Chapter 1 PS: Manifolds and Smooth Maps

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Section 1.1: *Definitions*

Problem 1.1.3: Let $X \subset \mathbb{R}^N, Y \subset \mathbb{R}^M, Z \subset \mathbb{R}^L$ and $f : X \to Y, g : Y \to Z$ be smooth. Show $g \circ f : X \to Z$ is smooth and that if they're both diffeomorphisms, so is $g \circ f$

Proof If $O \subset Z$ is open, then $g^{-1}(O) \subset Y$ is open as g is continuous. Similarly, $f^{-1}(g^{-1}(O)) \subset X$ is too. Thus $g \circ f$ is smooth. If they're diffeomorphisms, then as bijectivity is transitive and similar logic extends to g^{-1}, f^{-1} as they're both smooth, the statement follows. $\square$

Problem 1.1.5: Show that every k-dimensional vector subspace V of $\mathbb{R}^N$ is a manifold diffeomorphic to $\mathbb{R}^k$ and that all linear maps on V are smooth.

Note: If $\phi : \mathbb{R}^k \to V$ is a linear isomorphism, then the corresponding coordinate functions are linear functionals on V, termed ***Linear Coordinates.***

Proof Let $\mathcal{B}_V = \{v_1, \cdots, v_k\}$. Then $\forall v \in V, \exists a_1, \cdots, a_k \ni v = \sum a_i v_i$. Define $\phi : \mathbb{R}^k \to V, \phi(a_1, \cdots, a_k) = \sum a_i v_i$. As $\mathcal{B}_V$ forms a basis, such is surjective as $\mathcal{B}_V$ spans V and injective due to linear independence. Thus there's an inverse, which is obvious by decomposing v into its linear combination, then sending its coefficients. Both have partials w.r.t. a_i for all orders, and thus they're both smooth. So $V \cong_D \mathbb{R}^k$. As every linear map on a FDVS is continuous, follows. $\square$

Problem 1.1.11: Show that one can't parameterize $S^k, k \geq 1$ by a single parameterization.

Proof BWOC, say $\phi : S^k \to U \subset \mathbb{R}^k$ is a diffeomorphism. U is open, thus fails to be compact via Heine-Borel. But S^k is closed and bounded, thus compact by Heine-Borel. Homeomorphisms, thus diffeomorphisms, preserves compactness, a contradiction. $\square$

Problem 1.1.12: A ***Stereographic Projection*** is the function $\pi : S^2 - \{N\} \to \mathbb{R}^2$ for north pole $N = (0, 0, 1)$. So $\pi(p)$ is a point where the N-p line intersects the xy plane. Prove that π is a diffeomorphism.

Proof Say $p = (x, y, z) \in S^2 - \{N\}$. Parameterize the line as $\psi(t) = (xt, yt, zt + (1-t))$. Observe that $\psi(0) = (0, 0, 1)$ and $\psi(1) = (x, y, z)$. Then such intersects the xy plane when $0 = zt + 1 - t = (z-1)t + 1 \Rightarrow t = \frac{-1}{z-1} = \frac{1}{1-z}$. Note the only point s.t. $z = 1$ is N, though $N \notin S^2 - \{N\}$ anyway. So $(x, y) \in \mathbb{R}^2 - \{(0, 0)\}$. Thus, $\pi(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z})$

Such is smooth except when $z = 1$, which is excluded anyway. Now, any 2 points have a unique line segment connecting them. So $\forall x = (x_1, x_2) \in \mathbb{R}^2$, the line connecting N and x must be unique. Case 1: Say x lies outside the unit circle. As the upper hemisphere is concave down, $\overline{Nx}$ must intersect $S^2 - \{N\}$. Case 2: Say x lies inside the unit circle. Then the line segment fully lies in S^2. But the line is unbounded, thus must leave, hence intersects, $S^2 - \{N\}$. Case 3: Say x lies on the unit circle. Then $x \in S^2 - \{N\}$. So π is surjective.

