Asymmetric Super-Heston-rough volatility model with Zumbach effect as a scaling limit of quadratic Hawkes processes

Srikanth K Iyer
(Joint work with Priyanka Chudasama)

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE, BANGALORE

February 13, 2024

Motivation

“The hunt for a “perfect” statistical model of financial markets is still going on.”

– Pierre Blanc, Jonathan Donier, Jean-Phillippe Boucard (2015).

Motivation

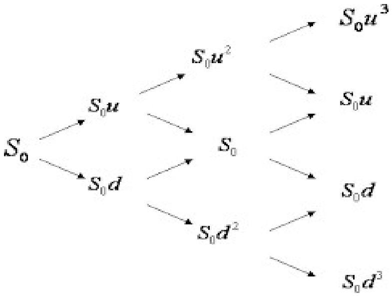
"The hunt for a "perfect" statistical model of financial markets is still going on."

– Pierre Blanc, Jonathan Donier, Jean-Phillippe Boucard (2015).

Aim: Microstructural foundations for macroscopic models.

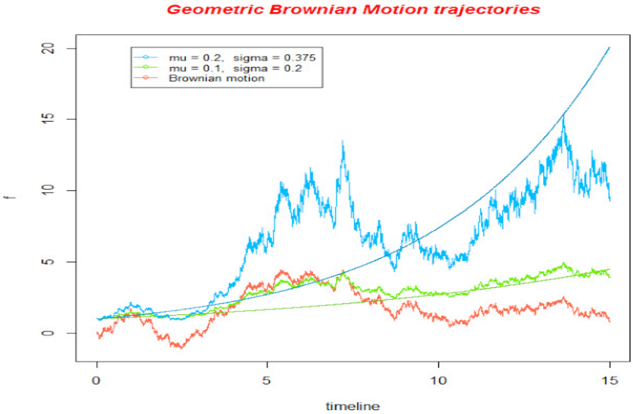
Multi-period Binomial Model

Multi-period Binomial Model



Scaling Limit

$$\frac{dS_t}{S_t} = \mu\, dt \;+\; \sigma dW_t; \qquad S_t = S_0 \exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t\right).$$



First Stylized Facts

1. Highly endogenous markets

First Stylized Facts

1. Highly endogenous markets
2. Absence of statistical arbitrage

First Stylized Facts

1. Highly endogenous markets
2. Absence of statistical arbitrage
3. Buying/Selling Asymmetry

Hawkes Process Framework for the Microstructure

Bacry, Delattre, Hoffmann, Muzy (2013)

$$P_t = N_t^1 - N_t^2$$

Hawkes Process Framework for the Microstructure

Bacry, Delattre, Hoffmann, Muzy (2013)

$$P_t = N_t^1 - N_t^2$$

N_t^i - Hawkes processes with intensity

$$\begin{pmatrix} \lambda_t^1 \\ \lambda_t^2 \end{pmatrix} = \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix} + \int_0^t \Phi(t-s) \begin{pmatrix} dN_s^1 \\ dN_s^2 \end{pmatrix}.$$

$$\Phi = \begin{pmatrix} \phi_1 & c\phi_2 \\ \phi_2 & \phi_1 + (c-1)\phi_2 \end{pmatrix}$$

$$K = \int_0^\infty \Phi(t)dt; \qquad \text{stability: } \rho(K) < 1.$$

Getting the Stylized Facts

Euch, Fukasawa, Rosenbaum (2018)

1. Highly endogenous markets: $\rho(K) \to 1$. Near unstable regime; stochastic volatility.

Getting the Stylized Facts

Euch, Fukasawa, Rosenbaum (2018)

1. Highly endogenous markets: $\rho(K) \to 1$. Near unstable regime; stochastic volatility.
2. Absence of statistical arbitrage: $\mu^1 = \mu^2$; row sums of Φ identical.

