Optimizing Multi-Hop Liquidity in Constant Product AMM

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1 Introduction

DEX liquidity is a key factor in assessing the loan to value ratio (LTV, aka liquidation threshold or collateral factor) of a collateral asset. Hence, it is sometime possible to improve the LTV of an asset by increasing its DEX liquidity, e.g., by providing liquidity mining incentives and/or using the project treasury.

In a multi-collateral multi-debt lending markets such as Aave and Compound v2, it is often required to improve the token's liquidity not only vs ETH or USDC, but also vs another token.

We recently launched a DEX liquidity recommendation system, which shows the % increase that is needed to facilitate higher LTVs. While the interpretation of the recommendation is straight forward when, e.g., the token is traded only vs ETH, and only the liquidity vs ETH needs to be boosted, it becomes more complex when the two tokens are not traded directly towards one another.

In this write-up we show how token to token liquidity can be increased in an optimal way, with respect to additional needed capital.

In the next section we bring the necessary definitions for liquidity and constant product AMMs, and then we prove that the best recipe for liquidity increase.

2 Definitions

Slippage. A slippage for a qty q is the change of price that results when selling a qty q of certain src token to certain dest token in the market. Formally, for a token x and y, the price of a trade of input qty Δx to output qty Δy is $P(\Delta x) = \frac{\Delta y}{\Delta x}$. The slippage for a quantity q is defined as $P(1)/P(q) - 1$.
Liquidity. For a given threshold, e.g., 5%, we the liquidity of two token pair is the maximum quantity with slippage smaller than the threshold (e.g., slippage < 0.05).
Constant product AMM. In a constant product AMM with inventory size x and y (for x token and y token respectively), the trade price is dictates by the formula $(x + \Delta x)(y - \Delta y) = xy$.

3 Slippage in constant product AMMs

In $xy = k$ AMMs, the price for 1 (wei) is

$$P(1) = y/x \tag{1}$$

By the constant product invariant we have:

$$(x + \Delta x)(y - \Delta y) = xy \tag{2}$$

and therefore:

$$\Delta y = y - \frac{xy}{x + \Delta x} = \frac{y\Delta x}{x + \Delta x} \tag{3}$$

Recall that $P(\Delta x) = \Delta y/\Delta x$ hence:

$$P(\Delta x) = \frac{y}{x + \Delta x} \tag{4}$$

From Equations 1 and 4 we get:

$$\frac{P(1)}{P(\Delta x)} = \frac{x + \Delta x}{x} = 1 + \frac{\Delta x}{x} \tag{5}$$

And finally, we get that the slippage for qty q is

$$\textit{slippage(q)} = q/x \tag{6}$$

where x is the current amount of balance of token x (also note that the USD value of x and y is the same, so this formula is symmetric for x and y values).

Hence, a simple corollary is that if one wants to double the liquidity for a given AMM reserve and a given slippage, then one needs to double the balance of x and y, and would need and additional funds in the equivalent amount of $2x$ (or equivalently $2y$) for that.

4 Multi-hop DEX.

Consider the case where we have token A that has an AMM for A/ETH pair, and a token B with an AMM for B/ETH pair. In such a scenario, in order to sell token A to token B, one first must sell A to ETH, and then sell the ETH to B. As in the previous section, doubling the sizes of the two AMMs will double the liquidity. However, in many cases, there is a more capital efficient way to do it.

Let x_1 and y_1 be the token A balance and ETH balance in the AMM for token A to ETH, and x_2 and y_2 be the ETH balance in the token B balance in the Token B to ETH AMM. Then a trade from token A adds Δx_1 to the first AMM, in return the Δy_1 of ETH. Then adds $\Delta x_2 = \Delta y_1$ to the second BAMM, and outputs Δy_2 of token B.

