A Game Theoretic Model of Chicken Bonds Dynamic

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1 Introduction

Chicken bonds were recently introduced by [1] as a novel mechanism to bootstrap liquidity. At its core it allow users, through a sequence of actions, to *bond* their TKN into $bTKN$, that generates higher yield. The manuscript of [1] precisely depict the Chicken Bonds mechanism, however the fair market price of $bTKN$ is left as an open question for future research.

In this paper, we use a game theoretic framework to answer this question (subject to the some relaxing assumption). Our framework assumes fully rational agents (players), who only aim to maximize their profits, and a fully liquid world, e.g., the agents can always get a hold of additional TKN, if needed. We then show an equilibrium state in which the agents never redeem or sell their $bTKN$, and also never chicken out. Further, in this equilibrium, all bonding operations are done in a single time point. In such equilibrium, it is possible to reason about the $bTKN$ supply, and the sizes of the different buckets. And thus, we get an recipe to calculate the fair price of $bTKN$. In addition, we get a framework to reason about how new equilibrium would be formed in the event that one of our assumptions temporarily does not hold. E.g., if a user did redeem (due to lack of market liquidity for $bTKN$), chickened out (due to change in market yields), or if the permanent bucket was increased by an external actor.

In the next section we present the formal definitions of our framework, and the assumptions we take. In Section 3 we present a certain dynamic and prove that under these assumptions, this dynamic is an equilibrium. And in Section 4 we show how to calculate the price of $bTKN$ in the obtained equilibrium.

In order to demonstrate the theoretical analysis, we setup a Jupyter Notebook simulation environment to generate example figures and graphs. The simulation code is available at [2].

2 Definitions and Assumptions

Plays, Strategies, and utility function. The set of actions is the allowed operations (bond, chicken-in/out, redeem, sell, and no operation) along with the corresponding quantities for each operation.

A finite (respectively infinite) play is a finite (resp. infinite) sequence of actions for each player (agent).

A player strategy, denoted by σ, is a recipe for the next action for every finite play sequence. I.e., it is a function that decides on the next move of the player, given the history of the play so far.

Given n strategies $\sigma_1, \ldots, \sigma_n$ for n player, denote $\overrightarrow{\sigma} = (\sigma_1, \ldots, \sigma_n)$ to be a *strategy profile*. A *play* according to strategy profile $\overrightarrow{\sigma}$ is a sequence of actions such that in every round, player i plays according to σ_i.

To asses the utility of every player we make the assumption that world is composed on of TKN and $bTKN$ (e.g., assets like USDC or ETH are not applicable), and we further assume there is a global discount factor $d < 1$ such that the utility of holding a TKN in time $t + 1$ is d-times the utility of holding it in time $t + 1$. Further, that the underlying utility of $bTKN$ at the end of the game, is the amount of redeemable TKN it gives.

Formally, we first definite the utility of a game that is played for finite amount of time, and then extend it to infinite rounds game.

Assumption 1. *Let ρ be a finite play of t rounds, according to strategy $\overrightarrow{\sigma}$. And for every player i, let $TKN(i)$ be his TKN balance, and $bTKN(i)$ be his bTKN balance. Further, at the end of the finite play each bTKN is redeemable to $r \cdot TKN$. Then the utility of player i, denoted by $U_i(\overrightarrow{\sigma}, t)$ is*

$$d^t(TKN(i) + r \cdot bTKN(i))$$

And for an infinite play the utility is

$$\lim_{t\to\infty} U_i(\overrightarrow{\sigma}, t)$$

Given a utility function and a strategy profile $\overrightarrow{\sigma}$, we say that $\overrightarrow{\sigma}$ is an *equilibrium* if no player can increase his utility by unilaterally changing his own strategy.

In order to claim that the fair market value of $bTKN$ is the one that is obtained in an equilibrium state we make the next assumption:

Assumption 2. *All players are rational and motivated by profits, and they do not form adversarial coalitions among them.*

The next assumption is needed to simplify our mathematical model:

Assumption 3. *The discount factor d is identical to all players. Further, denote by r the risk free interest rate for TKN, then $d = \frac{1}{1+r}$, and the APY for the pending, reserve and permanent bucket is also r.*

The next assumption suggests that the market is fully liquid

Assumption 4. *It is always possible to borrow TKN with an yearly interest rate of r, without any need for a collateral.*

Our final assumption is that after bonding, there is a single point in time where it is best to chicken in the full bonding amount.

Assumption 5. *Let ρ by a play such that player i bonded M tokens at time t_0 and $q_p = M$ (i.e., he is the only user in the pending bucket), and further, we assume that no other player will ever bond, then there exist a time $t_1 >= t_0$, such that chickening in all of the M tokens in time t_1 maximizes player i utility.*

We note that this assumption is known not to be true in general, and in Section 5 we depict relevant cases where this assumption fully holds or almost holds.

We can replace Assumption 5 by a more relaxed version, namely:

Assumption 6. *There always exist an M and a strategy profile that bonds M TKN at $t = 0$, and chicken in at different time points, and every user does not profit or lose from participating (i.e., the utility is the same as just holding TKN), and moreover, not single player can profit more by changing his chicken in time.*

We note that Assumption 5 implies the correctness of the above assumption (the strategy is to chicken in in t_1), and hence this assumption is more relaxed.

3 An equilibrium strategy profile

In this section, we present a subset of strategy profiles, and rely on the assumptions from Section 2 to prove that this subset contains at least one equilibrium strategy profile. For simplicity, we assume that in the beginning of the play, the pending bucket is empty. Given q_r and q_d, the sizes of the reserve and permanent buckets (respectively), let M be the maximal amount of TKN that can be bonded at time $t = 0$ and then chicken-in (at certain different times) that generate yield (utility) that is not worse than holding the M tokens (and get r interest rate on them).

Let $\overrightarrow{\sigma_M}$ be the infinite set of strategy profiles such that at time $t = 0$ the above happens (M tokens are bonded, and M is maximal), and after that no additional bonding operations are ever done.

We first show that such maximal M exist (i.e., that $\overrightarrow{\sigma_M}$ is not an empty set). Surely, for $M = 0$ the utility of bonding is identical to hold. Hence we need to prove that for a big enough M the utility of bonding is negative.

Claim 1. *For every q_r, q_d, α and $r > 0$, there exist some M such that for every $N > M$ the utility of bonding N tokens, is worse than holding N tokens, regardless to the different chicken-in times.*

Proof. Intuitively, this holds since for a ridiculously big enough M, the APY on M, even for a 1 second bonding time, will be higher than the size of q_d, and if the user will decide to chicken before 1 second elapsed, then he will contribute 99% of his deposit to the permanent bucket, and would lose even more.