Getting the Stylized Facts

Euch, Fukasawa, Rosenbaum (2018)

1. Highly endogenous markets: $\rho(K) \to 1$. Near unstable regime; stochastic volatility.
2. Absence of statistical arbitrage: $\mu^1 = \mu^2$; row sums of Φ identical.
3. Buying/Selling Asymmetry: $c \neq 1$.

The Heston Model as the Scaling Limit

$$dP_t = \frac{1}{1 - (\| \phi_1 \|_1 - \| \phi_2 \|_1)} \sqrt{\frac{2}{1+c}} \sqrt{V_t}\, dW_t$$

$$dV_t = \kappa(v_0 - V_t)\, dt + \eta \sqrt{V_t}\, dB_t$$

$$d\langle W, B \rangle_t = \frac{1-c}{\sqrt{2(1+c^2)}}\, dt.$$

The Heston Model as the Scaling Limit

$$dP_t = \frac{1}{1 - (\| \phi_1 \|_1 - \| \phi_2 \|_1)} \sqrt{\frac{2}{1+c}} \sqrt{V_t}\, dW_t$$

$$dV_t = \kappa(v_0 - V_t)\, dt + \eta \sqrt{V_t}\, dB_t$$

$$d\langle W, B \rangle_t = \frac{1-c}{\sqrt{2(1+c^2)}}\, dt.$$

- Leverage Effect: $c > 1$; negative correlations between price returns and vol increments at macro scale.

The Heston Model as the Scaling Limit

$$dP_t = \frac{1}{1 - (\| \phi_1 \|_1 - \| \phi_2 \|_1)} \sqrt{\frac{2}{1+c}} \sqrt{V_t}\, dW_t$$

$$dV_t = \kappa(v_0 - V_t)\, dt + \eta \sqrt{V_t}\, dB_t$$

$$d\langle W, B \rangle_t = \frac{1-c}{\sqrt{2(1+c^2)}}\, dt.$$

- Leverage Effect: $c > 1$; negative correlations between price returns and vol increments at macro scale.
- Stochastic volatility only in the near critical regime.

One More Stylized Fact

- Volatility is rough (Gatheral *et. al.* 2018); log-vol $\approx$ fractional BM with Hurst parameter of order 0.1
- Meta orders, split by trading algorithms; Leads to long range correlations
- Tail of the largest eigenvalue:

$$\int_t^\infty (\phi_1 + c\phi_2)(s)ds \sim Ct^{-\alpha}, \alpha \in \left(\frac{1}{2}, 1\right) \dots$$

Scaling Limit is the Rough Heston Model

$$V_t = V_0 + \int_0^t (t-s)^{\alpha-1}\kappa(c - V_s)ds + \eta\sqrt{V_s}dB_s$$

TRA: The Zumbach Effect (2010)

- TRA: Financial Time Series Statistically Asymmetric when past and future interchanged

TRA: The Zumbach Effect (2010)

- TRA: Financial Time Series Statistically Asymmetric when past and future interchanged
  1. past returns negatively affect future vols but not other way around

TRA: The Zumbach Effect (2010)

- TRA: Financial Time Series Statistically Asymmetric when past and future interchanged
  1. past returns negatively affect future vols but not other way around
  2. past large scale realized vols more correlated with future small scale realized vols than vice-versa

Encoding the Zumbach Effect

- Quadratic Hawkes Process: Blanc, Donier, Bouchaud (2015).

Encoding the Zumbach Effect

- Quadratic Hawkes Process: Blanc, Donier, Bouchaud (2015).
- Price trends affect volatility

Encoding the Zumbach Effect

- Quadratic Hawkes Process: Blanc, Donier, Bouchaud (2015).
- Price trends affect volatility

$$\lambda_t = \mu + \int_0^t \phi(t-s)dN_s + Z_t^2; \qquad Z_t = \int_0^t k(t-s)dP_s$$

$$P_t = \sum_{i=1}^{N_t} \xi_i; \qquad \xi_i \overset{iid}{\sim} \pm 1 \text{ w.p. } \frac{1}{2}.$$

- A univariate Hawkes process is being used here; positive and negative price trends have same impact on volatility. Martingale structure inside the quadratic term.

