# Risk Pools: The Coin Toss Game

"A miracle is any event with a probability less than twenty percent”

**E. Fermi**

This article is for those who believe in miracles, believe that miracles exist in the world around us, but despite this, seek to explain them.

## 0x_Introduction

This article is dedicated to analyzing the coin toss game, which serves as a demonstration of the capabilities and essence of the “Risk Pools” project.

To explain the game and select parameters, a bit of theory on random processes is necessary.

## 1x_Random Processes

Any natural phenomenon we come across is typically explicable by a random process. Random processes are quite diverse, which is why mathematicians have introduced a classification (the classification here is not complete, it is given for illustrative purposes only). Let's figure this out together. As is known, random processes are characterized by the concept of a distribution function. We won't delve into this term in more detail, we'll just say that this function has certain parameters.

For example, a normal distribution function has a variance (or standard deviation, which is the width of the well-known bell curve). We call a random process stationary if the parameters of the distribution function do not depend on time, otherwise, the process is non-stationary.

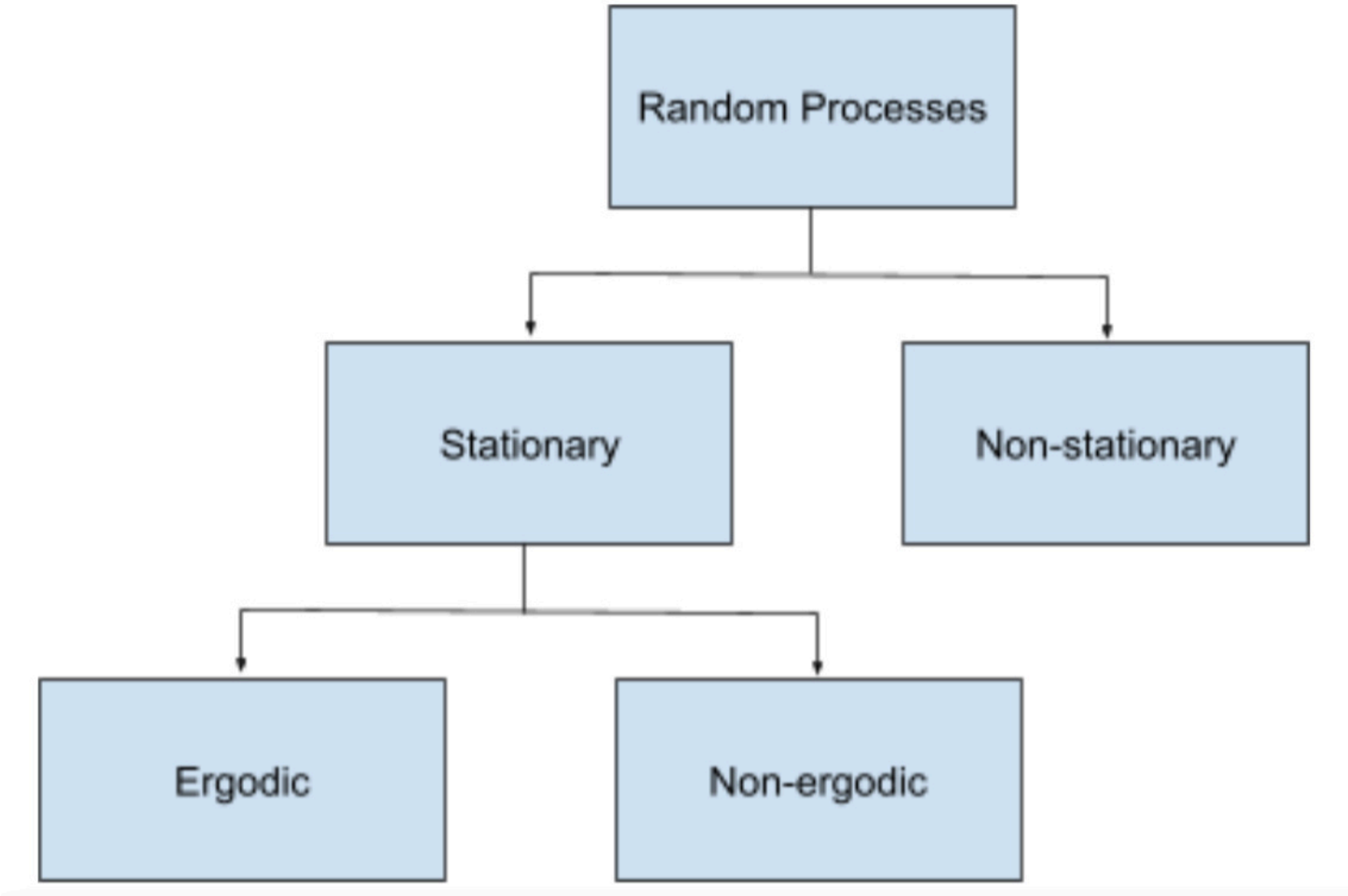



Figure 1. Classification of Random Processes

In turn, we divide stationary processes into two types: ergodic and non-ergodic. This division is less obvious than the first but no less important and we will try to explain it using an example without delving into mathematics.

We offer you the following processes: at first, we throw a die and guess the number, and with a friend, we toss a coin and try to guess it too, but in the case of the coin, we make a bet. For example, if heads - we get $50, and if tails, we pay the friend $50. You have conducted this experiment many times. In the minds of most people, these are generally the same random processes, our consciousness easily substitutes in both cases us with a die and a coin in a multitude of us as if in parallel universes simultaneously making throws and indeed, what's the difference whether I throw a coin 1000 times or 1000 people throw a coin once?