BWOC, say the N-x line intersects $S^2 - \{N\}$ at least twice. That rules out case 3, as the line is fully in the sphere, intersects where $x \in S^2 - \{N\}, ||x||^2 = 1$, then has a monotonically increasing norm, thus never will intersect $S^2 - \{N\}$ again. That also rules out case 2 due to concavity. That leaves case 1. Take N-x, and slice a plane where the z-component is arbitrary. "Rotate" the plane (yaw) until it's merely an xz plane. Thus the arc can be described as $z = \sqrt{1 - x^2}$ and line as $z = 1 + \frac{x}{\alpha}, \alpha > 1$. Then $\sqrt{1 - x^2} = 1 + \frac{x}{\alpha} \Rightarrow 1 - x^2 = \frac{x^2}{\alpha^2} + \frac{2x}{\alpha} + 1 \Rightarrow 0 = x^2(1 + \frac{1}{\alpha^2}) + \frac{2x}{\alpha} = x(x(1 + \frac{1}{\alpha^2}) + \frac{2}{\alpha})$. By rotating, y was forced to be 0, so if $x = 0$, that leaves $z = 1$, a contradiction. So the eqn has only 1 soln. But recall: we assume the line intersects $S^2 - \{N\}$ at least twice. So these distinct intersections must lie on the line, thus are on the plane and remains when rotated. Yet they're simultaneously on the arc: a contradiction. So π is injective.

As the components of π are rational, partials of all orders exist. Instead of using the InFT, we construct the inverse explicitly. Let $(u, v) \in \mathbb{R}^2$ and $p = (x, y, z) \in S^2 - \{N\}$. Parameterize as $l(t) = N + t((u, v, 0) - N) = (tu, tv, 1 - t)$. Then $(x, y, z) = (tu, tv, 1 - t) \Rightarrow x^2 + y^2 + z^2 = t^2u^2 + t^2v^2 + (1 - t)^2 = 1 \Rightarrow t^2(u^2 + v^2) + (1 - 2t + t^2) = t^2(u^2 + v^2 + 1) - 2t + 1 = 1 \Rightarrow t(t(u^2 + v^2 + 1) - 2) = 0 \Rightarrow t = 0$ or $\frac{2}{u^2+v^2+1}$. But $t = 0$ gives N and let $r = u^2 + v^2$. Thus $\pi^{-1}(u, v) = (\frac{2u}{r+1}, \frac{2v}{r+1}, \frac{r-1}{r+1})$. That's rational, thus partials of all orders exist. But then π is a global diffeomorphism, so $S^2 - \{N\} \cong_D \mathbb{R}^2$ $\square$

Problem 1.1.13: Generalize by defining a diffeomorphism $S^k - \{N\} \to \mathbb{R}^k$

Proof Let $p = (x_1, \cdots, x_{k+1}) \in S^k - \{N\}$. That suggests $\pi(x) = (\frac{x_1}{1-x_{k+1}}, \cdots, \frac{x_k}{1-x_{k+1}})$. Note that the inverse has a similar construction: $\pi^{-1}(u) = (\frac{2u_1}{||u||^2+1}, \cdots, \frac{2u_k}{||u||^2+1}, \frac{||u||^2-1}{||u||^2+1})$, thus its existence means bijectivity. Both are rational, and thus are smooth. So $S^k - \{N\} \cong_D \mathbb{R}^k$ $\square$

Problem 1.1.16: The ***Diagonal*** Δ in $X \times X$ is the set of points of form (x, x). Show that Δ is diffeomorphic to X, thus Δ is a manifold iff X is.

Proof Define $\phi : X \to X \times X, \phi(x) = (x, x)$. Observe $\phi(x) = \phi(y) \Rightarrow (x, x) = (y, y) \Rightarrow x = y$. Surjectivity is trivial. Clearly ϕ and its inverse has partials of all orders, thus $\Delta \cong_D X$ $\square$

Section 1.2: *Derivatives and Tangents*

Problem 1.2.3: Let V be a vector subspace of $\mathbb{R}^N$. Show that $T_x(V) = V$ if $x \in V$

Proof By Q1.1.5, recall that $\exists k \in \mathbb{N} \ni V \cong_D \mathbb{R}^k$ which there's a linear isomorphism, say ψ. Define $\phi : \mathbb{R}^k \to V, \phi(u) = x + \psi(u)$. Note that ψ, thus ϕ, is smooth. Yet as ψ is linear, follows $D\phi_0(u) = \psi(u) \Rightarrow T_x(V) = Im(D\psi) = V$ $\square$

Problem 1.2.4: Say $f : X \to Y$ is a diffeomorphism. Prove that $\forall x \in X, Df_x$ defines an isomorphism of tangent spaces.

