CME 252: Support Vector Machines

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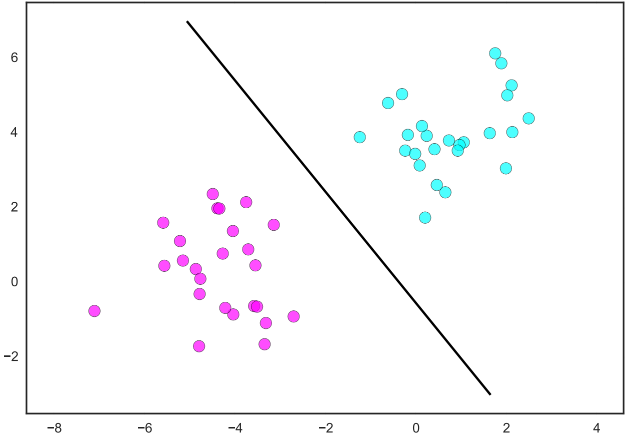
Intro

Outline

Support Vector Machines

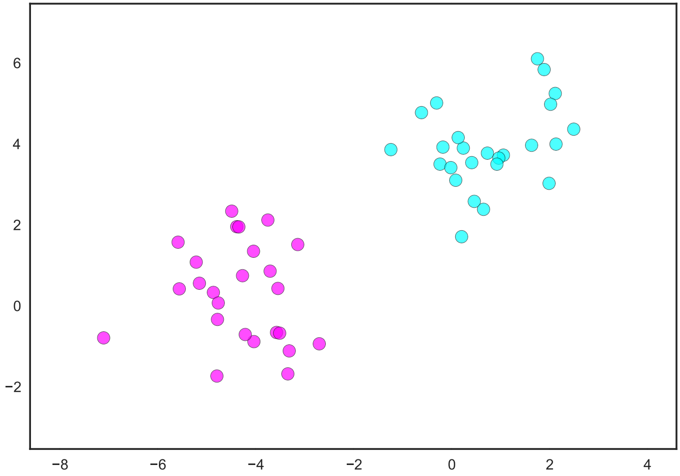
- find a hyperplane to separate data points into two classes
- use hyperplane to classify new (unseen) points



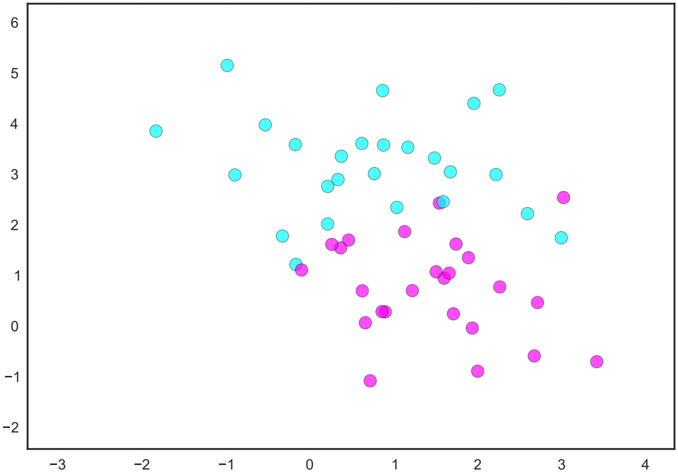
Scenarios

- assume data falls into one of these categories:
  - strictly linearly separable
  - approximately (not strictly) linearly separable
  - approximately non-linearly separable (hyperplanes won't work)