In other words, the calculation of probability over an ensemble (1000 people like me throw once) replaces the calculation of probability over time (I throw 100 times). This substitution is quite convenient because it is easier to calculate average expectations of probability. But in the coin toss case, this is incorrect. Why? Because you (I risk assuming) do not print dollars, i.e., a short series of throws may zero out your wallet, and you will not have more money to return your situation to the expected - a fair game (zero mathematical expectation) for a coin toss.

And how, you may ask, about the average mathematical expectation? Well, it seems like 0 but over the ensemble, i.e., from thousands of players, there will be 1 or 2 who will continue to throw heads time after time and precisely because of their phenomenal success - the mathematical expectation rushes to zero.

What do you think is the probability of becoming one of them? And we come to the interesting conclusion that there are processes, for example, throwing dice without bets when the probability over an ensemble equals the probability over time, and tossing coins with bets when you cannot replace the probability over time with the probability over an ensemble. The former are called ergodic, the latter are non-ergodic.

Below, we will return to our experiment with the coin. Now let's discuss an example of the implementation of our protocol.

# 2x_Theory of the Coin Toss Game

The essence of the coin toss game is quite simple - if the game is conducted fairly and our coin is completely ideal then obviously, tails fall out half the time, heads the other half. Next, we will call a throw a trial where the result of the trial equals 0 in the case of tails and 1 in the case of heads.

$$P_i = \lim_{N\to\infty} \frac{N_i}{N} = 0,5 \quad (1)$$

Where:

N - the total number of trials;

Ni - the number of trials with “i” result  (zero or one);

Pi - Probability of obtaining an “i” result type (zero or one).

Now let's complicate the concept of a trial a bit and under a trial, we will understand a series of two throws. Then the types of results are as follows:

| **Trial time** | **Description** | **Probability** |
| --- | --- | --- |
| i=0 | Tails fell out both times | P1=0,25 |
| i=1 | One heads, one tails | - |
| i=2 | Heads fell out both times | P1=0,25 |

In general, the following formula can express this:

$$P_i = \lim_{N\to\infty} \frac{N_i}{N_S} \quad (2)$$

Where:

Ns - number of trial series;

N - length of each series;

