# CME 252: Support Vector Machines

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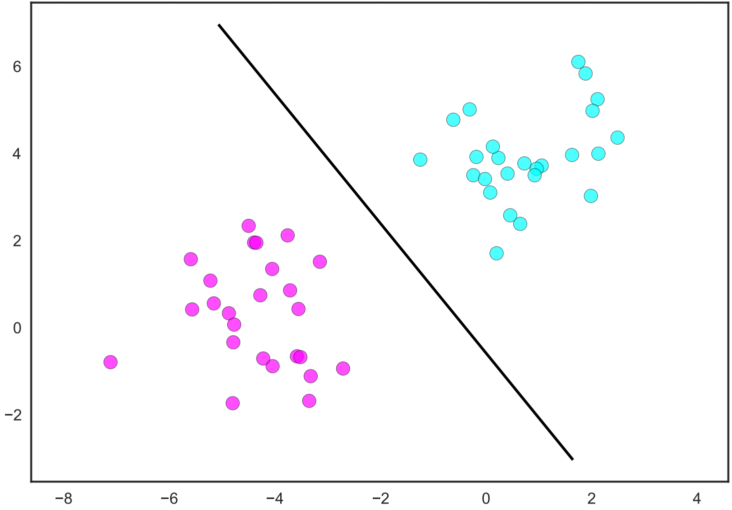
# Introduction

# Outline

# Support Vector Machines

- many related/overlapping names:
  - maximum margin classifier
  - support vector classifier
  - (robust) linear discrimination/classification
  - support vector machine
- I won't always use the right name
- we'll start with:
  - find a hyperplane to separate data points into two classes
  - use hyperplane to classify new (unseen) points

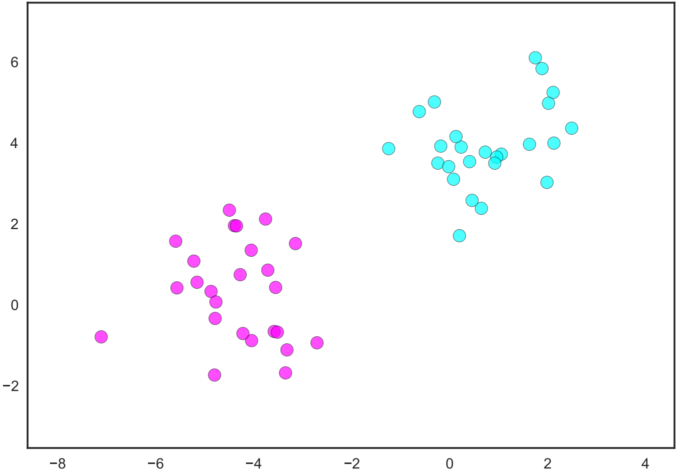
# Support Vector Machines



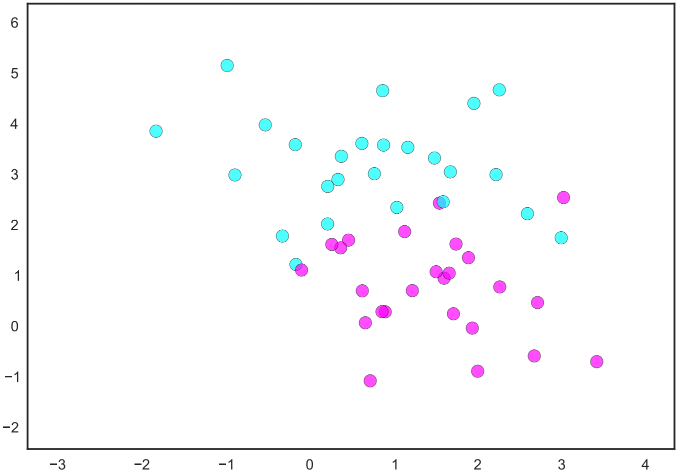
# Scenarios

- classify data in increasingly sophisticated scenarios:
  - strictly linearly separable
  - approximately (not strictly) linearly separable
  - approximately non-linearly separable (hyperplanes won't work)