Proof Note that $Df_x : T_x(X) \to T_{f(x)}(Y)$. As f is a diffeomorphism, its inverse is smooth. Now, $D(f^{-1} \circ f)_x = D(id_X)_x = id_{T_x(X)} = Df^{-1}_{f(x)} \circ Df_x$ by definition and Chain Rule. Similarly, $D(f \circ f^{-1})_x = id_{T_{f(x)}(Y)} = Df_x \circ Df^{-1}_{f(x)}$. Thus Df_x has an inverse. As all derivatives as linear, follows that $T_x(X) \cong_I T_{f(x)}(Y)$ $\square$

Problem 1.2.5: Prove that $\mathbb{R}^k \ncong_D \mathbb{R}^l \iff k \neq l$

Note: They have the same cardinality: uncountable. Thus there's a bijection.

Proof Forward is trivial via contrapositive. For backwards, assume BWOC that there's diffeomorphism $f : \mathbb{R}^k \to \mathbb{R}^l$. By Q1.2.4, Df_x defines an isomorphism $T_x(\mathbb{R}^k) \cong_I T_{f(x)}(\mathbb{R}^l)$. But by Q1.2.5, those are trivial subspaces, so $\mathbb{R}^k \cong_I \mathbb{R}^l$. Yet isomorphism preserves dimension, a contradiction. $\square$

Problem 1.2.6: $T_{(a,b)}(S^1)$ is a 1D subspace of $\mathbb{R}^2$. Calculate.

Solution Geometrically, that's a line tangent at $(a, b) \in S^1$. Recall the gradient is normal, thus orthogonal to the tangent. Thus $T_{(a,b)}(S^1) = \{v \in \mathbb{R}^2 | Df(a, b) \cdot v = 0\}$. As $f(a, b) = a^2 + b^2 - 1 \Rightarrow Df(a, b) = (2a, 2b)$, thus $(2a, 2b) \cdots (v_1, v_2) = 2av_1 + 2bv_2 = 0$. So: [***Answer***] $\boxed{T_{(a,b)}(S^1) = \{v \in \mathbb{R}^2 | av_1 + bv_2 = 0\}}$

Problem 1.2.8: What's the tangent space to paraboloid $x^2+y^2-z^2=a$ at $(\sqrt{a},0,0), a>0$?

Solution By similar logic, $Df(x,y,z)=(2x,2y,-2z)$. Then $2xv_1+2yv_2-2zv_3=0 \Rightarrow xv_1+yv_2-zv_3=xv_1+yv_2-\sqrt{x^2+y^2-a}v_3=0$. Observe that if v_2,v_3 were arbitrary, then v_1 is determined as long as $x\neq 0$. But at $(\sqrt{a},0,0)$, follows that $v_1=\frac{0v_2-(a+0^2-a)v_3}{\sqrt{a}}=0$. Thus for paraboloid P, then [***Answer***] $\boxed{T_{(\sqrt{a},0,0)}(P)=\{v|v_2,v_3\in\mathbb{R},v_1=0\}\cong\mathbb{R}^2}$

Problem 1.2.12: ***Curve*** c in a manifold X is a smooth $t\to c(t)$ of an interval of $\mathbb{R}\to X$. The ***Velocity Vector*** of c at time t_0 is $\frac{dc}{dt}(t_0):=Dc_{t_0}(1)\in T_{x_0}(X), x_0=c(t_0)$ and $Dc_{t_0}:\mathbb{R}\to T_{x_0}(X)$

(**a**) If $X=\mathbb{R}^k$ and $c(t)=(c_1(t),\cdots,c_k(t))$, check that $\frac{dc}{dt}(t_0)=(c_1'(t_0),\cdots,c_k'(t_0))$

Proof Recall that $Dc_{t_0}(h)$ is the directional derivative of c at t_0 in h, so $Dc_{t_0}(h)=(\frac{d}{dt}c_1(t_0),\cdots,\frac{d}{dt}c_k(t_0))\cdot h=h\cdot(c_1'(t_0),\cdots,c_k'(t_0))$. Thus for $h=1$, follows. $\square$

(**b**) Prove that every vector in $T_x(X)$ is the velocity vector of some curve in X and conversely.

