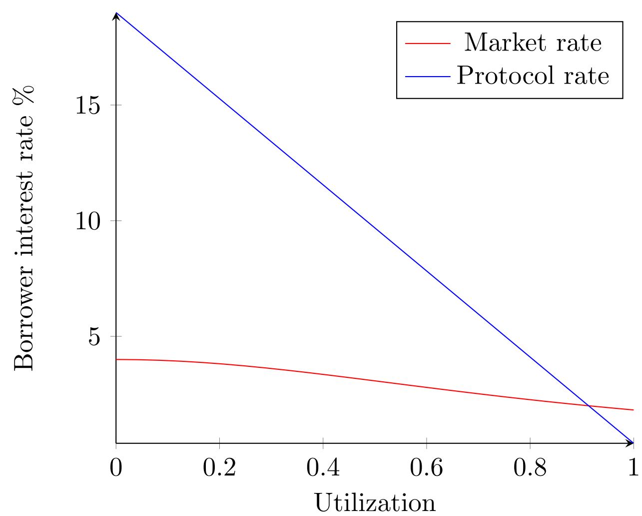
We note that the market dynamic is such that when the borrow interest rate is lower than the market expectations (and equivalently the supply interest rate is also lower) then the utilization will increase, as more borrowers will borrow (or alternatively suppliers will withdraw their deposits). However, when the borrow (supply) interest rate is higher then borrowers will repay their debt (suppliers will deposit more) and the utilization will decrease. Hence, the intersection of the protocol interest rate with the expected interest rate is the equilibrium point. However, the same equilibrium would have been reached with $I_b(u) = 2$





An intersection at the same point would also hold if we select a monotonically decreasing function, namely, $I_b(u) = 19 - 17 \cdot \sqrt{6/5} \cdot u$





However, in this case, the protocol interest rate is initially higher than the market demand, hence the equilibrium will be formed at the initial utilization 0. Unless, from some reason, the system will start in a condition where the utilization is $\sqrt{5/6}$.

4 Non linear model

In the most general model, we have a $B(I_b)$, $S(I_s)$ and $I_s = I_b \cdot u$. In the reminder of the section we write I instead of I_b. The only equation that holds is

$$\frac{B(I)}{S(I \cdot u)} = u \tag{16}$$

Hence,

$$B(I) = S(I \cdot u) \cdot u \tag{17}$$

and if we multiply both sides of the equations by I we get that

$$B(I) \cdot I = S(I \cdot u) \cdot I \cdot u \tag{18}$$

Recall that we aim to maximize I_s, meaning to maximize $I \cdot u$. As S is monotonically non decreasing function, a maximal value for $I \cdot u$ is obtained when a maximal value for $S(I \cdot u) \cdot I \cdot u$ is obtained. By Equation 18 we get that our goal is to find a value for I that maximizes $B(I) \cdot I$. If $B(I)$ is differentiable, then the maximum value is obtained when it's derivative is 0. Recall that

$$\frac{d(B(I) \cdot I)}{dI} = B(I) + B'(I) \cdot I \tag{19}$$

And thus, the maximum value is obtained when

$$I = -\frac{B(I)}{B'(I)} \tag{20}$$

We denote by I^* the optimal I, for which Equation 20 holds. I.e.,

$$I^* = -\frac{B(I^*)}{B'(I^*)} \tag{21}$$

The Taylor expansion of B give us

$$B(0) = B(I^*) - B'(I^*) \cdot I^* + \sum_{n=2}^{\infty} \frac{B^{(n)}(I^*)}{n!}(-I^*)^n \tag{22}$$

From Equation 20 we get that $B(I^*) - B'(I^*) = 2B(I^*)$ and therefore, by Equation 22 we get that

$$B(I^*) = \frac{B(0) - \sum_{n=2}^{\infty} \frac{B^{(n)}(I^*)}{n!}(-I^*)^n}{2} \tag{23}$$

We conjecture that in many cases, the right hand part of the sum is negligible w.r.t the size of $B(0)$, but we leave it to future work to characterise the demand functions that adhere to this conjecture.

5 Simulation model

We build a simulator that takes as input

- Initially supply at time t = 0.
- Arbitrary supply and borrow demand functions (as a function of the supply and borrow interest rates).
- Arbitrary protocol's borrow interest function, as a function of current supply and borrow sizes.

The simulator outputs the market dynamics by repeating the repeating the following iteration steps:

1. Calculate the current borrow interest rate, given current supply and borrow, and the predefined protocol's interest rate.
2. If the demand for borrow, according to the protocol's interest rate, is higher than current borrow, increase the borrow, until the demand matches the new interest rate.
3. If the demand for supply, according to the protocol's interest rate, is higher than current supply, increase the supply, until the demand matches the new interest rate.
4. If supply or borrow were increased in the last iteration, go to step 1.

It is an easy observation that in each iteration, the supply and borrow numbers that are obtained adhere to the demand curve and to the protocol predefined interest function. Further, if at a certain iteration, the supply and borrow did not increase, then an equilibrium point was reached, as neither suppliers, nor borrowers have the incentive to unilaterally increase their supply or borrow.

To handle arbitrary input functions, the simulator incrementally changes the supply and borrow sizes with small steps, until intersection point is met. It outputs the equilibrium result, and show the distance from that optimal outcome, had a different interest function would have been chosen.

6 Limitations of the model

The model assume linear demand curves for supply and borrow. While surely the demand for supply (borrow) does not decrease (increase) as the interest rate goes up, in practice it might resembles a step function, with sharp increases whenever the offered interest rate is higher than what the competitive protocols offer. A linear approximation would still have some merit in this case, however the results will be less granular.

Our analysis shows the optimal interest rate is independent of the supply demand curve. However it relies on the result that the optimal utilization is $\sqrt{b/a}$. We note that when $b > a$ the result is bigger than 1, which is not possible. In such case the optimal utilization is 1, and different results are obtained.

The demand curves for supply and borrow could change over time, and we didn't discuss the best mechanisim to adjust the protocol interest rate curve.

Our analysis ignores the negative consequences of high utilization, however we believe high utilization can be mitigate in similar ways it is mitigated today, namely, to decide the interest rate according to a different curve, once high utilization is reached.