We deffer the formal proof of the claim to the appendix. □

Next, we prove that a player cannot unilaterally improve his utility by doing a chicken out operation.

Claim 2. *For every strategy profile in $\overrightarrow{\sigma_M}$ a user a user cannot unilaterally improve his utility by doing a chicken out operation.*

Proof. The user would have profit more by never bonding, in contradiction to the definition of $\overrightarrow{\sigma_M}$. □

We now prove that a player would not benefit by deviating from $\overrightarrow{\sigma_M}$ and doing a redeem operation.

Claim 3. *For every strategy profile in $\overrightarrow{\sigma_M}$ a user a user cannot unilaterally improve his utility by doing a redeem operation.*

Proof. Redeem is only possible when the $bTKN$ supply is not empty, which in turns means that the permanent bucket is not empty. Hence, the interest rate on the redeemable amount is strictly lower than the interest rate of the reserve bucket (as it get additional yield from the permanent bucket). Since we assume that the user can borrow TKN with an interest rate of r, then he would be better of not to redeem, and borrow the TKN instead. □

We show that it does not make sense to do additional bonding at a later time $t > 0$.

Claim 4. *For every strategy profile in $\overrightarrow{\sigma_M}$ a user a user cannot unilaterally improve his utility by doing additional bonding at time $t > 0$.*

Proof. Intuitively, as the strategy profile does not contain a redeem operation, it follows that the size of q_r is only growing, and as the permanent bucket is not empty, it grows faster than $(1+r)^t$. Hence, bonding at a time later than $t = 0$ will result in lower $bTKN$ supply, and the as the APY on $bTKN$ is always higher than on TKN, the proof follows.

We give a formal proof in the appendix. □

Finally, we show that sell and buy operations can be ignored.

Claim 5. *For every strategy profile in $\overrightarrow{\sigma_M}$ two players, p_1 and p_2 cannot unilaterally improve their utility by having player p_1 selling $bTKN$ to player p_2.*

Proof. As this operation is zero sum, if one player strictly profits, it would mean that the other player lose. □

We are now ready to show a specific strategy profile that is an equilibrium.

Claim 6. *Let $\overrightarrow{\sigma_M}^* \in \overrightarrow{\sigma_M}$ be a strategy profile such that the chicken in time t_c is identical to every player and equal to the optimal chicken in time for the play that a single user bonded M tokens, as described in Assumption 5, or the strategy that is described in Assumption 6. Further, M is the maximal amount of tokens such that the utility of every player is identical to the one that is obtained by never bonding. Then $\overrightarrow{\sigma_M}^*$ is an equilibrium.*

Proof. Towards a contradiction, assume there exist a player i that could improve his utility by deviating from the strategy profile. The claims we previously proved suggest the deviation does not include chicken-out, sell, redeem or later bonding operations. Assumption 5 (or Assumption 6 excludes the option to chicken in at a later time. Hence, the only remaining operation is to increase his initial bonding amount at time $t = 0$. As all players bond and chicken in at the same time, then if player i improved his utility, then so did the other players, which contradicts the assumption that M is maximal. □

4 Finding a fair market price

Using the assumptions of 2 and the claims in 3, we can follow this approach to find the fair market price:

1. Given α, q_r, q_d, S and $r > 0$, if we know that additional bonding will never happen, as it is not profitable to bond even for a small amount M, then *fair price* is

$$p_f(t) = \frac{(q_d + q_r) * (1 + r)^t}{S} \text{ for all } t$$

   This stems from the utility definition in Section 2, and indeed the utility in such case, as t approaches infinity is $q_d + q_r$.

2. For the small amount M, we can calculate the optimal chicken in time T^*
3. This will also give us the total supply in bTKN for any time $t > T^*$
4. As we also know that no more bonding/chickening in happens after T^*, we in fact know the fair price per bTKN for any time $t > T^*$
5. Using this, we can derive the current value of our utility/PNL function: if this is larger than 0, we can increase our M and repeat steps 2-4 until we derive at a non-positive value for our utility/PNL function

Given q_r, q_d, S, α and $r > 0$ we will hence derive at the maximum M of claim 1, which then sets our system into equilibrium and hence allows to derive at a fair price p_f.

Note that this uses the assumption that chicken in of parts/multiple bonds adding up to M happen at the same, optimal time. As shown in section 5 , this mustn't always be the case, yet we are able to present cases where this assumption is reasonable, which is also confirmed by simulations that showed that deviating from the optimal chicken in time gives negligible gains in utility for a lot of parametrizations.

4.1 The PNL/utility function

We need to define the utility function for bonding. Every TKN holder will look at their potential profit from entering a bond and eventually chickening in. More specifically, they would want to maximize their present value for their straetgy. Assume we have $\alpha, q_r, q_d, S > 0$ as defined in the paper and the assumptions from section 2. Then the PNL/utility function of this consideration for any given t is:

$$\begin{aligned} PNL_M(t) &= \frac{1}{(1+r)^t}\left(s_M(t)p_{f_M}(t) - M(1+r)^t\right) \\ &= \frac{1}{(1+r)^t}\left(\frac{M}{p_{r_M}(t)}\frac{t}{t+\alpha}p_{f_M}(t) - M(1+r)^t\right) \end{aligned}$$

where

- M the bonded TKN amount, i.e. the size of the pending bucket q_p as defined in the paper
- $p_r(t)$ is the redemption price $p_r = \frac{q_{r_M}(t)}{S_M(t)}$. $q_{r_M}(t)$ is the size of the reserve pool, which is given by $q_r + (q_r + q_d + M) * ((1+r)^t - 1)$ as all returns are accrued into it. $S_M(t)$ is the bTKN supply at time t, which equals $S(0)$ as long as no one chickened in until t
- $p_{f_M}(t)$ is the fair price of bTKN at time t (under the assumption that no further bonding will ever happen), which we can denote as $p_{f_M}(t) = \frac{v_{f_M}(t)}{S_M(t)}$, which is: the present value of all bTKN at time t divided by a function that represents the (future) total supply discounted to time t.
  Given our assumptions $v_{f_M}(t)$ is a perpetual bond with interest rate r. As this is also our discount rate $v_{f_M}(0) = \frac{(q_r+q_d+M)r}{r} = (q_r + q_d + M)$ or in general: $v_{f_M}(t) = (q_r + q_d + M) * (1 + r)^t$.
  This value gets split by an increasing amount of bTKN supply, each time a bonder chickens in, which

is why this supply function $\tilde{S_M}(t)$ is different from $S(t)$ used for $p_r(t)$. More concretely, $\tilde{S_M}(t)$ can be written as sum of the initial bTKN supply S and the chickened in supply $s_M(t)$ times some discount factor $d_{f_M}(t)$:

$$\tilde{S_M}(t) = d_{f_M}(t)(S + s_M(t))$$

This discount factor equals 1 after all bonders have chickened in. Note that this indeed creates a circular dependency: the decision to bond depends on the fair price, which depends on the future supply of bTKN which depends on the supply accrued when bonders chicken in which depends on the decision to bond in the first place.