Proof Forward: Let $v\in T_x(X)$. As X is a manifold, $\exists\phi:U\subset\mathbb{R}^k\to X\ni\phi(0)=x, T_x(X)=\{D\phi_0(u)|u\in\mathbb{R}^k\}$. So $\exists u\in\mathbb{R}^k\ni v=D\phi_0(u)$. Define $c(t):=\phi(tu)$. Observe $c(0)=\phi(0)=x$ and that $\frac{dc}{dt}(0)=D\phi_0(u)=v$

Backward: Time-translate curve c s.t. $c(0)=x$. Define $\tilde{c}(t):=\phi^{-1}(c(t))$. Then $\tilde{c}(0)=\phi^{-1}(x)=0\in\mathbb{R}^k$. Thus $\frac{dc}{dt}(0)=D\phi_0(\tilde{c}'(0))\in T_x(X)$ as $\tilde{c}'(0)\in\mathbb{R}^k$ and $D\phi_0:\mathbb{R}^k\to T_x(X)$ $\square$

Addendum: Thus $T_x(X)=\{\text{velocity vectors of smooth curves c in X}\ni c(0)=x\}$

Section 1.3: *InFT and Immersions*

Problem 1.3.3: Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be a local diffeomorphism. Prove that Imf is an open interval, yet $\mathbb{R}^1 \cong_D f(\mathbb{R}^1)$

Proof By definition, $\forall x \in \mathbb{R}, \exists$ neighborhoods $x \in U_x, f(x) \in V_x \ni U_x \cong_D V_x$. However, f remains bijective, smooth, and its inverse also smooth, on any subset of neighborhoods. By definition, $\exists$ open $x \in O_x \subset U_x \ni O_x \cong_D f(O_x)$. But then $f(\mathbb{R}) = \cup_{x\in\mathbb{R}} f(O_x)$ is the union of open is open in $\mathbb{R}$. As $\mathbb{R}$ is connected and f is continuous, $f(\mathbb{R})$ is connected. But the only connected, open subset is an open interval. Now, f is still smooth at each point and is bijective. So $\exists f^{-1}$ and is locally smooth, thus $\mathbb{R} \cong_D f(\mathbb{R})$ □

Problem 1.3.4: Construct a local diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ that isn't a diffeomorphism onto its image.

Solution Recall InFT holds at x iff f is smooth and Df_x is an isomorphism. So the only way f can't be diffeomorphism is that f can't be bijective. Recall $f(y) = (cosy, siny)$ maps $\mathbb{R} \to S^1$, but Df isn't square. So try [***Answer***] $\boxed{f(x, y) = (e^x cosy, e^x siny)}$ Then $\mathbb{R}^2 \to S^1$ and $det[Df_{(a,b)}] = e^{2x} > 0$. So by InFT, f is a local diffeomorphism $\forall (a, b) \in \mathbb{R}^2$. But then $f(x, y) = f(x, y + 2\pi)$, so f can't be a global diffeomorphism.

Problem 1.3.5: Prove that an injective local diffeomorphism $f : X \to Y$ is a diffeomorphism of X onto an open subset of Y.

Proof Per Q1.3.3, local diffeomorphism also means $\forall x \in X, \exists$ open $x \in U_x \subset X \ni U_x \cong_D f(U_x)$. As diffeomorphisms are open maps (for f^{-1} is continuous), follows $f(X) = \cup_{x\in X} f(U_x)$ is open as the union of open sets. f is smooth, injective and $f : X \to f(X)$ is surjective by definition. Thus the inverse is well-defined, hence "gluing" together all the local smooth inverses is also well-defined. Thus $X \cong_D f(X)$ □

Problem 1.3.6: If f and g are immersions...

Note: Recall $f : M \to N$ is an ***Immersion*** if M, N are manifolds with $Df_x : T_x(M) \to T_{f(x)}(N)$ is injective.