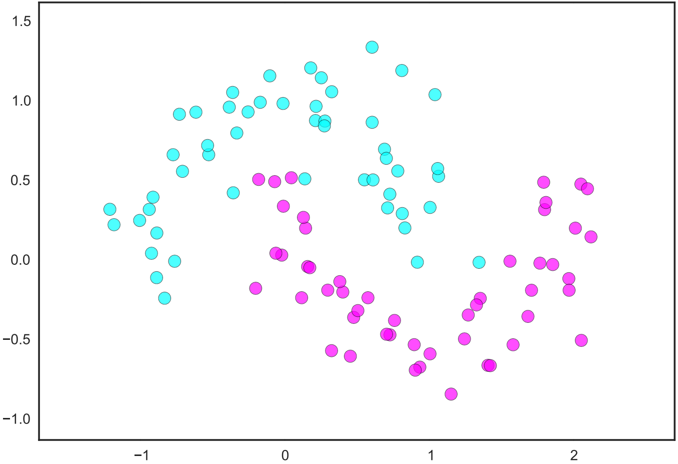
# Strictly Linearly Separable Data



# Approximately Linearly Separable Data



# Approximately Non-linearly Separable



# Linearly Separable Problem

# Outline

# Linearly Separable Problem

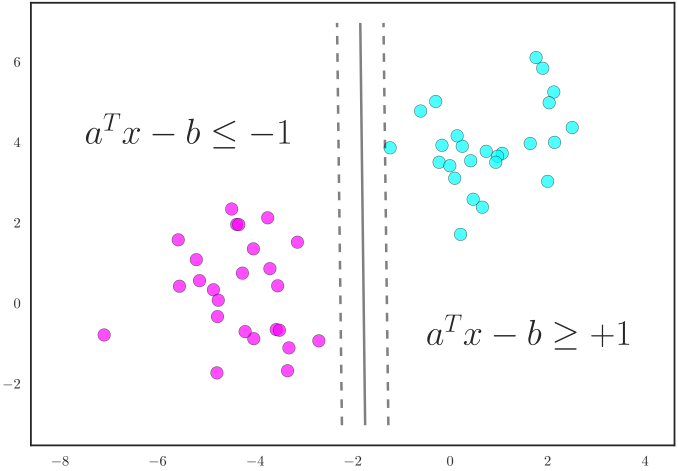
- data: $x_i \in \mathbf{R}^n$ with labels $y_i \in \{+1, -1\}$ for $i = 1, \ldots, N$
- assume **strictly** linearly separable
- find hyperplane $\{x \mid a^T x = b\}$ that separates points by label

$$
\begin{aligned}
a^T x_i - b > 0 &\text{ if } y_i = +1 \\
a^T x_i - b < 0 &\text{ if } y_i = -1
\end{aligned}
$$

- **rescale** $a$, $b$ so that

$$
\begin{aligned}
a^T x_i - b \geq +1 &\text{ if } y_i = +1 \\
a^T x_i - b \leq -1 &\text{ if } y_i = -1
\end{aligned}
$$

# Linearly Separable Problem



# Linearly Separable Problem

- for all $i$, rewrite constraints as

$$y_i \left( a^T x_i - b \right) \geq 1$$

- get **feasibility** problem

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & y_i \left( a^T x_i - b \right) \geq 1 \text{ for } i = 1, \ldots, N \end{array}$$

with variables $a \in \mathbf{R}^n$, $b \in \mathbf{R}$

# CVXPY for Separable Problem

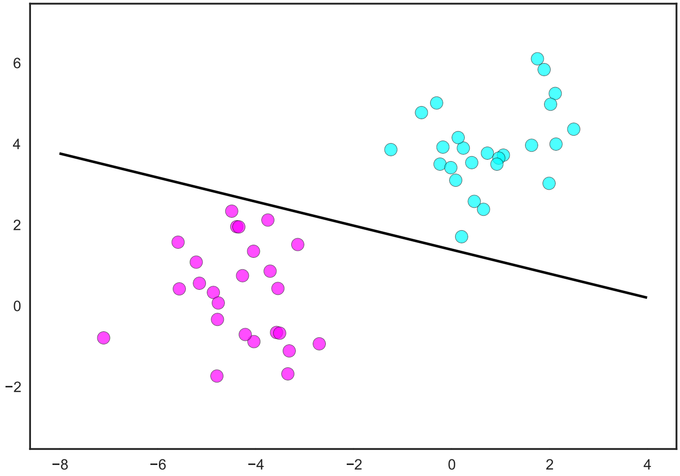
```
a = Variable(n)
b = Variable()

obj = Minimize(0)
constr = [mul_elemwise(y, X*a - b) >= 1]
Problem(obj, constr).solve()
```