Super-Heston-Rough-Volatility with Zumbach Effect: Dandapani, Jusselin, Rosenbaum (2021)

Scaling Limit in the near unstable regime:

$$dP_t = \sqrt{V_t}\, dW_t$$

$$V_t = V_0 + C_1 \int_0^t (t-s)^{\alpha-1}(\theta(s) - V_s + Z_s^2)\, ds + C_2 \int_0^t (t-s)^{\alpha-1} \sqrt{V_s}\, dB_s$$

$$Z_t = \int_0^t k(t-s)\sqrt{V_s}\, dW_s$$

- W, B independent Brownian motions; $\alpha \in (\frac{1}{2}, 1)$.

Super-Heston-Rough-Volatility with Zumbach Effect: Dandapani, Jusselin, Rosenbaum (2021)

Scaling Limit in the near unstable regime:

$$dP_t = \sqrt{V_t}\, dW_t$$

$$V_t = V_0 + C_1 \int_0^t (t-s)^{\alpha-1} (\theta(s) - V_s + Z_s^2)\, ds + C_2 \int_0^t (t-s)^{\alpha-1} \sqrt{V_s}\, dB_s$$

$$Z_t = \int_0^t k(t-s) \sqrt{V_s}\, dW_s$$

- W, B independent Brownian motions; $\alpha \in (\frac{1}{2}, 1)$.
- Stochastic Volatility even in the stable regime.

Super-Heston-Rough-Volatility with Zumbach Effect: Dandapani, Jusselin, Rosenbaum (2021)

Scaling Limit in the near unstable regime:

$$dP_t = \sqrt{V_t}\, dW_t$$

$$V_t = V_0 + C_1 \int_0^t (t-s)^{\alpha-1} (\theta(s) - V_s + Z_s^2)\, ds + C_2 \int_0^t (t-s)^{\alpha-1} \sqrt{V_s}\, dB_s$$

$$Z_t = \int_0^t k(t-s) \sqrt{V_s}\, dW_s$$

- W, B independent Brownian motions; $\alpha \in (\frac{1}{2}, 1)$.
- Stochastic Volatility even in the stable regime.
- Super: Enhanced volatility tails.

Encoding Asymmetric Impact of Price Trends

$$P_t = N_t^1 - N_t^2$$

Encoding Asymmetric Impact of Price Trends

$$P_t = N_t^1 - N_t^2$$

$$\begin{pmatrix} \lambda_t^1 \\ \lambda_t^2 \end{pmatrix} = \begin{pmatrix} \mu \\ \mu \end{pmatrix} + \int_0^t \Phi(t-s) \begin{pmatrix} dN_s^1 \\ dN_s^2 \end{pmatrix} + \left(\int_0^t \kappa(t-s) \begin{pmatrix} dM_s^1 \\ dM_s^2 \end{pmatrix} \right)^2$$

$$\Phi = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_1 \end{pmatrix}$$

$$\kappa = \begin{pmatrix} k_1 & -k_2 \\ -k_2 & k_1 \end{pmatrix}; \qquad k_1 > k_2 > 0$$

$$M_t^i = N_t^i - \int_0^t \lambda_s^i \, ds, \qquad i = 1, 2.$$

Stability

$$\text{Stability: } \sum_{i=1}^{2} [\| \phi_i \|_1 + \| k_i \|_2^2] \; < \; 1$$

Stability

$$\text{Stability: } \sum_{i=1}^{2} [\| \phi_i \|_1 + \| k_i \|_2^2] \ < \ 1$$

$$k_1(k_2) = \frac{(\sqrt{\alpha_1} \pm \sqrt{\alpha_2})}{2} k$$

Stability

$$\text{Stability: } \sum_{i=1}^{2} [\| \phi_i \|_1 + \| k_i \|_2^2] \; < \; 1$$

$$k_1(k_2) = \frac{(\sqrt{\alpha_1} \pm \sqrt{\alpha_2})}{2} k$$

$$\alpha = \sqrt{\alpha_1 \alpha_2}; \;\; \phi = \phi_1 - \phi_2 \;\; \overline{\phi} = \phi_1 + \phi_2$$