(a) Show that $f \times g$ is too.

Proof As both are smooth, then $f \times g$ are too.

I claim that for manifolds X, Y, then $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$. PROOF: Consider f that sends L. to R. by $f(v) = (D(\pi_X)_{(x,y)}(v), D(\pi_Y)_{(x,y)}(v))$ for canonical projections π_X, π_Y from $X \times Y$ to X and Y, respectively. The concatenation of linear is linear. Similarly defined $g : T_x(X) \times T_y(Y) \to T_{(x,y)}(X \times Y), g(v, w) = D(l_X)_x(v) + D(l_Y)_y(w)$ where $l_X : X \to X \times Y$ sends $X \to X \times \{x\}$ slice and similarly for l_Y. Then $\pi_X \circ l_X = id_X, \pi_Y \circ l_Y = id_Y$. Thus:

$$\begin{aligned}(f \circ g)(v, w) &= f(D(l_X)_x(v) + D(l_Y)_y(w)) \\ &= (D(\pi_X)_{(x,y)}[D(l_X)_x(v) + D(l_Y)_y(w)], D(\pi_Y)_{(x,y)}[D(l_X)_x(v) + D(l_Y)_y(w)]) \\ &= (D(\pi_X \circ l_X)_x(v) + D(\pi_X \circ l_Y)_y(w), D(\pi_Y \circ l_X)_x(v) + D(\pi_Y \circ l_Y)_y(w)) \\ &= (v, w)\end{aligned}$$

Hence f is surjective. As both have the same dimension, f is a linear isomorphism. ■

I now claim $D(f \times g)_{(x,y)}(v, w) = (Df_x(v), Dg_y(w))$. PROOF: Say $f : M \to N, g : M' \to N', (f \times g)(x, y) = (f(x), g(y))$ for $x \in M, y \in M'$. Then $f \times g : M \times M' \to N \times N'$. Then $D(f \times g)_{(x,y)} : T_{(x,y)}(M \times M') \to T_{(f(x),g(y))}(N \times N') \cong_I T_{f(x)}N \times T_{g(y)}N'$. Thus both differentials map to the same codomain. Now, $J(f \times g)_{(x,y)} = \begin{bmatrix} Jf_x & Jf_y \\ Jg_x & Jg_y \end{bmatrix} = \begin{bmatrix} Jf_x & 0 \\ 0 & Jg_y \end{bmatrix} = (Jf_x, Jf_y)$ ■

Now, $D(f \times g)_{(x,y)}(v, w) = (0, 0) \Rightarrow Df_x(v) = 0, Df_g(w) = 0 \Rightarrow (v, w) = 0$ as f and g are immersions. Thus $f \times g$ is too. □

(**b**) Show that $g \circ f$ is too.

Proof Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. Recall the Chain Rule: $D(g \circ f)_x = Dg_{f(x)} \circ Df_x$. Observe the map is $T_x(X) \xrightarrow{Df_x(X)} T_{f(x)}(Y) \xrightarrow{Dg_{f(x)}(Y)} T_{(g \circ f)(x)}(Z)$. As injectivity is transitive, such follows. □

(**c**) Show that restricting f to any submanifold of its domain is an immersion.

Proof Say $f : M \to N$ with submanifold $S \subset M$. Let $f|_S = f \circ i$ for inclusion $i : S \to M$. As both are immersions, follows by Q1.3.6(**b**) above. □

(**d**) When $dimX = dimY$, show that $f : X \to Y$ is an immersion iff it's a local diffeomorphism.

Proof I claim that if X is a finite-dimensional smooth manifold, then $dimX = dimT_x(X), \forall x \in X$. PROOF: Say $dimX = n$. Then $\exists$ open $x \in U \subset X \ni \psi : U \to \psi(U) \subset \mathbb{R}^n$ for which ψ is a diffeomorphism. But then $D\psi_x : T_x(X) \to T_{\psi(x)}(\mathbb{R}^n) \cong \mathbb{R}^n$ by proof of Q1.2.3. As $D\psi_x$ is a valid isomorphism, follows $dimT_x(X) = n$ ■

Forward: By definition, $\forall x \in X, Df_x$ is a linear injection. Yet $dimT_x(X) = dimX = dimY = dimT_{f(x)}(Y)$. By the Rank-Nullity Theorem, Df_x is also surjective. Thus Df_x is an isomorphism, so follows by the InFT.