Using above, if we assume all bonds chicken in at the same time, we can write the present value of our utility function as

$$PNL_M(t) = \frac{s_M(t)(q_r + q_d + M)}{(S + s_M(t))} - M \tag{1}$$

To get a better understanding of the different parameters at work, lets consider a single bond, such that the present value of the PNL function is

$$\begin{aligned} PNL_M(t) &= M\left(\frac{t}{t+\alpha}\frac{S(q_r+q_d+M)}{S\frac{((q_r+(q_r+q_p+M)*((1+r)^t-1))(t+\alpha)+t*M)}{(q_r+(q_r+q_p+M)*((1+r)^t-1)(t+\alpha)}} - 1\right) \\ &= M\left(\frac{t(q_r+q_d+M)}{(q_r+(q_r+q_p+M)*((1+r)^t-1))(t+\alpha)+Mt} - 1\right) \end{aligned}$$

This function always starts negative as $PNL(0) = -M$ and approaches this value for large t:

$$\lim_{t\to\infty} PNL_M(t) = \lim_{t\to\infty} M\left(\underbrace{\frac{t(q_r+q_d+M)}{(q_r+(q_r+q_p+M)*((1+r)^t-1))(t+\alpha)+Mt}}_{\approx\frac{1}{(1+r)^t}\to 0} - 1\right)) = -M$$

For large t the value accrual of bTKN saturates whilst the opportunity cost of earning interest for M TKN continously grows by $(1+r)^t$, which is why the present value eventually approaches M for $t \to \infty$. The question is if the PNL actually ever becomes non-negative, as only then bonding is a reasonable option. This depends obviously on M, α and r. The other parameters are the ratio of q_r to M and $\frac{q_d}{M}$, which we see when rewriting the PNL using $Q := q_r + q_d + M$:

$$PNL_M(t) = M\left(\frac{tQ}{(q_r + Q*((1+r)^t-1))(t+\alpha)+Mt} - 1\right) \tag{2}$$

Extracting M in numerator and denominator we get:

$$\begin{aligned} PNL_M(t) &= M\left(\frac{\frac{Q}{M}t}{M\left(\left(\frac{q_r}{M}+\frac{Q}{M}((1+r)^t-1)\right)(t+\alpha)+t\right)} - 1\right) \\ &= M\left(\frac{t\left(\frac{q_r}{M}+\frac{q_d}{M}+1\right)}{\left(\frac{q_r}{M}(1+r)^t+(\frac{q_d}{M}+1)((1+r)^t-1)\right)(t+\alpha)+t} - 1\right) \end{aligned}$$

We will use $\alpha = 0.1, r = 0.05, q_r = 10000, q_d = 5000$ and $M = 1000$ if we don't change the parameter:

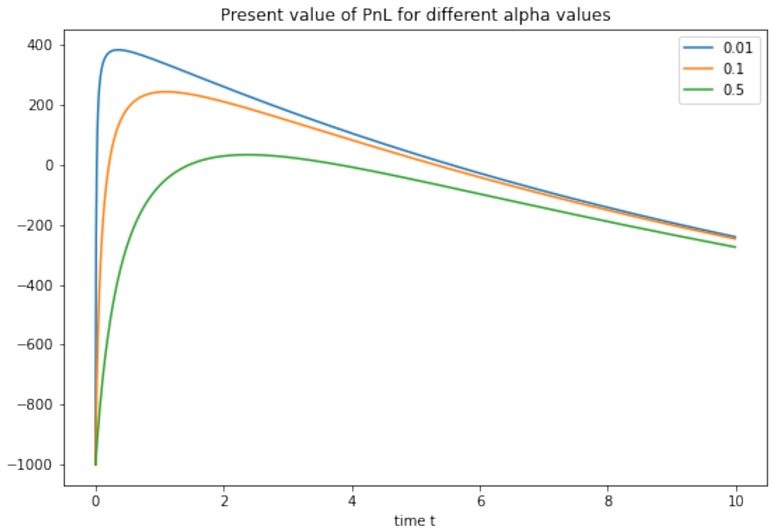

Figure 1

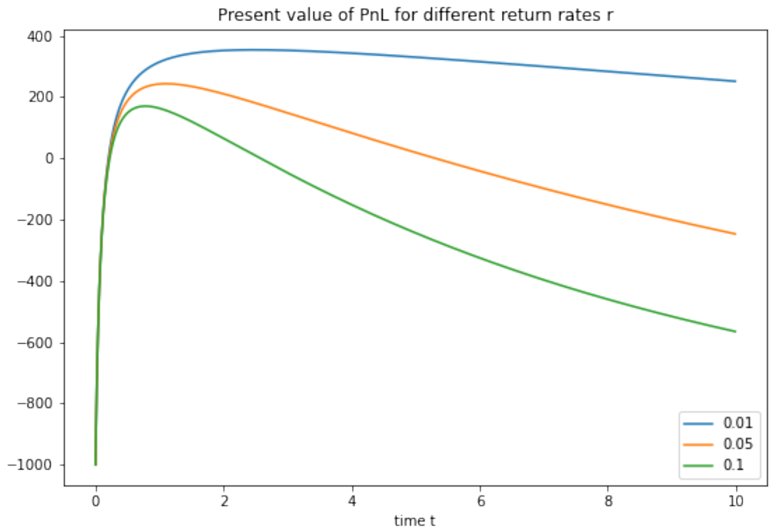

Figure 2

Higher α means the increase of bTKN accrual saturates faster - consequently, we see lower maximal gain when bonding and chickening in at t, in the case of $\alpha = 0.5$ (green line) we barely even reach a positive value.

Higher return rates r increase the denominator of our PNL function and hence decrease our PNL function as visible in Figure 2. The different M values in figure 3 are shown relative to the sum of reserve and permanent bucket, i.e. $q_r + q_d = 15000$.