# Which Separator?

# Outline

# Which Separator?



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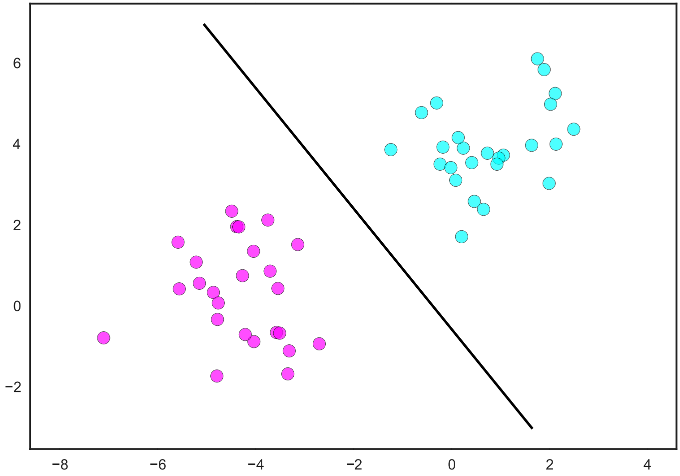

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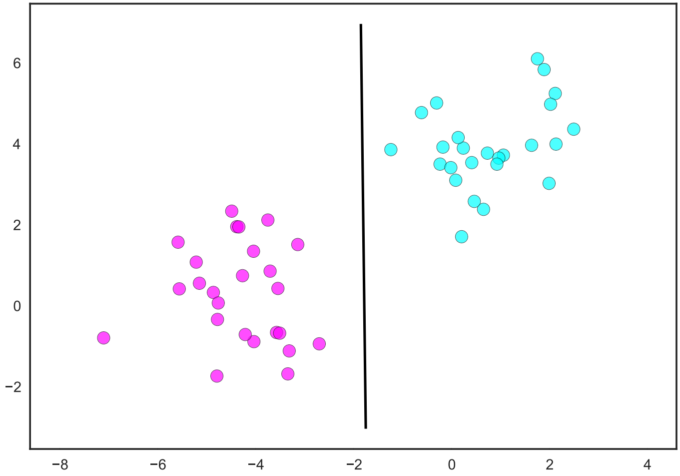

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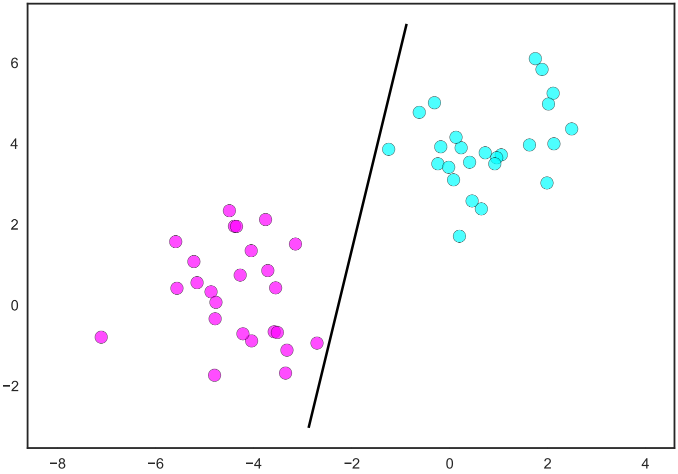


# Maximum Margin Classifier

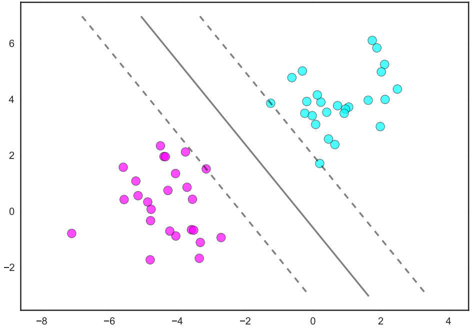
# Outline

# Maximum Margin Classifier

- infinitely many choices for separating hyperplane
- choose one which maximizes **width** of separating **slab**

$$\{x \mid -1 \le a^T x - b \le +1\}$$

- "maximum margin" or "robust linear" classifier



# Maximum Margin Classifier

- **margin**, or width of separating slab

$$\{x \mid -1 \le a^T x - b \le +1\}$$

  is $2/\|a\|_2$ (via linear algebra)
- suggests optimization problem

$$\begin{array}{ll} \text{maximize} & 2/\|a\|_2 \\ \text{subject to} & y_i \left(a^T x_i - b\right) \ge 1 \text{ for } i = 1, \ldots, N \end{array}$$

- but not convex!