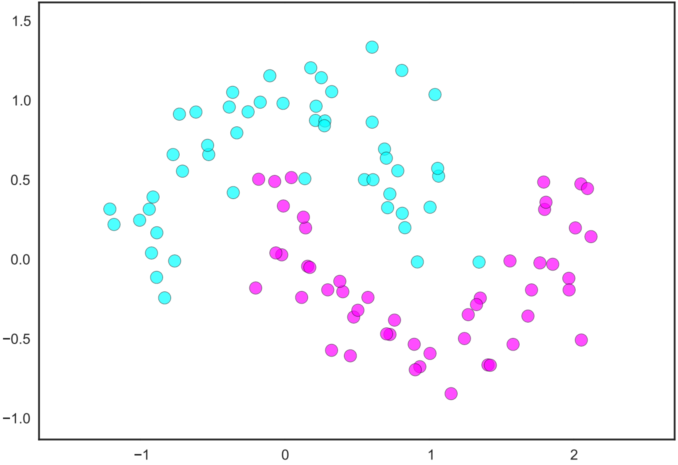
Strictly Linearly Separable Data



Approximately Linearly Separable Data



Approximately Non-linearly Separable



Linearly Separable Problem

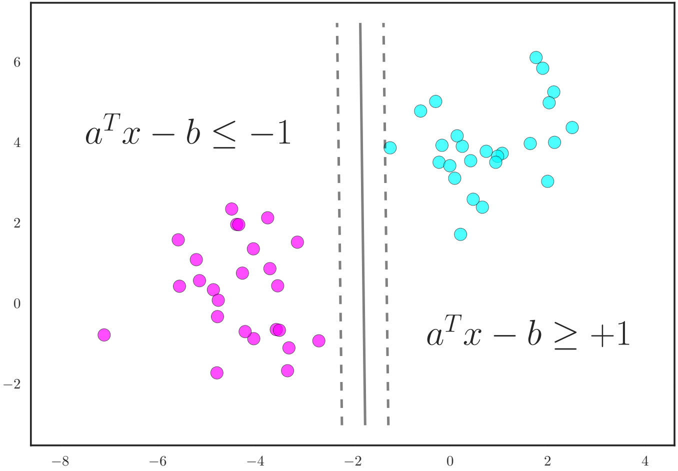
- data: $x_i \in \mathbf{R}^n$ with labels $y_i \in \{+1, -1\}$ for $i = 1, \ldots, m$
- assume **strictly** linearly separable
- find hyperplane $\{x \mid a^T x = b\}$ that separates points by label

$$a^T x_i - b > 0 \text{ if } y_i = +1$$
$$a^T x_i - b < 0 \text{ if } y_i = -1$$

- **rescale** a, b so that

$$a^T x_i - b \geq +1 \text{ if } y_i = +1$$
$$a^T x_i - b \leq -1 \text{ if } y_i = -1$$

Linearly Separable Problem



Linearly Separable Problem

- for all i, rewrite constraints as

$$y_i \left(a^T x_i - b\right) \geq 1$$

- get **feasibility** problem

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & y_i \left(a^T x_i - b\right) \geq 1 \text{ for } i = 1, \ldots, m \end{array}$$

with variables $a \in \mathbf{R}^n$, $b \in \mathbf{R}$

CVXPY for Separable Problem

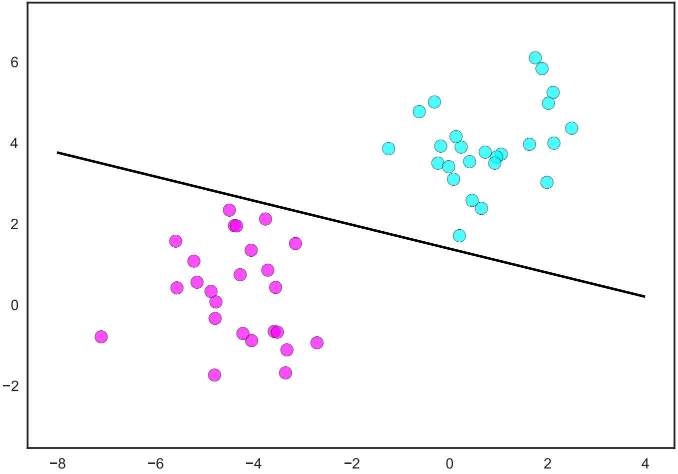
```
a = Variable(n)
b = Variable()

obj = Minimize(0)
constr = [mul_elemwise(y, X*a - b) >= 1]
Problem(obj, constr).solve()
```

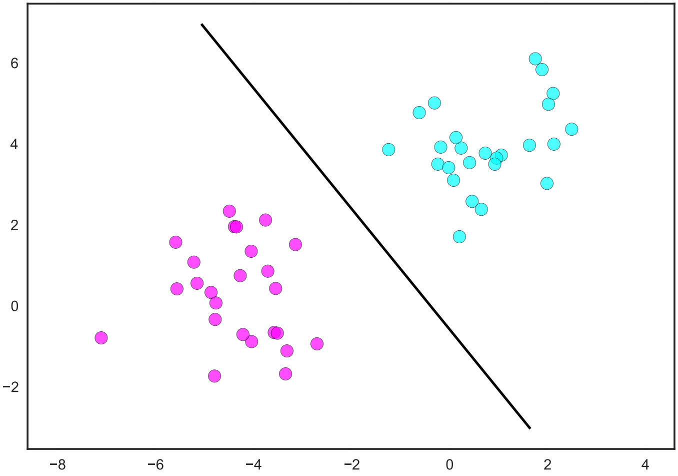
Which Separator?

Outline

Which Separator?



Which Separator?