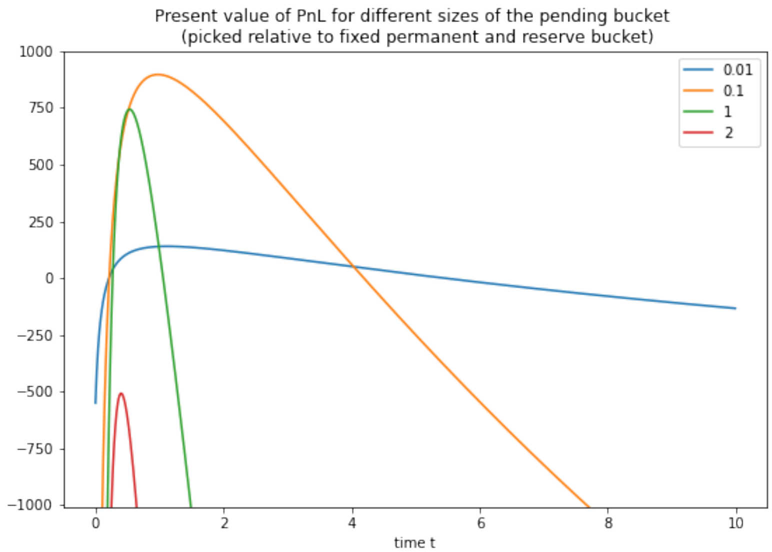

Figure 3

Large amounts of M increase the cost of bonding whilst also increasing the accrued value of bTKN. As stated in 1 at some point that must lead to situations like the red curve, i.e. the present value staying negative for all t, but conversely it doesn't mean that smaller M will result in higher present value as we see when comparing the other three curves.

When the reserve is large in comparison to the bonded amount M, we increase the denominator of our PNL function and hence see lower/flatter curves in figure 4. Lastly, higher ratios of the permanent bucket to the bonded amount help increasing the PNL values, which we see in figure 5.

4.2 The optimal chicken in time

One candidate for the optimal chicken in time is the local maxium of our present value PNL function, which we can calculate via its derivative. We use the notation of 2:

$$\frac{dPNL_M(t)}{dt} = MQ * \frac{\alpha * (q_r + Q * ((1+r)^t - 1)) - Qt(t+\alpha)(1+r)^t \ln(1+r)}{((q_r + Q * ((1+r)^t - 1))(t+\alpha) + Mt)^2} \tag{3}$$

We can find the root numerically solving the equation

$$\alpha * (q_r + Q * ((1+r)^t - 1)) - Qt(t+\alpha)(1+r)^t \ln(1+r) = 0$$

When we assume small r we can find an analytical solution using the approximations $\ln(1+r) \approx r$ and $(1+r)^t \approx 1 + rt$, which is the root of

$$t^3 + \frac{t^2}{r} * (1 + \alpha r) - \frac{q_r}{Q} \frac{\alpha}{r^2} = 0$$

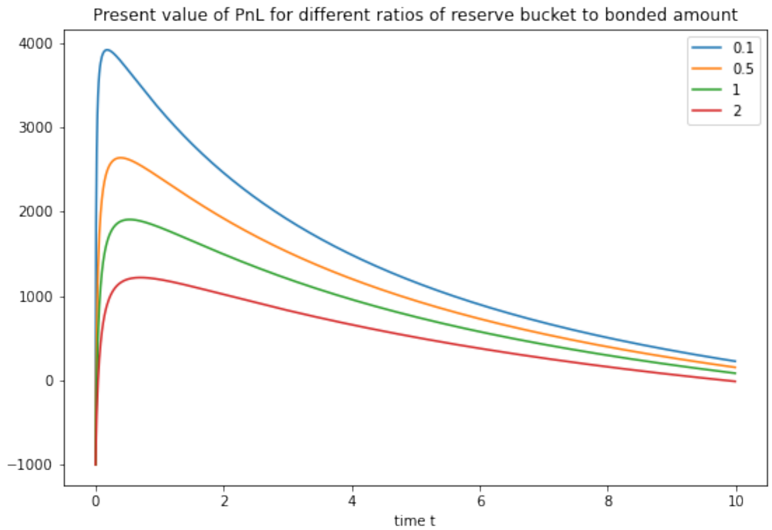

Figure 4

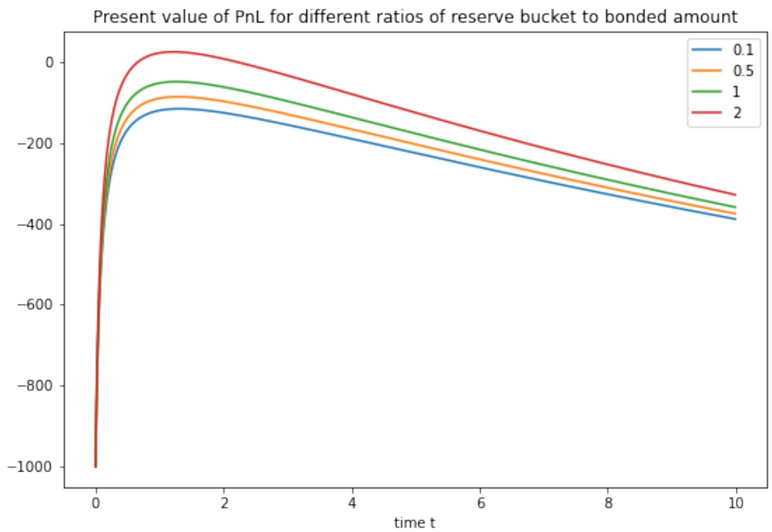

Figure 5

However, the analytical solution is also rather complex, so we just reference it (see here).

We can consider this optimum also as follows: Once bonded, all that one can do is chicken in or chicken out, hence our cost is fixed whilst the chicken in time determines how much supply of the total bTKN supply we accrue. As mentioned in our assumptions (2), we can assume that chickening out and redemptions don't happen and hence the optimal time to chicken in is the time where we maximize the share of bTKN supply.

Let's revisit the supply of bTKN one accrues after t given bonded amount M

$$s_M(t) = M\frac{t}{t+\alpha} * \frac{S_M(t)}{q_r + (q_r + q_p + q_d) * ((1+r)^t - 1)} \tag{4}$$

where we assume $S_M(t) = S$, the intitial supply of bTKN, which means no one has chickened in yet. Consequently the share of bTKN supply when chicken in is

$$p_M(t) = \frac{s_M(t)}{S_M(t) + s_M(t)} = \frac{Mt}{(q_r + (q_r + q_p + q_d) * ((1+r)^t - 1))(t+\alpha) + Mt}$$

The relationship to the present value PNL (2) is clear, when we take the derivative:

$$\frac{dp_M(t)}{dt} = M\frac{\alpha * (q_r + Q * ((1+r)^t - 1)) - Qt(t+\alpha)(1+r)^t \ln(1+r)}{((q_r + Q * ((1+r)^t - 1))(t+\alpha) + Mt)^2} \qquad JupyterLink$$

This has the same root as the derivative of the present value PNL (3) and hence the same optimal chicken in time T^*.