Backward: By definition, $\forall x \in X, \exists$ neighborhood $x \in U_x \ni U \to f(U)$ is a diffeomorphism. But then $id_{T_x(X)} = D(f^{-1} \circ f)_x = D(f^{-1})_{f(x)} \circ Df_x$, and similarly $id_{T_{f(x)}(Y)} = Df_x \circ D(f^{-1})_{f(x)}$. Thus Df_x is invertible, and hence injective. Thus f is an immersion. $\square$

Problem 1.3.8: Let $f : \mathbb{R}^1 \to \mathbb{R}^2, f(t) = (cosh(t), sinh(t))$

(**a**) Show that f is an embedding.

Proof Note that $Df(t) = (sinh(t), cosh(t))$. As $sinh(t)$ is injective, so is f and $Df(t)$. Now, for compact $K \subset \mathbb{R}^2$, such is bounded and closed by Heine-Borel. Then $\exists M > 0 \ni \forall x \in K, ||x|| \leq M$. But $||f(t)||^2 = cosh^2(t) + sinh^2(t) = cosh^2(t) + cosh^2(t) - 1 \leq M^2 \Rightarrow cosh^2(t) \leq \frac{M^2+1}{2}$. This serves an upper bound as cosh is symmetric, $cosh(t) \geq 1$, and increases when $0 < t \to \infty$. Thus $f^{-1}(K) \subset [-T, T]$ for some $T > 0$. As f is continuous, the preimage of closed is closed. So by Heine-Borel, $f^{-1}(K)$ is compact, thus f is proper and hence an embedding. $\square$

(**b**) Prove that Imf is one nappe of the hyperbola $x^2 - y^2 = 1$

Note: A ***Nappe*** is 1 of 2 opposing sheets that makes a cone together, like ><

Proof Observe that $cosh^2(t) - sinh^2(t) = (\frac{e^t+e^{-t}}{2})^2 - (\frac{e^t-e^{-t}}{2})^2 = \frac{e^{2t}+e^{-2t}+2}{4} - \frac{e^{2t}+e^{-2t}-2}{4} = 1$. Thus Imf lies in the hyperbola. Note that $y = \pm\sqrt{x^2-1}$, which is ill-defined for $|x| < 1$. But as $cosh(t) \geq 1, \forall t \in \mathbb{R}$, follows that f only traces the right nappe. $\square$

Problem 1.3.10: Let $f : X \to Y$ be a smooth map that's injective on a compact submanifold Z of X. Say $\forall x \in Z, Df_x : T_x(X) \to T_{f(x)}(Y)$ is an isomorphism. Then f maps Z diffeomorphically onto $f(Z)$. Prove that f maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y.

Note: If Z is a single point, then that's just InFT, hence this generalizes InFT.

Proof By InFT, f is a local diffeomorphism $\forall x \in Z$, say $f|_{U_x} : U_x \to V_x$ for open neighborhoods. Then $\{U_x\}$ open covers Z, so by compactness, such admits a finite subcover. Define $U := \cup_{i=1}^N U_{x_i}$. Now U is a open neighborhood of Z.

Now for arbitrary $i \neq j$, let $W_{ij} := \{y \in Y | \exists U_{x_i} \ni p \neq q \in U_{x_j}, f(p) = f(q)\}$. Then there's sequences (repeats allowed) $p_n \in U_{x_i}, q_n \in U_{x_j} \ni f(p_n) = f(q_n) = y_n$. As f is continuous, $f(Z)$ is compact, thus sequentially compact (**when assuming Y is Hausdorff**). Hence $\exists y_{n_k} \to y \in f(Z) \Rightarrow \exists z \in Z \ni f(z) = y$. By construction, $p_{n_k} = f^{-1}(y_{n_k}) \subset U_{x_i}, q_{n_k} \in f^{-1}(y_{n_k}) \subset U_{x_j}$. So then by sequential compactness, both have convergent subsequences to say p, q. By sequential continuity, $f(p) = f(q) = f(z)$. As f is injective, $p = q = z$. Yet f is a local diffeomorphism near z, thus injective, forcing $p_n = q_n$ for large n. That contradicts $W_{ij} \neq \emptyset$, thus f must be injective on U.