Stability

$$\text{Stability: } \sum_{i=1}^{2} [\| \phi_i \|_1 + \| k_i \|_2^2] \ < \ 1$$

$$k_1(k_2) = \frac{(\sqrt{\alpha_1} \pm \sqrt{\alpha_2})}{2} k$$

$$\alpha = \sqrt{\alpha_1 \alpha_2}; \ \ \phi = \phi_1 - \phi_2 \ \ \overline{\phi} = \phi_1 + \phi_2$$

$$\text{Stability: } \| \overline{\phi} \|_1 + \frac{(\alpha_1 + \alpha_2)}{2} \| k \|_2^2 < \ 1$$

Scaling

$$P^T_{tT} = N^{(1,T)}_{tT} - N^{(2,T)}_{tT}, \qquad t \in [0,1], \; T \geq 0.$$

Scaling

$$P^T_{tT} = N^{(1,T)}_{tT} - N^{(2,T)}_{tT}, \qquad t \in [0,1],\ T \geq 0.$$

$$M^{(i,T)}_t = N^{(i,T)}_t - \int_0^t \lambda^{(i,T)}_s \, ds; \qquad M^T_t = M^{(1,T)}_t - M^{(2,T)}_t$$

Scaling

$$P^T_{tT} = N^{(1,T)}_{tT} - N^{(2,T)}_{tT}, \qquad t \in [0,1],\ T \geq 0.$$

$$M^{(i,T)}_t = N^{(i,T)}_t - \int_0^t \lambda^{(i,T)}_s \, ds; \qquad M^T_t = M^{(1,T)}_t - M^{(2,T)}_t$$

$$\lambda^T_{tT} := \lambda^{(1,T)}_{tT} - \lambda^{(2,T)}_{tT} = \int_0^t \phi^T(T(t-s))\sqrt{T}\, d\left(\frac{M^T_{tT}}{\sqrt{T}}\right)$$
$$+ \int_0^t \phi^T(T(t-s)) T \lambda^T_{Ts}\, ds + \alpha \left[\int_0^t k^T(T(t-s))\sqrt{T}\, d\left(\frac{M^T_{tT}}{\sqrt{T}}\right)\right]^2$$

Scaling

$$P^T_{tT} = N^{(1,T)}_{tT} - N^{(2,T)}_{tT}, \qquad t \in [0,1],\ T \geq 0.$$

$$M^{(i,T)}_t = N^{(i,T)}_t - \int_0^t \lambda^{(i,T)}_s \, ds; \qquad M^T_t = M^{(1,T)}_t - M^{(2,T)}_t$$

$$\lambda^T_{tT} := \lambda^{(1,T)}_{tT} - \lambda^{(2,T)}_{tT} = \int_0^t \phi^T(T(t-s))\sqrt{T}\, d\left(\frac{M^T_{tT}}{\sqrt{T}}\right)$$
$$+ \int_0^t \phi^T(T(t-s))T\lambda^T_{Ts}\, ds + \alpha\left[\int_0^t k^T(T(t-s))\sqrt{T}\, d\left(\frac{M^T_{tT}}{\sqrt{T}}\right)\right]^2$$

$$\phi^T(.) = \phi\left(\frac{\cdot}{T}\right)\frac{\beta}{T},;\ \ \bar{\phi}^T(.) = \phi\left(\frac{\cdot}{T}\right)\frac{\bar{\beta}}{T}; \qquad k^T(.) = k\left(\frac{\cdot}{T}\right)\frac{\sqrt{\gamma}}{T}$$