Let's take the parametrization from our charts shown earlier, i.e. $\alpha = 0.1, r = 0.05, q_r = 10000, q_d = 5000$ and $M = 1000$ and $\alpha = 0.1$. In this case numerical and approximated optimum are quite close (1.1005 vs. 1.086) and the optimal present value of the PNL differs by less than 0.01%:

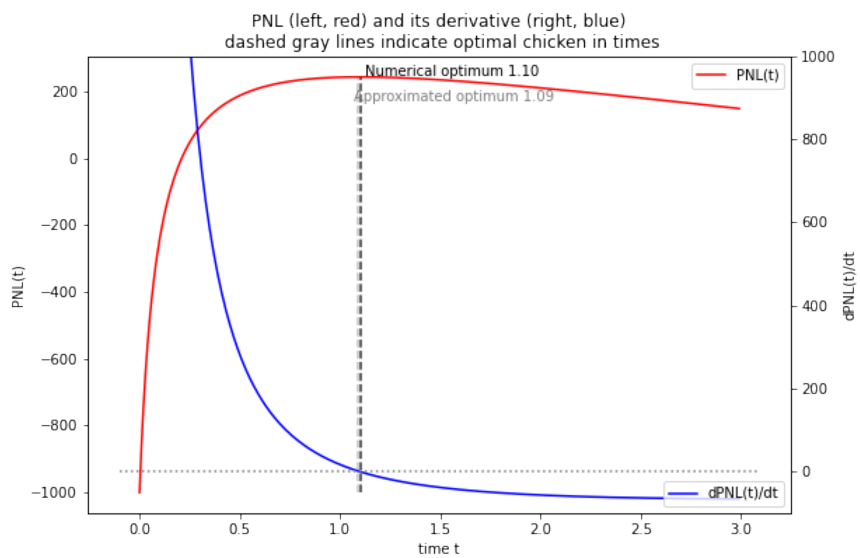

Figure 6

The formula of the approximation shows that the optimal chicken in time depends on α, r and $q_r/(q_r + q_d + M)$. If we again take our setting from earlier (see last paragraph) then we can see in 7 that

1. Higher values of α result in later optimal chicken in times
2. Higher ratios of q_r to $(q_r + q_d + M)$ result in steeper curves and hence also later optimal chicken in times (as also visible in 1)

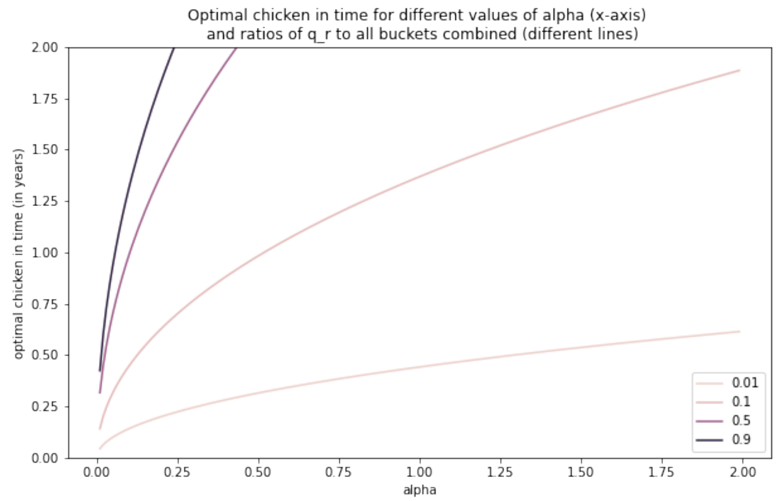

Figure 7

4.3 Determining the fair bTKN price

So far we considered the bonding and chicken in times if one bonds the amount M. As visible in the charts of section 4.1 and following the claims in section 3, more players can bond, or more TKN can be bonded as long as the utility function, the present value of the PNL function is not negative. As we assume that all bonders are going to chicken in at the same, optimal time T^*, we can derive what maximal M^* will be bonded at $t = 0$. If we assume, TKN holders at $t = 0$ know about the optimal chicken in time T^* derived in the last section, then they also know the total supply shortly after that time, i.e. at $T^* + \epsilon$:

$$S(T^* + \epsilon) = S + s_M(T^*)$$

where S is the initial bTKN supply at $t = 0$ and $s_M(T^*)$ is the supply function (4) when M TKN chicken in at T^*. This value is:

$$s_M(T^*) = M\frac{T^*}{T^* + \alpha} * \frac{S}{q_r + (q_r + M + q_d) * ((1 + r)^{T^*} - 1)} \tag{5}$$

At that time, we know that the fair value of all bTKN supply is then again the present value of a perpetual bond:

$$v_f(T^*) = (q_r + q_d + M) * (1 + r)^{T^*}$$

As we know the total supply, we can calculate the fair price for all times $t > T^*$:

$$\begin{aligned} p_{f_M}(t) &= \frac{v_{f_M}(t)}{S + s_M(T^*)} \\ &= \frac{(q_r + q_d + M) * (1 + r)^t}{S + M\frac{T^*}{T^*+\alpha} * \frac{S}{q_r+(q_r+M+q_d)*((1+r)^{T^*}-1)}} \\ &= \frac{Q(1 + r)^t * (\frac{M}{Q} + (1 + r)^{T^*} - 1)}{S * (\frac{q_r}{Q} + (1 + r)^{T^*} - 1 + \frac{M}{Q} * \frac{T^*}{T^*+\alpha})} \end{aligned}$$

As r is also our discount rate, the current fair price is

$$p_{f_M}(0) = \frac{1}{(1+r)^t} p_{f_M}(t)$$
$$= \frac{Q(\frac{q_r}{Q} + (1+r)^{T^*} - 1)}{S * (\frac{q_r}{Q} + (1+r)^{T^*} - 1 + \frac{M}{Q} * \frac{T^*}{T^*+\alpha})}$$