Ni - number of trials with an "i" result

Pi - Probability of obtaining an "i" result

Let's make one more complication by extending the series of trials to N - and N can be any large number. Now, our series consists of N throws, while obviously, the number of possible outcomes of these trials is N+1. Now let's ask how many ways a certain result can be realized (for example, k times tails will fall out and N-k times heads). Combinatorics provides the answer to this query:

$$\Omega_k = \frac{N!}{k!(N-k)!} \quad (3)$$

If we want to understand how many ways there are to realize combinations, then:

$$\Sigma\Omega_k = 2^N \quad (4)$$

Formulas (3) and (4) are presented without proof, you can familiarize yourself with their proofs in a course on combinatorics. Dividing (3) by (4) we find the probability of obtaining the result k:

$$P_k = \frac{N!}{2^N k!(N-k)!} \quad (5)$$

Assuming that N - is much larger than one, we can use Stirling's formula:

$$\ln ln\,(n!) \;=\; n \ln ln\left(\frac{n}{e}\right) \quad (6)$$

Using this formula, we can transform the factorials in (5), and then we get the following formula for determining the probability of trial K:

$$P_k = \sqrt{\frac{2}{\pi N}} \exp exp\left(\frac{-2n^2}{N}\right) \quad (7)$$

Where:

n= (k-N/2) - deviation of the result from the average.

This formula is called the Gaussian distribution or normal distribution of probability. Thus, we proved that there is a normal distribution of probability of realizing one result or another in the coin toss game (again, noting that by trial we mean a series of throws). The fact that the distribution is normal allows us to set up an experiment by tossing a coin, where we can use the normal law of distribution as a function of the distribution of a random variable.