With U open, recall that each V_{x_i} is open, so $f(U) = f(\cup_{i=1}^N U_{x_i} = \cup_{i=1}^N f(U_{x_i}) = \cup_{i=1}^N V_{x_i}$. As $f|_U$ is injective, smooth, and has local smooth inverses, the inverse is well-defined and smooth on $f(U)$. That generalizes InFT. $\square$

Section 1.4: *Submersions*

Problem 1.4.1: If $f : X \to Y$ is a submersion and $U \subset X$ is open, show $f(U)$ is open in Y.

Proof By the Local Submersion Theorem, $\forall x \in X, \exists \phi_x : V_x \subset X \to \mathbb{R}^n, \phi_x(x) = 0$ and $\psi_x : W_x \subset Y \to \mathbb{R}^m \ni \psi_x(f(x)) = 0 \ni \psi \circ f \circ \phi^{-1}(x_1, \cdots, x_n) = (x_1, \cdots, x_m)$

Note that each V_x, W_x are open. As $x \in V_x \cap U$ is open, say such contains $B_\epsilon(x)$. As the diagram is commutative and ϕ_x, ψ_x are both diffeomorphisms, for canonical submersion π_x, follows that $f(x) \in (\psi_x^{-1} \circ \pi_x \circ \phi_x)(B_\epsilon(x)) \subset f(U)$ is indeed open. Yet x was arbitrary, thus $f(U)$ is open. $\square$

Problem 1.4.2: Let X be compact and Y connected.

(**a**) Show that every submersion $f : X \to Y$ is surjective.

Proof Let $\{O_i\}$ open cover X. As X is compact and f is continuous, follows that $f(X)$ is compact. So then $\forall x \in X$, choose open neighborhood $U_x \subset X$. By Q1.4.1, follows that $f(U_x)$ is open, thus $f(X) = \cup_{x \in X} f(U_x)$ is too. Yet compact implies closed in Hausdorff spaces, and the only clopen sets in a connected space is the space itself or the emptyset. But $X \neq \emptyset$, thus $f(X) = Y$ $\square$

(**b**) Show there's no submersions of compact manifolds into Euclidean spaces.

Proof BWOC, say $f : X \to \mathbb{R}^n$ is a submersion with X compact. Note that $\mathbb{R}^n$ is connected. But by (**a**) above, $f(X) = \mathbb{R}^n$ is compact, a contradiction. $\square$

Problem 1.4.3: Show that $\gamma(t) = (t, t^2, t^3)$ embeds $\mathbb{R} \to \mathbb{R}^3$. Find 2 independent functions that globally define the image. Are your functions independent on all of $\mathbb{R}^3$, or just on an open neighborhood of the image?

Proof Such is clearly injective (see 1st component) and immersion as $D\gamma = (1, 2t, 3t^2)$ is injective (see 2nd component). Let $\gamma^{-1} : \gamma(\mathbb{R}) \to \mathbb{R}, \gamma^{-1}(t, \sqrt{t}, \sqrt[3]{t}) = (t)$. That's continuous, thus the preimage of compact is compact.

Now let $x = t, y = t^2, z = t^3 \Rightarrow x^2 = y, x^3 = z \Rightarrow y - x^2 = 0 = z - x^3$. Define [***Answer***] $\boxed{g_1(x, y, z) = y - x^2, g_2(x, y, z) = z - x^3}$ Observe that they vanish at the image, namely $\gamma(\mathbb{R}) = \{(x, y, z) | g_1(x, y, z) = 0 = g_2(x, y, z)\}$. Observe that $D(g_1) = (-2x, 1, 0), D(g_2) = (-3x^2, 0, 1)$. The Jacobian is thus $\begin{bmatrix} -2x & 1 & 0 \\ -3x^2 & 0 & 1 \end{bmatrix}$. Clearly the rank is 2, thus they're LI everywhere in $\mathbb{R}^3$ $\square$

Problem 1.4.5: Define $f : \mathbb{R}^3 \to \mathbb{R}, f(x,y,z) = x^2 + y^2 - z^2$

(**a**) Check that 0 is the only critical value of f.