However, this is only true, if we know that there is no additional bonding in $t = 0$, i.e. if M is the maximum amount such that the PNL/utility function as defined in 1 is not positive. Per 1 we know that such M^* exists. This means, that M^* is bonded in $t = 0$, since all bonders know that those will chicken in at the optimal time T^*, resulting in the final bTKN supply $S + s_{M^*}(T^*)$ after which the fair price of bTKN is always

$$p_{f_{M^*}}(t) = \frac{Q^*(1+r)^t * (\frac{M^*}{Q^*} + (1+r)^{T^*} - 1)}{S * (\frac{q_r}{Q^*} + (1+r)^{T^*} - 1 + \frac{M^*}{Q^*} * \frac{T^*}{T^*+\alpha})} \text{ for } t > T^*$$

using $Q^* = q_r + q_d + M^*$. Then no more bonding is reasonable and we reach an equilibrium state. The question remains, how to calculate this maximal M^*. As outlined in the beginning of this section we can approach this iteratively by starting with small M, calculating optimal chicken in time and derive the fair price to check if the PNL/utility function is still positive until we arrive at M^*. We do so first in our example setting again:

Let's come back to our setting of $\alpha = 0.1, r = 0.05, q_r = 10000$ and $q_d = 5000$. Let's further assume the current bTKN supply is 5000

It turns out, that the maximal TKN amount to bond is $M^* = 87407$. We got to this using above approach, which looks as follows if we start increasing M stepwise:

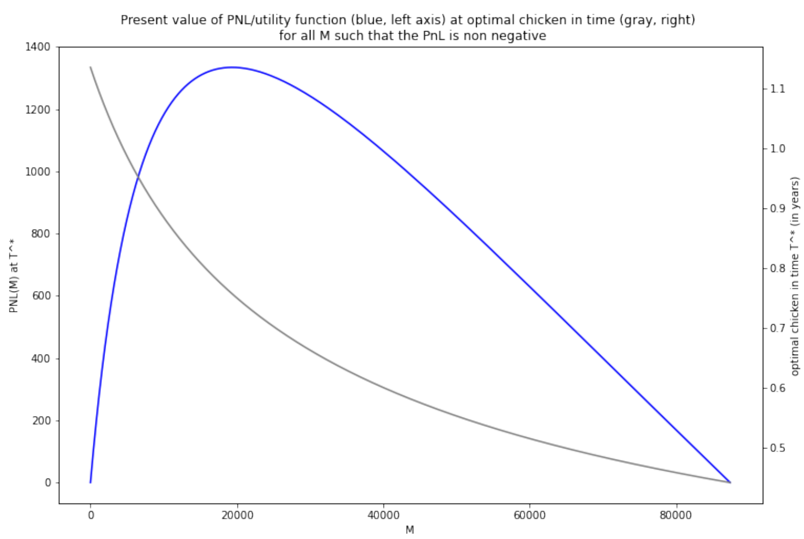

Figure 8

As we increase M the optimal chicken in time (gray line, right axis) decreases (as the ratio $\frac{q_r}{q_r+q_d+M}$ decreases, see the considerations in 7). Also, eventually claim 1 comes into effect and our PNL function turns negative (blue line, right axis), which is at $M^* = 87407$.

For all M we calculate the fair price at the optimal chicken in time, but as laid out, we can discount it then to the time 0, which results in this graph of fair price of bTKN at time 0 for the different values of M:

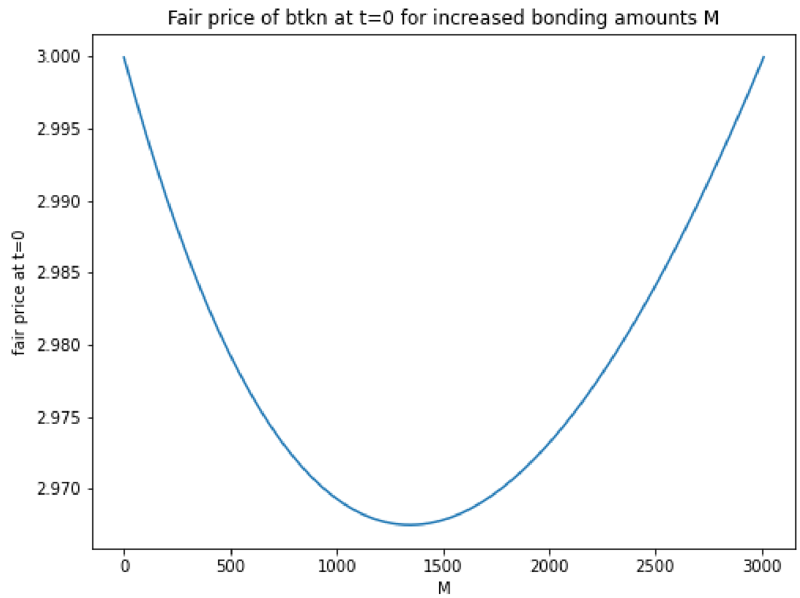

Figure 9

It is reasonable that the fair price discounted to $t = 0$ for our maximum M is equal to the fair market price to $M = 0$: If no one bonds, the fair bTKN price discounted to $t = 0$ is always $\frac{q_d+q_r}{S}$. As the present value for some $M > 0$ has positive present value over holding the TKN and not bonding, users bond. As more bonds/larger amounts are bonded we increase the value but also the future supply of bTKN (as we assume users will chicken in). This has a negative effect on the current fair price of bTKN at first, but eventually the total bonded amount M gets large enough to increase it. There is however a point, where additional bonding does not generate positive present value for the PNL, which is exactly when the original far bTKN price of $t = 0$ without any bonding is reached.

Last, let's take a look at the ex-ante PNL curve for our M^*, which potential bonders would look at - all bonders know the optimal chicken in time T^* and hence bonding makes sense as long as the PNL for this value is non-negative, where the smallest value is obviously 0, which we see in Figure 10.

5 Is chickening in at the optimum really optimal?

One of the assumptions in the last section was that all bonds chicken in at the same time T^*, which we derived as local maximum of the present value of the PNL, which is also where we achieve the maximum of the accrued supply as share of the total bTKN supply for the given bonding amount M. As it turns out, one can split a bonding amount M into several bonds M_i that chicken in at different times $t_i \neq T^*$ and achieve a higher share of bTKN supply.

Let's consider the case with two bonds $M_1 = M_2$ with $2M_1 = M$ that bonded at $t = 0$. One can obviously generalize the below discussion to n different bonds and bonding times.