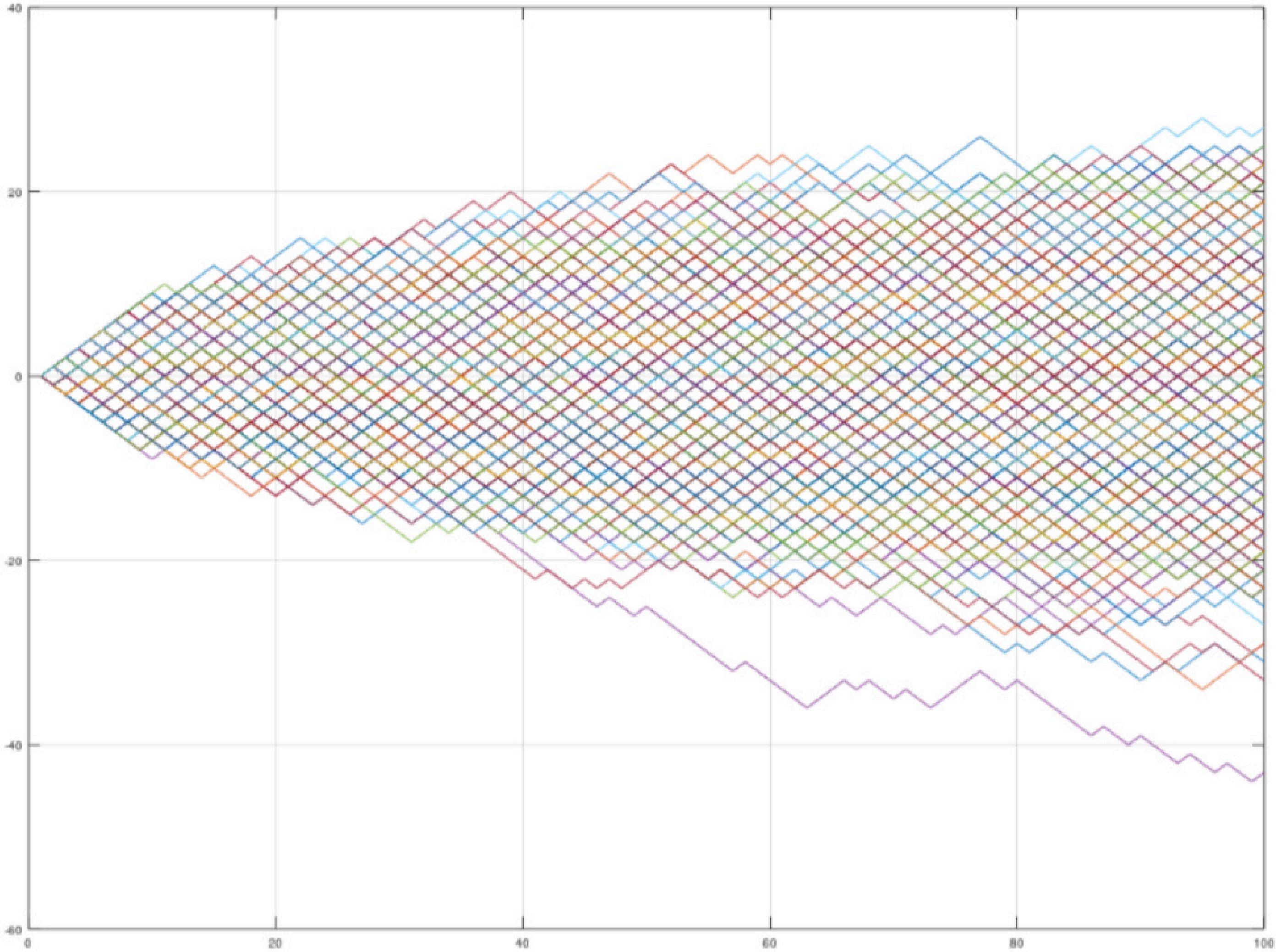
# 3x_A Place for Miracles

Is there a place for miracles in nature? Let's respond to this query with the aid of an experiment and the article's epigraph as guides. In the sections above, we discussed the classification of random processes and became acquainted with one of them - the coin toss game. Furthermore, we showed that the probabilities of different outcomes in this game obey the normal law of distribution under certain assumptions. Now we are ready to conduct experiments.

Suppose we have two players and an absolutely symmetrical coin. We have Player A, he has a certain amount of money - SA, there is Player B, he has an amount SB. The rules are simple: Player A throws the coin and if it tails, then Player A passes one dollar to Player B. The question is, who among the players and with what probability will win and, most importantly, in how many throws? To answer this question, we will not develop the theory of this game, we will simply conduct simulation experiments using the Monte Carlo method (our estimates of 1000 experiments are quite sufficient to assess the result).

**First Experiment**

Suppose that Player A's balance is 10 dollars and Player B's balance is -10 dollars. Then our simulation gives such a result:



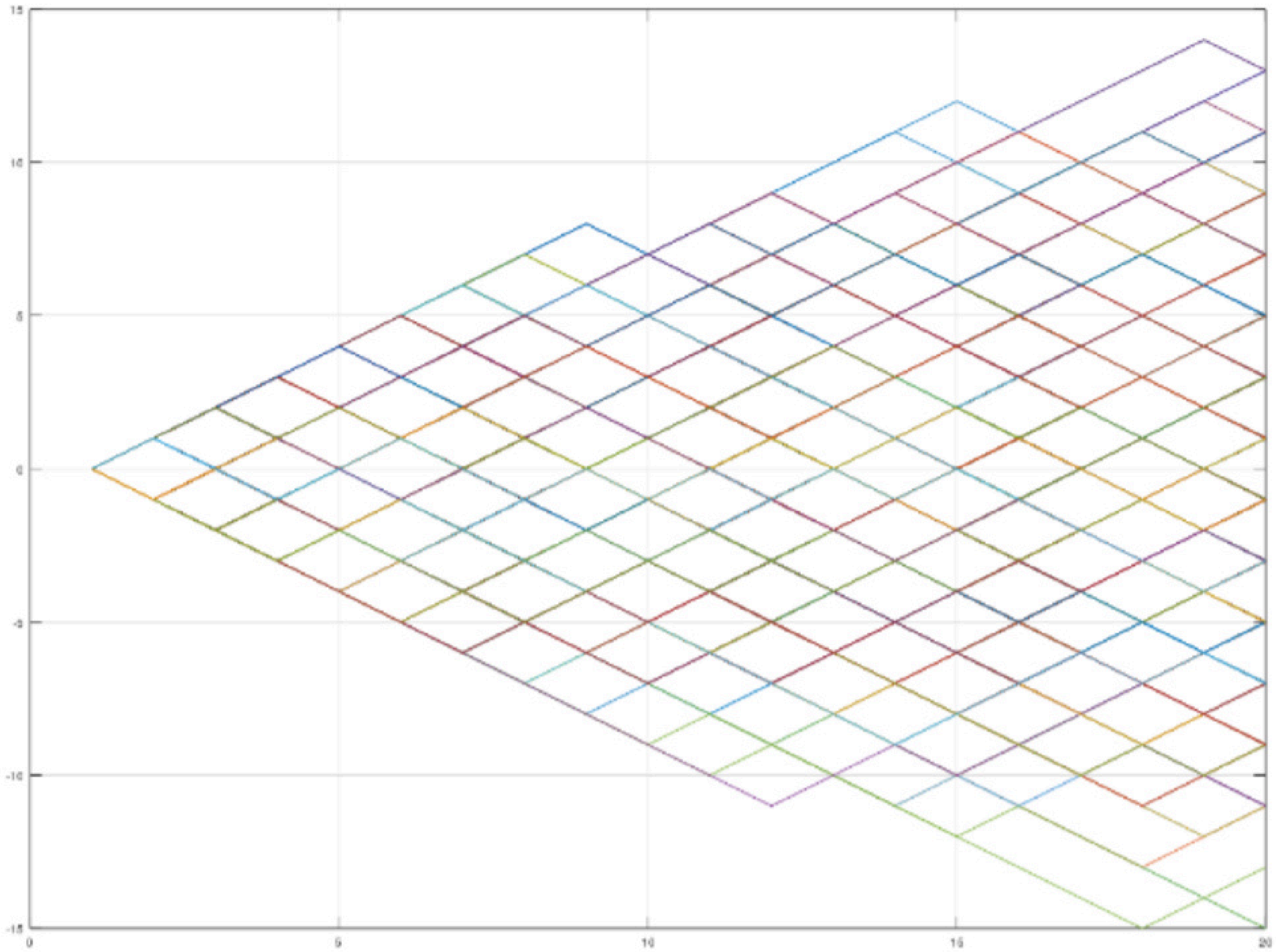

The average probability of Player A losing was 51.81% (at a theoretical 50%), and Player B – 48.18% (at a theoretical 50%), while the victory occurred on moves 71-72 on average.

From this experiment, we can draw two conclusions:

1. The results of our algorithm are close to the theory, so it can be used to assess further.
2. The game under these assumptions is of an equal nature.

**Second Experiment**

Let's make the following assumption Player A's balance is 1 dollar, and Player B's balance is still 10 dollars. The non-ergodicity of the process, which we talked about above, comes into play in this experiment. Let’s see what impact it has on the results:




The average probability of Player A losing was 97.22% (at a theoretical 90.91%), and that of Player B was 2.78% (at a theoretical 9.09%), with the loss occurring on the 7th move on average.

The results of the experiment may seem paradoxical. Let's conduct it a few more times:

| Simulation number | Probability of A losing | Probability of B losing | On which move does the loss occur on average |
|---|---|---|---|
| 1 | 97,22% | 9,09% | 7,31 |
| 2 | 98,56% | 1,43% | 7,13 |
| 3 | 98,44% | 1,56% | 7,15 |
| 4 | 98,35% | 1,65% | 6,86 |
| 5 | 98,08% | 1,92% | 6,97 |

This is the principle on which any casino is built, i.e., the casino initially has a larger balance than you, and even in a fair game (with a probability of 50% to 50% - as in coin tossing), the casino wins. So the player who has a larger balance wins, and he wins more and more as the game continues. Further, these players we will call the Pool Holders. Referring to the article's epigraph, winning the Pool Holder is indeed a miracle. The popularity of the casinos demonstrates that people believe in miracles.

# 4x_Conclusion

Based on the theory above, we analyzed the coin toss game and created the corresponding risk pool. We invite you to join it. Having analyzed the game, we were able to select game parameters in such a way that the game is interesting for both parties - and for players and pool holders. We draw attention to the fact that this is just a model game based on the simplest random process. In the future, we will offer models and risk pools of random processes in the world around us.