# Maximum Margin Classifier

- reformulate:

$$\text{maximize } 2/\|a\|_2 \iff \text{minimize } \|a\|_2$$

gives

$$\begin{array}{ll} \text{minimize} & \|a\|_2 \\ \text{subject to} & y_i \left(a^T x_i - b\right) \geq 1 \text{ for } i = 1, \ldots, N, \end{array}$$

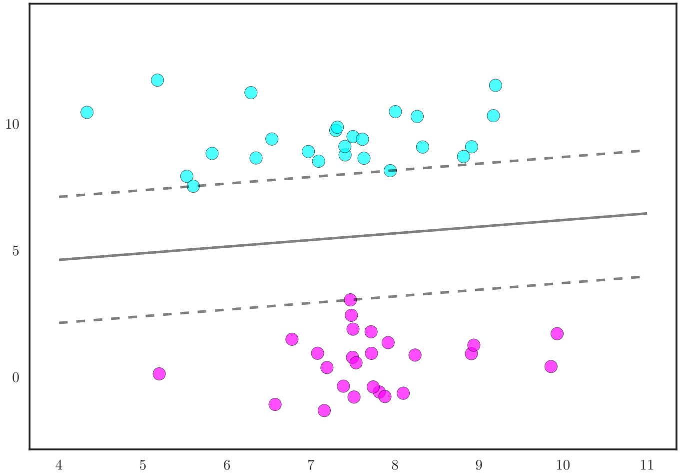
the **maximum margin classifier** (MMC) problem

# Maximum Margin Classifier in CVXPY

```
a = Variable(n)
b = Variable()

obj = Minimize(norm(a))
constr = [mul_elemwise(y, X*a - b) >= 1]
Problem(obj, constr).solve()
```

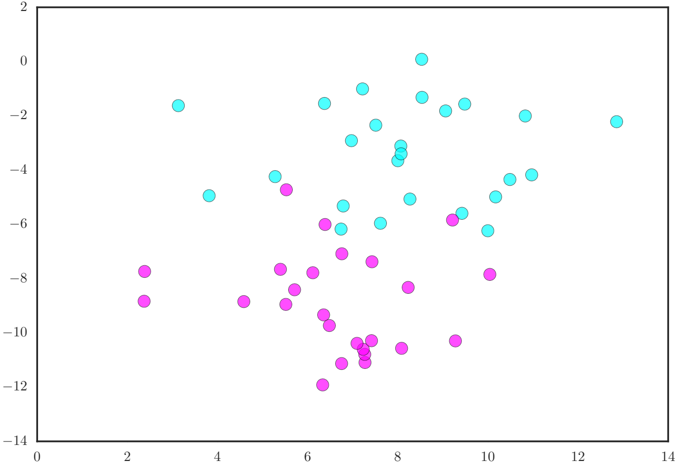
# Maximum Margin Classifier



# Non-separable Linear Classification

# Outline

# Non-separable Linear Classification



# Non-separable Linear Classification

- no separating hyperplane exists
- try finding linear separator

```
obj = Minimize(0)
constr = [mul_elemwise(y, X*a - b) >= 1]
prob = Problem(obj, constr)
prob.solve()
```

- results in `prob.status == 'infeasible'`

# Sparse Violation Classifier

# Outline

# Sparse Violation Classifier

- idea: "relax" constraints to make problem feasible
- add **slack** variables $u \in \mathbf{R}_+^N$ to allow data points to be on "wrong side" of hyperplane

$$y_i \left( a^T x_i - b \right) \geq 1 - u_i, \quad u_i \geq 0$$

  - $u_i = 0$: $x_i$ on **right** side of hyperplane
  - $0 < u_i < 1$: $x_i$ on **right** side, but **inside slab** $\{x \mid -1 \leq a^T x - b \leq +1\}$
  - $u_i > 1$: $x_i$ on **wrong** side of hyperplane

# Sparse Violation Classifier

- $u$ gives measure of how much constraints are violated
- for large $u$ can make **any** data feasible
- want $u$ "small"; minimize its sum

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^T u \\
\text{subject to} & y_i \left( a^T x_i - b \right) \geq 1 - u_i \text{ for } i = 1, \ldots, N \\
& u \geq 0
\end{array}
$$

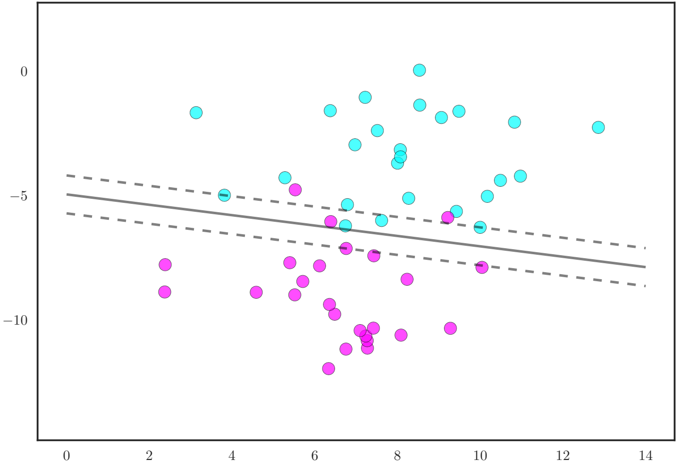
- I'll call it **sparse violation classifier** (SpVC)
- $\mathbf{1}^T u = \|u\|_1$, since $u \geq 0$; good **heuristic** for separator with few (sparse) violations