Scaled Intensity

$$\lambda^T_{tT} = \beta \int_0^t \phi(t-s)\lambda^T_{Ts} ds \,+\, \int_0^t \phi(t-s)\frac{1}{\sqrt{T}}\, d\left(\frac{M^T_{tT}}{\sqrt{T}}\right)$$
$$+\, \alpha\gamma \left[\int_0^t k(t-s)\, d\left(\frac{M^T_{tT}}{\sqrt{T}}\right)\right]^2 .$$

Assumptions

(i) $0 < \overline{\beta} + (\alpha_1 + \alpha_2)\gamma < 1$, $\left\|k^2\right\|_1 = \|\phi\|_1 = 1$.

(ii) The function $k \in L^{2+\epsilon}$ for some $\epsilon > 0$ and for any $0 \leq t \leq \hat{t} \leq 1$,

$$\int_0^t |k(\hat{t} - s) - k(t - s)|^2 < C|\hat{t} - t|^r$$

for some $r > 0$ and $C > 0$ and

$$\frac{1}{\eta}\int_0^1 |k(t)|^2 t^{-2\eta} dt + \int_0^1 \int_0^1 \frac{|k(t) - k(s)|^2}{|t - s|^{1+2\eta}} dsdt, \text{ is finite.}$$

for some $\eta \in (0, 1)$.

Scaling Limit in the Stable Regime

Theorem

The family of processes

$$\left\{ \left(\frac{P^T_{tT}}{T}, \frac{P^T_{tT} - T\Lambda^T_t}{\sqrt{T}} \right) : t \in [0, 1] \right\}_{T>0}$$

is C-tight. Any subsequential limit (X, M) *satisfies*

$$X_t = \int_0^t V_s ds \text{ and } M_t = \int_0^t \sqrt{V_s} dB_s,$$

$$V_t = \beta \int_0^t \phi(t-s) V_s ds + \alpha\gamma \left(\int_0^t k(t-s)\, dM_s \right)^2,$$

$$\overline{V}_t = 2\mu + \overline{\beta} \int_0^t \phi(t-s) \overline{V}_s ds + \frac{(\alpha_1 + \alpha_2)}{2} \gamma \left(\int_0^t k(t-s) \sqrt{\overline{V}_s} dB_s \right)^2.$$

Nearly unstable criteria

For, non-linear Hawkes processes to be stable the mean intensity should converges to a finite constant.

$$\mathbb{E}[\lambda_t^T] \leq \frac{2\mu_T}{1-\left(\left\|\overline{\phi}^T\right\|_1+\frac{(\alpha_1+\alpha_2)}{2}\left\|k^T\right\|_2^2\right)}.$$

Let $a_T = \left\|\overline{\phi}^T\right\|_1 + \frac{(\alpha_1+\alpha_2)}{2}\left\|k^T\right\|_2^2$.

- Stable case: $a_T < 1$ for all $T \in [0, \infty]$.
- Nearly unstable criteria: $a_T < 1$ for all $T \in [0, \infty)$ and $a_T \to 1$ as $T \to \infty$.

Scaling

$$\frac{1-a_T}{2\mu_T} P^T_{tT} = \frac{1-a_T}{2\mu_T}(N^{(1,T)}_{tT} - N^{(2,T)}_{tT}), \qquad t \in [0,1], \;\; T \geq 0.$$

$$\begin{aligned}\lambda^{*T}_{tT} &:= \frac{1-a_T}{2\mu_T}\lambda^T_{tT} = \frac{1-a_T}{2\mu_T}\int_0^t \phi^T(T(t-s))T\lambda^T_{Ts}ds \\ &+ \frac{1-a_T}{2\mu_T}\left(\int_0^t \phi^T(T(t-s))\, dM^T_{Ts} + \alpha\left[\int_0^t k^T(T(t-s))\, dM^T_{Ts}\right]^2\right).\end{aligned}$$

$$M_{tT} = M^{(1,T)}_{tT} - M^{(2,T)}_{tT}; \phi^T(.) = \beta_T\phi(.),$$

$$\overline{\phi}^T(.) = \overline{\beta}_T\phi(.), k^T(.) = k\left(\frac{.}{T}\right)\sqrt{\frac{1-a_T}{T}}.$$

Here, $|\beta_T| < \overline{\beta_T} < 1$, $|\beta_T| = \frac{1}{c_1(1-a_T)+1}$, $\overline{\beta_T} = \frac{1}{c_2(1-a_T)+1}$ and $c_2 > c_1 > 0$ are constants.