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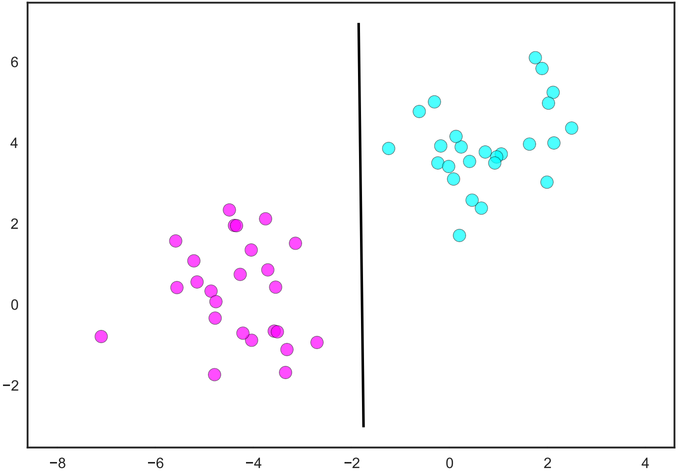
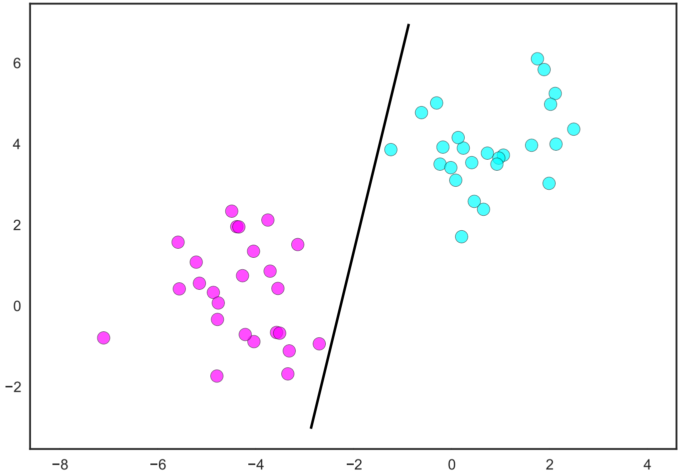


Which Separator?



Maximum Margin Classifier

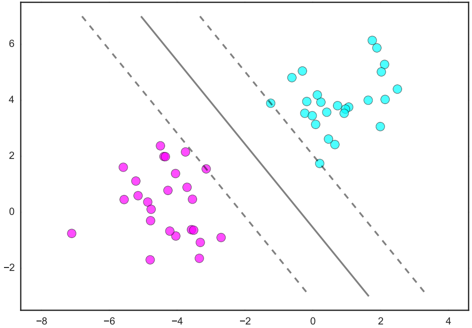
Outline

Maximum Margin Classifier

- infinitely many choices for separating hyperplane
- choose one which maximizes **width** of separating **slab**

$$\{x \mid -1 \leq a^T x - b \leq +1\}$$

- "maximum margin" or "robust linear" classifier



Maximum Margin Classifier

- width of separating slab

$$\{x \mid -1 \le a^T x - b \le +1\}$$

  is $2/\|a\|_2$ (via linear algebra)
- suggests optimization problem

$$\begin{array}{ll} \text{maximize} & 2/\|a\|_2 \\ \text{subject to} & y_i \left(a^T x_i - b\right) \ge 1 \text{ for } i = 1, \ldots, m \end{array}$$

- but not convex!

Maximum Margin Classifier

- reformulate:

$$\text{maximize } 2/\|a\|_2 \iff \text{minimize } \|a\|_2$$

  gives

$$\begin{array}{ll} \text{minimize} & \|a\|_2 \\ \text{subject to} & y_i \left(a^T x_i - b\right) \geq 1 \text{ for } i = 1, \ldots, m, \end{array}$$

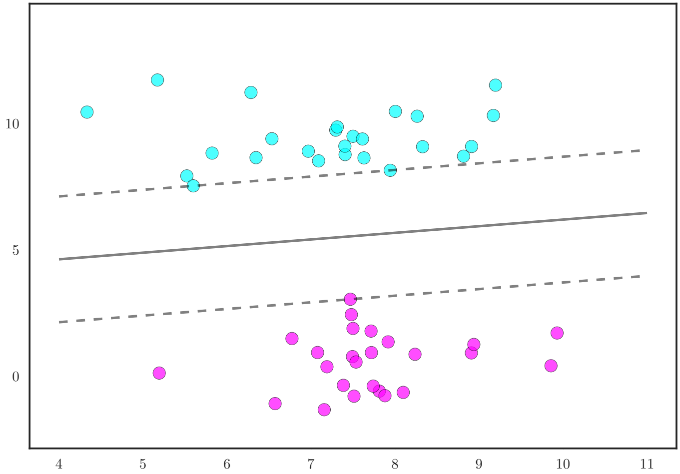
  the **maximum margin classifier** problem

Maximum Margin Classifier in CVXPY

```
a = Variable(n)
b = Variable()

obj = Minimize(norm(a))
constr = [mul_elemwise(y, X*a - b) >= 1]
Problem(obj, constr).solve()
```

Maximum Margin Classifier



Nonseparable linear classification

- relaxed feasibility problem
- l1 penality to minimize misclassificaiton: pure LP
- tradeoff between classification and width of slab: SOCP

Hinge loss

- reformulate as hinge loss objective
- general loss function form... $l(Ax + b)$

logistic

- change loss function to get logistic loss
- other loss functions

regularization

- regularize to get sparse classifier. . .

nonlinear discrimination

- adding features
- polynomial discrimination any different?
- rbf kernel? radial basis function
- kernel methods and relationship with convex opt. . .

algorithms

- note that so far, we have said **nothing** about **how** to compute a supporting vector
- we have focused on modeling
- that's OK, we're focusing on modeling
- algorithms involve duality and optimality conditions

scikitlearn comparison

- make sure it matches up with python SVM formulation
- maybe even do a timing comparison. . .

data science perspective

- cleaning and centering data
- sparse predictors