# CVXPY

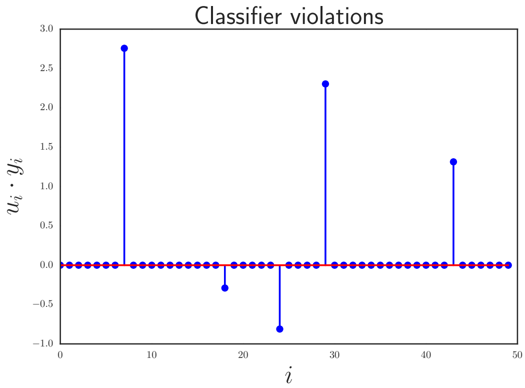
```
a = Variable(n)
b = Variable()
u = Variable(N)

obj = Minimize(sum_entries(u))
constr = [mul_elemwise(y, X*a - b) >= 1 - u, u >= 0]
Problem(obj, constr).solve()
```

# Example



# Example




- "$+$" class has 3 misclassified points
- "$-$" class has 2 correctly classified, but inside slab

# Support Vector Classifier

# Outline

# Support Vector Classifier

- idea: combine aspects of last two classifiers
  - sparse violations of SpVC
  - robustness of large separating slab in MMC
- optimize both:

$$
\begin{array}{ll}
\text{minimize} & \|a\|_2 + \rho \mathbf{1}^T u \\
\text{subject to} & y_i \left( a^T x_i - b \right) \geq 1 - u_i \text{ for } i = 1, \ldots, N \\
& u \geq 0
\end{array}
$$

- $\rho > 0$ trades-off between margin $2/\|a\|_2$ and classification violations $\mathbf{1}^T u$

# Loss Functions

# Outline

# Hinge Loss

- in SpVC, it follows from $y_i \left(a^T x_i - b\right) \geq 1 - u_i$, $u_i \geq 0$, that

$$u_i = \begin{cases} 0 & y_i \left(a^T x_i - b\right) \geq 1 \\ 1 - y_i \left(a^T x_i - b\right) & y_i \left(a^T x_i - b\right) < 1 \end{cases}$$

- rewrite as $u_i = \ell_h \left[y_i \left(a^T x_i - b\right)\right]$, where

$$\ell_h(z) = \begin{cases} 0 & z \geq 1 \\ 1 - z & z < 1 \end{cases}$$

is the **hinge loss** function, equivalently: $\max(0, 1 - z)$ or $(1 - z)_+$

# Hinge Loss Problem

- note that $\ell_h$ is convex, so we can rewrite SpVC as the **equivalent problem**

$$\text{minimize} \quad \sum_{i=1}^{N} \ell_h \left[ y_i \left( a^T x_i - b \right) \right]$$

- unconstrained (non-differentiable) convex problem
- in CVXPY:

```
def hinge(z):
    return pos(1-z)

r = mul_elemwise(y, X*a - b)
obj = Minimize(sum_entries(hinge(r)))
Problem(obj).solve()
```

# other

- show indicator function to show same as feasibility problem
- logistic loss is logistic regression
- other random loss functions
- infinity norm of violations for max violation

# Non-separable Linear Classification

- relaxed feasibility problem
- l1 penality to minimize misclassificaiton: pure LP
- tradeoff between classification and width of slab: SOCP

# Hinge loss

- reformulate as hinge loss objective
- general loss function form... $l(Ax + b)$

# logistic

- change loss function to get logistic loss
- other loss functions

# regularization

- regularize to get sparse classifier...

# nonlinear discrimination

- adding features
- polynomial discrimination any different?
- rbf kernel? radial basis function
- kernel methods and relationship with convex opt...

# algorithms

- note that so far, we have said **nothing** about **how** to compute a supporting vector
- we have focused on modeling
- that's OK, we're focusing on modeling
- algorithms involve duality and optimality conditions

# scikitlearn comparison

- make sure it matches up with python SVM formulation
- maybe even do a timing comparison...

# data science perspective

- cleaning and centering data
- sparse predictors