Scaled Intensity

$$\lambda^{*T}_{tT} := \frac{1-a_T}{2\mu_T}\lambda^T_{tT} = \int_0^t \frac{(1-a_T)T\psi_T(T(t-s))}{\sqrt{2\mu_T(1-a_T)T}}\, dM^{*T}_s$$
$$+ \frac{1-a_T}{2\mu_T}\alpha(Z^T_{tT})^2 + \int_0^t (1-a_T)T\psi_T(T(t-s))\left(\frac{\alpha}{2\mu_T}(Z^T_{sT})^2\right) ds.$$

$$M^{*T}_{tT} = \sqrt{\frac{1-a_T}{2\mu_T T}}(M^{(1,T)}_{tT} - M^{(2,T)}_{tT}), Z^T_{tT} = \sqrt{2\mu}\int_0^t k(t-s)dM^{*T}_s,$$

$\psi_T := \sum_{i\geq 1}(\phi^T)^{\circledast i}$, where $\circledast$ is convolution operator.

Assumptions

- $a_T := \left\| \overline{\phi}^T \right\|_1 + \frac{(\alpha_1+\alpha_2)}{2} \left\| k^T \right\|_2^2 \to 1$ as $T \to \infty$.

Assumptions

- $a_T := \left\|\overline{\phi}^T\right\|_1 + \frac{(\alpha_1+\alpha_2)}{2}\left\|k^T\right\|_2^2 \to 1$ as $T \to \infty$.
- $\phi^T(\cdot) = \beta_T \phi(\cdot), \qquad \overline{\phi}^T(\cdot) = \overline{\beta}_T \phi(\cdot), \qquad \|\phi\|_1 = 1.$
  $k^T(.) = k\left(\frac{\cdot}{T}\right)\sqrt{\frac{1-a_T}{T}}, \qquad \|k\|_2^2 = 1.$

Assumptions

- $a_T := \left\|\overline{\phi}^T\right\|_1 + \frac{(\alpha_1+\alpha_2)}{2}\left\|k^T\right\|_2^2 \to 1$ as $T \to \infty$.
- $\phi^T(\cdot) = \beta_T \phi(\cdot), \qquad \overline{\phi}^T(\cdot) = \overline{\beta}_T \phi(\cdot), \qquad \|\phi\|_1 = 1.$
  $k^T(.) = k\left(\frac{\cdot}{T}\right)\sqrt{\frac{1-a_T}{T}}, \qquad \|k\|_2^2 = 1.$
- $\tilde{\alpha} x^{\tilde{\alpha}} \int_x^{+\infty} \phi(s)ds \to K$ as $x \to +\infty; \qquad \tilde{\alpha} \in (\frac{1}{2}, 1).$

Assumptions

- $a_T := \left\|\overline{\phi}^T\right\|_1 + \frac{(\alpha_1+\alpha_2)}{2}\left\|k^T\right\|_2^2 \to 1$ as $T \to \infty$.
- $\phi^T(\cdot) = \beta_T\phi(\cdot), \qquad \overline{\phi}^T(\cdot) = \overline{\beta}_T\phi(\cdot), \qquad \|\phi\|_1 = 1.$
  $k^T(.) = k\left(\frac{\cdot}{T}\right)\sqrt{\frac{1-a_T}{T}}, \qquad \|k\|_2^2 = 1.$
- $\tilde{\alpha}x^{\tilde{\alpha}}\int_x^{+\infty}\phi(s)ds \to K$ as $x \to +\infty$; $\qquad \tilde{\alpha} \in (\frac{1}{2}, 1)$.
- Let $\delta = K\frac{\Gamma(1-\tilde{\alpha})}{\tilde{\alpha}}$. As $T \to \infty$