From the $xy = k$ invariant of Equation 2 we get

$$(x_1 + \Delta x_1)(y_1 - \Delta y_1) = x_1 y_1 (x_2 + \Delta x_2)(y_2 - \Delta y_2) = x_2 y_2 \tag{7}$$

Hence

$$\Delta y_1 = \frac{y_1 \Delta x_1}{x_1 + \Delta x_1} \Delta y_2 = \frac{y_2 \Delta x_2}{x_2 + \Delta x_2} \tag{8}$$

As $\Delta x_2 = \Delta y_1$ we get

$$\Delta y_2 = \frac{y_1 y_2 \Delta x_1}{x_2(x_1 + \Delta x_1) + y_1 \Delta x_1} = \frac{y_1 y_2 \Delta x_1}{x_2 x_1 + \Delta x_1 (y_1 + x_2)} \tag{9}$$

Moreover

$$P(\Delta x_1) = \frac{\Delta y_2}{\Delta x_1} = \frac{y_1 y_2}{x_! x_2 + \Delta x_1 (y1 + x_2)} \tag{10}$$

The price for the basic unit $P(1)$ is the price of A to ETH times the price of ETH to B, hence:

$$P(1) = y_1/x_1 \cdot y_2/x_2 \tag{11}$$

From Equations 10 and 11 we get:

$$\frac{P(0)}{P(\Delta x_1)} = 1 + \Delta x_1 (\frac{y_1}{x_1} \cdot 1/x_2 + 1/x_1) \tag{12}$$

Hence

$$\textit{slippage(q)} = q(\frac{y_1}{x_1} \cdot 1/x_2 + 1/x_1) \tag{13}$$

Finally, as $\frac{1}{x_1} = \frac{y_1}{y_1 x_1}$ we get

$$\textit{slippage(q)} = q\frac{y_1}{x_1}(1/x_2 + 1/y_1) \tag{14}$$

Where y_1 is the ETH balance of the first AMM and x_2 is the ETH balance of the second AMM. Note that $\frac{y_1}{x_1}$ is simply the market price of token A wrt ETH, and it does not depend on the size of the AMM. Hence, to maximize the liquidity (and minimize the slippage) we wish to minimize $(1/x_2 + 1/y_1)$.

For this purpose, we assume we have a budget of β ETH. And we decide to add α ETH to x_2 and the reminder to y_1. And we want to find the α that minimizes

$$\frac{1}{x_2 + \alpha} + \frac{1}{y_1 + \beta - \alpha} \tag{15}$$

subject to $0 \leq \alpha \leq \beta$. Comparing the derivative to 0, we get that the minimal value is obtained at

$$\alpha = \frac{y_1 - x_2 + \beta}{2} \tag{16}$$

For this value of α we get that the ETH balance of both reserves is equal. However, this is only possible to obtain under the constrain that $0 \leq \alpha \leq \beta$. In particular, if $x_2 \geq y_1 + \beta$, then the minimum is obtained for $\alpha = 0$, namely, when all the funds are added only for the second reserve. Similarly, if $y_1 \geq x_2 + \beta$, then the minimum is obtained when all the funds are added only for the first reserve.

5 Numerical Example

Consider two reserves, a reserve for token A vs ETH, with x and y balance of 1000 ETH each, and a reserve for token B vs ETH, with x and y balance of 100 ETH each. Hence $\frac{1}{x_2} + \frac{1}{y_1} = 1/100 + 1/1000 = 0.011$. If we want to improve the liquidity by a factor of 2, then in the naive way we would double the balances in the two reserves, which would require additional $(2 \cdot 100 + 2 \cdot 1000) = 2200$ ETH, and we would get $\frac{1}{x_2} + \frac{1}{y_1} = 1/200 + 1/2000 = 0.0055$.

Instead, we solve the equation

$$\frac{1}{100+\alpha} + \frac{1}{1000+\beta-\alpha} = 0.0055 \tag{17}$$

And get that the above holds for $\beta = \alpha = 122.222222$. Hence, by adding $2 \cdot 122.222222 \approx 245$ ETH we get the same token A to token B liquidity as we would get by naively adding 2200 ETH.