Together with the variables and assumptions from the previous sections the accrued bTKN supply of M_1 at time t is as we defined in 4:

$$s_{M_1}(t) = M_1 \frac{t}{t+\alpha} * \frac{S}{q_r + (q_r + q_d + M) * ((1+r)^t - 1)} \tag{6}$$

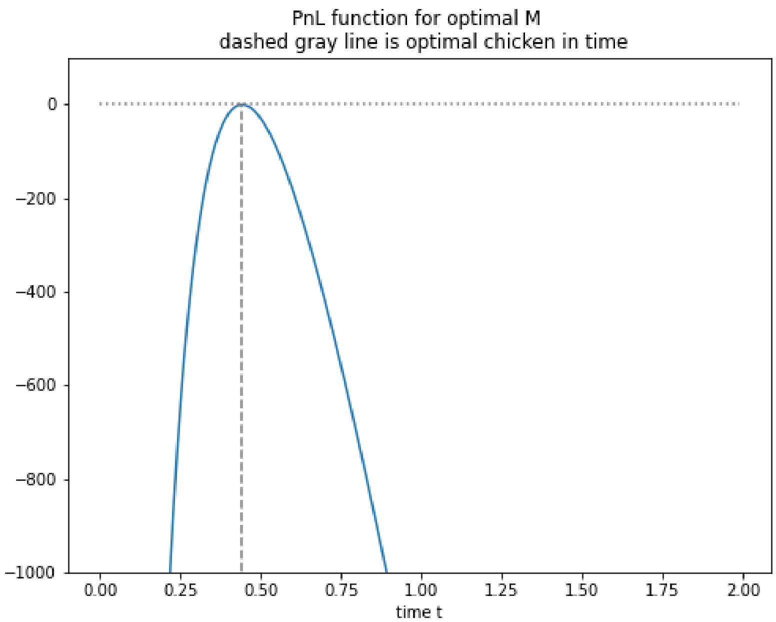

Figure 10

Note that the right factor is the inverse of the redemption price $p_{r_M}(t) = \frac{q_r(q_r+q_d+M)*((1+r)^t-1)}{S}$, i.e. the ratio of TKN supply in the reserve to the total bTKN supply. Let's assume now that M_1 gets chickened in at some time t_1. This means, M_1 is added to q_r, which changes the above formulas for M_2. Specifically for $t > t_1$, the inverse of the redemption price changes:

$$p_{r_M}^{-1}(t) = \frac{S + s_{M1}(t_1)}{q_r + (q_r + q_d + M) * ((1+r)^t - 1) + M_1 \frac{t_1}{t_1+\alpha}}$$

Rewriting this with $M_1 \frac{t}{t+\alpha} = x$ and $q_r + (q_r + q_d + M) * ((1+r)^t - 1) = q_{r_M}(t)$ we get

$$p_{r_M}^{-1}(t) = \frac{S + \frac{xS}{q_{r_M}(t_1)}}{q_{r_M}(t) + x} = \frac{S(1 + \frac{x}{q_{r_M}(t_1)})}{q_{r_M}(t)(1 + \frac{x}{q_{r_M}(t)})}$$

$\frac{S}{q_{r_M}(t)}$ is the inverse redemption price before t_1 (see 6). Since $q_{r_M}(t_1) < q_{r_M}(t)$ for $t_1 < t$ it follows that $p_{r_M}^{-1}(t_1) < p_{r_M}^{-1}(t)$ when M_1 gets chickened in at t_1. Whilst the redemption price at time of chickening in t_1 stays the same, it decreases going forward for M_2 compared to the price if it would get chickened in at t_1 as well. Put simply, by not chickening in M_2 when M_1 is chickened in, the supply accrual for M_2 gets a boost as the inverse redemption price after t_1 is higher compared to the situation w/o M_1 chickening in.

The formula for the bTKN accrual of M_2 is hence

$$s_{M_2}(t) = M_2 \frac{t}{t+\alpha} \frac{S + s_{M_1}(t_1)}{q_r + (q_r + q_d + M) * ((1+r)^t - 1) + M_1 \frac{t_1}{t_1+\alpha}} \tag{7}$$

Using $\alpha(t) := \frac{t}{t+\alpha}$, we can write the ratio of s_2 to s_1 as

$$\frac{s_2}{s_1} = \frac{\alpha(t_2)}{\alpha(t_1)} * \frac{((1+r)^{t_1} - 1) * (q_r + q_d + M) + q_r + M_1\alpha(t_1)}{((1+r)^{t_2} - 1) * (q_r + q_d + M) + q_r + M_1\alpha(t_1)}$$

Let's assume small r and use $(1+r)^t \approx 1+rt$ and denote $Y := \frac{M_1\alpha(t_1)+q_r}{q_r+q_d+M}$, to see when this ratio can be larger than 1:

$$\begin{aligned}
&\frac{s_2}{s_1} \approx \frac{\alpha(t_2)}{\alpha(t_1)}\frac{(rt_1+Y)}{(rt_2+Y)} > 1\\
&\Leftrightarrow \alpha(t_2)(rt_1+Y) > \alpha(t_1)(rt_2+Y)
\end{aligned}$$

As $\alpha(t_2) > \alpha(t_1)$ for all $t_2 > t_1$, it is easy to see that there are cases when this holds (e.g. when $rt_1 << Y$ and $rt_2 << Y$), even when t_1 is the optimal chicken in time. Let's assume s_1 is the accrued bTKN supply at optimal chicken in time and we have above situation, i.e. $\frac{s_2}{s_1} = x > 1$. Then the ratio of supply shares of using s_1 and s_2 (chicken in at two distinct times) vs. $2s_1$ (chicken in all at the optimal time) is also larger 1, i.e. one is better of chickening in at distinct times:

$$\begin{aligned}
(1+x)S &> 2S\\
\Leftrightarrow S(1+x) + 2s_1(1+x) &> 2S + 2s_1(1+x)\\
\Leftrightarrow (1+x)(S+2s_1) &> 2(S+s_1(1+x))\\
\Leftrightarrow \frac{(1+x)(S+2s_1)}{2(S+s_1(1+x)} &> 1\\
\Leftrightarrow \frac{s_1(1+x)}{S+s_1(1+x)}\frac{(S+2s_1)}{2s_1} &> 1
\end{aligned}$$

The left fraction is the supply share using distinct chicken in times, the right one is the inverse of the supply share using the optimal chicken in time, hence their ratio is larger 1.

We also ratified our calculations with numerical simulation. An in [3] we show that for a single player it is better to chicken in at two different times.

6 Conclusion

The Chicken Bond protocol can give rise to complex dynamics and equilibrium classes. In this write-up we described the dynamic of a relatively simple equilibrium which give rise to the following dynamic:

1. After certain bootstrapping period people do not bond anymore, and the pnl for additional bonding is not positive.
2. This steady state changes only once one of the following happens: (i) market interest rate changes (ii) someone redeems his $bTKN$ as there is not enough market liquidity for him to sell it. (iii) the permanent bucket inventory is increased by an external player who subsidizes it.
3. At this point additional M tokens are bonded and later chicken in, with minimal acceptable profit (in theory 0). Then we go back to 1.