$$(1-a_T)T^{\tilde{\alpha}} \to \lambda\delta, \qquad \mu_T T^{(1-\tilde{\alpha})} \to \frac{\mu^*}{\delta}, \qquad \overline{\mu}_T T^{(1-\tilde{\alpha})} \to \frac{\overline{\mu}^*}{\delta}.$$

$|\beta_T| < \overline{\beta_T} < 1$, $|\beta_T| = \frac{1}{c_1(1-a_T)+1}$, $\overline{\beta}_T = \frac{1}{c_2(1-a_T)+1}$ and $c_2 > c_1 > 0$ are constants.

Scaling Limit in the nearly-unstable regime

Theorem

$$\left(\frac{1-a_T}{2\mu_T}\left(\frac{P^T_{tT}}{T}\right), \sqrt{\frac{1-a_T}{2\mu_T}}\left(\frac{P^T_{tT}-T\Lambda^T_t}{\sqrt{T}}\right)\right)_{t\in[0,1]}$$ *is C-tight.*

Scaling Limit in the nearly-unstable regime

Theorem

$$\left(\frac{1-a_T}{2\mu_T} \left(\frac{P^T_{tT}}{T} \right), \sqrt{\frac{1-a_T}{2\mu_T}} \left(\frac{P^T_{tT} - T\Lambda^T_t}{\sqrt{T}} \right) \right)_{t\in[0,1]} \quad \textit{is C-tight.}$$

Any subsequential limit (X, M^*) *satisfies*

$$X_t = \int_0^t V_s\, ds, \qquad M^*_t = \int_0^t \sqrt{V_s}\, dB^1_s, \qquad \overline{M}^*_t = \int_0^t \sqrt{\overline{V}_s}\, dB^2_s,$$

$$Z^*_t = \int_0^t k(t-s)\, dM^*_s, \qquad \langle B^1, B^2 \rangle_t = \frac{V_t}{\overline{\overline{V}}_t}.$$

Scaling Limit in the nearly-unstable regime

Theorem

$$\left(\frac{1-a_T}{2\mu_T}\left(\frac{P^T_{tT}}{T}\right), \sqrt{\frac{1-a_T}{2\mu_T}}\left(\frac{P^T_{tT}-T\Lambda^T_t}{\sqrt{T}}\right)\right)_{t\in[0,1]} \quad \text{is C-tight.}$$

Any subsequential limit (X, M^*) *satisfies*

$$X_t = \int_0^t V_s\, ds, \qquad M^*_t = \int_0^t \sqrt{V_s}\, dB^1_s, \qquad \overline{M}^*_t = \int_0^t \sqrt{\overline{V}_s}\, dB^2_s,$$

$$Z^*_t = \int_0^t k(t-s)\, dM^*_s, \qquad \langle B^1, B^2\rangle_t = \frac{V_t}{\overline{V}_t}.$$

$$V_t = \int_0^t \frac{1}{c_1} f^{\tilde{\alpha},\lambda}(t-s)\left[\frac{1}{\sqrt{\lambda\overline{\mu}^*}}\, dM^*_s + \left(\frac{\mu^*}{\overline{\mu}^*} + \alpha(Z^*_s)^2\right) ds\right]$$

$$\overline{V}_t = \int_0^t \frac{1}{c_2} f^{\tilde{\alpha},\lambda}(t-s)\left[\frac{1}{\sqrt{\lambda\overline{\mu}^*}}\, d\overline{M}^*_s + \left(1 + \frac{(\alpha_1+\alpha_2)}{2}(Z^*_s)^2\right) ds\right]$$