An interesting observation is that if the minimal acceptable profit is 0, then the additional bonding increase the $bTKN$ supply without changing the fair market price of $bTKN$ (discounted to present value). This stems from the fact that the new bonders profit is 0, and thus the existing $bTKN$ holders didn't lose.

We proved the above under certain assumptions on an optimal chicken in time. We believe that in the general case Assumption 6 holds, and thus the dynamic is true in general, however we leave the proof of Assumption 6 to future work.

References

[1] Liquity Team, Chicken Bonds: Self-Bootstrapping Liquidity. 2022.

[2] Jupyter Notebook Simulation Environment. https://tinyurl.com/4f3e566m

[3] PnL Python Simulation Enviroment. https://tinyurl.com/2p9wy96b

A Proof of Claim 1

To prove above claim we use the definition of the utility function in 1:

$$PNL_M(t) = \frac{s_M(t)(q_r + q_d + M)}{d_{f_M}(t)(S + s_M(t))} - M$$

If there is a t_M for which $PNL_M(t_M) > 0$ then entering a bond with amount M has higher utility than holding it.

Proof. In order to prove the claim, we can assume that given $M > 0$ there exists at least one time $t_M b$ such that $PNL_M(t_M) > 0$. Otherwise there is no point for our claim, as in those cases one is always worse bonding vs. holding the tokens. We can fix this t_M as a chicken in time and are interested in the behaviour of

$$P\tilde{N}L_{t_M}(M) = PNL_M(t_M)$$

As stated earlier (and obvious with above formula): $P\tilde{N}L_{t_M}(0) = 0$. Furthermore, $P\tilde{N}L_{t_M}(M)$ is continous and differentiable. The maximum degree of that function is 2:

$$P\tilde{N}L_{t_M}(M) = \underbrace{\frac{s_{t_M}(M)}{(S + s_{t_M}(M))}}_{\text{degree } \leq 1} * \overbrace{(q_r + q_d + M)}^{\text{degree=1}} - \underbrace{M}_{\text{degree=1}}$$

which means we have at most one extreme point. We also see that the function eventually turns negative:

$$\lim_{M\to\infty} P\tilde{N}L_{t_M}(M) = \lim_{M\to\infty} \underbrace{\frac{s_{t_M}(M)}{(S + s_{t_M}(M))}}_{\leq 1} *(q_r + q_d) + M \underbrace{\left(\frac{s_{t_M}(M)}{(S + s_{t_M}(M))} - 1\right)}_{<0}$$

This means there must be a $M^* \geq M$ such that $PNL_{t^*_M}(M^*) = 0$ and hence the PNL turns negative for $N > M^*$, which proves the claim.

As M changes though, the chicken in time t_M might change as well. However, above reasoning holds as we can still assume a positive PNL at t_M and the limit is negative as well. Similarly, if we assume different chicken in times $\vec{t_M}$ across i bonds with $\sum_i^n M_i = M$, then the limit expression gets more complicated but still holds.

□

We are also showing an example for above case in 4.1, see figure 3.

B Proof of Claim 4

Towards a contradiction we assume that player 1 increase his utility by bonding at time $t_1 > 0$ and chickening in at time $t_1 + t_2$. Let t^* be the time at which the M $bTKN$ are chickened in by the rest of the player. We first prove it for the case where $t_1 + t_2 \leq t^*$, and then for the that case that $t_1 + t2 > t^*$.

Proof for $t_1 + t_2 \leq t^*$. Without loss of generality we assume that player 1 bond $(1+r)^{t_1}$ $bTKN$ at time t_1. Denote $\alpha(t) = t/(t+\alpha)$. After chickening in at t_2 his supply of $bTKN$ is

$$s_1 = \frac{S \cdot \alpha(t_2) \cdot (1+r)^{t_1}}{q_r \cdot (1+r)^{t_1+t_2} + (q_d + M)((1+r)^{t_1+t_2} - 1) + (1+r)^{t_1} \cdot ((1+r)^{t_2} - 1)}$$

After dividing the numerator and the denominator by $(1+r)^{t_1}$ we get:

$$s_1 = \frac{S \cdot \alpha(t_2)}{q_r \cdot (1+r)^{t_2} + (q_d + M)((1+r)^{t_2} - \frac{1}{(1+r)^{t_1}}) + ((1+r)^{t_2} - 1)}$$

As the numerator is independent of t_1 and the denominator is minimal when $t_1 = 0$, we get that s_1 is maximized when $t_1 = 0$.

After chickening the M tokens at time t^* the additional supply would be

$$s^* = \frac{(S + s_1) \cdot M \cdot \alpha(t^*)}{q_r \cdot (1+r)^{t^*} + (q_d + M)((1+r)^{t^*} - 1) + (1+r)^{t^*}}$$

We observe that the denominator of s^* does not depend on the value of t_1, hence for some constant k (independent of the value of t_1) we get that $s^* = k(S + s_1)$ In addition, the fair value of all the $bTKN$ after time t^*, discounted to time $t = 0$ is $q_r + q_d + 1 + M$, and it is also independent of t_1. We denote this value by v.

Hence, the fair value of player 1 is

$$\frac{v \cdot s_1}{S + s_1 + s^*} = \frac{v \cdot s_1}{S + s_1 + k(S + s_1)} = \frac{v \cdot s_1}{(1+k)S + (1+k)s_1}$$

This value is increasing as s_1 is increasing, and thus get the maximal value when $t_1 = 0$. The proof for the first case is completed.

Proof for $t_1 + t_2 > t^*$. Let t_1 and t_2 be time units such that bonding at time t_1 and chicken in at time $t_1 + t_2$ improves player 1 utility, and $t_1 + t_2 > t^*$.

After chickening in at t_2 his supply of $bTKN$ is

$$s_1 = \frac{S \cdot \alpha(t_2) \cdot (1+r)^{t_1}}{(q_r + M) \cdot (1+r)^{t_1+t_2} + (q_d)((1+r)^{t_1+t_2} - 1) + (1+r)^{t_1} \cdot ((1+r)^{t_2} - 1)}$$

By the same arguments as in the first scenario, s_1 gets the maximal $t_1 = 0$, the fair value, discounted to time $t = 0$ is independent of t_1, and the utility of player 1 is maximized when s_1 is maximal. Hence, the maximal utility is when $t_1 = 0$, and the proof for the second case is completed. $\square$