Proof Observe that $Df_{(x,y,z)}v = (2x, 2y, -2z)v$ can't be surjective onto $\mathbb{R}$ iff it's the origin. $\square$

(**b**) Prove that if a, b are either both $(+)$ or $(-)$, then $f^{-1}(a) \cong_D f^{-1}(b)$

Proof WLOG, say $a, b > 0$. Consider $a\lambda^2 = b, \lambda = \sqrt{b/a}$. Define $\psi(x,y,z) = (\lambda x, \lambda y, \lambda z)$. As λ is a constant, follows ψ is simply a linear diffeomorphism. Now let $(x,y,z) \in f^{-1}(a)$. Then $f(\psi(x,y,z)) = \lambda^2(x^2+y^2-z^2) = \lambda^2 a = b$, thus $\psi(f^{-1}(a)) \subset f^{-1}(b)$. Now let $(x', y', z') \in f^{-1}(b)$. Then as ψ is bijective $\exists (x,y,z) \ni (x', y', z') = \psi(x,y,z)$. Observe that $b = f(x', y', z') = f(\psi(x,y,z)) = \lambda^2 f(x,y,z) = \frac{b}{a} f(x,y,z) \Rightarrow a = f(x,y,z) \Rightarrow (x,y,z) \in f^{-1}(a)$. Thus $f^{-1}(b) = \psi(f^{-1}(a)) \Rightarrow f^{-1}(a) \cong_D f^{-1}(b)$ as ψ is a diffeomorphism. $\square$

(**c**) How does the topology of $f^{-1}(c)$ change as c passes through the critical value?

Solution In 3D desmos, graph $z \pm \sqrt{x^2+y^2-a}$. Observe that it changes from a hyperboloid $\to$ connected cones $\to$ disconnected cones when $a > 0 \to a = 0 \to a < 0$, which looks like cytokinesis.

Problem 1.4.7: Say y is a regular value of $f : X \to Y$ with X compact and $dimX = dimY$

(**a**) Show that $f^{-1}(y)$ is finite.

Proof By the Preimage Theorem as y is regular, then $f^{-1}(y)$ is a submanifold of X with $dim f^{-1}(y) = dimX - dimY = 0$. Thus, $\forall x \in f^{-1}(y)$, such has a chart where an open neighborhood U_x of x is diffeomorphic to $\mathbb{R}^0 = \{0\}$. Thus $|U_x| = 1$ by bijectivity. BWOC, say $|f^{-1}(y)| = \infty$. Then $\{x | x \in f^{-1}(y)\}$ minimally open covers itself, contradicting X being compact. $\square$

(**b**) Prove $\exists$ neighborhood $y \in U \subset Y \ni f^{-1}(U) = \sqcup_{i=1}^N V_i \ni x \in V_i$ is an open neighborhood and f allows $V_i \cong_D U$. That proves the ***Stack of Records Theorem***

Proof By (**a**), let $f^{-1}(y) = \{x_1, \cdots, x_n\}$. As y is regular, by Rank-Nullity, $Df_{x_i} : T_{x_i}(X) \to T_y(Y)$ is an isomorphism. By the InFT, $\exists x \in V_i' \subset X, y \in U_i \subset Y$ with both neighborhoods open $\ni f : V_i' \to U_i$ is a diffeomorphism. As $f^{-1}(y)$ is finite, $y \in U = \cap_{i=1}^n U_i$ is still open and V_i' can "shrink" around x_i to be pairwise disjoint for X is Hausdorff. As $U \subset U_i, \forall i$ and f still remains a local diffeomorphism, $x_i \in V_i := V_i' \cap f^{-1}(U)$ is open. As $V_i' \cap f^{-1}(U) \subset f^{-1}(U) \xrightarrow{f} U$, that means $f|_{V_i} : V_i \to U$ is a diffeomorphism. Yet $f(\sqcup(V_i' \cap f^{-1}(U))) = \sqcup f(V_i' \cap f^{-1}(U)) = [\cup f(V_i')] \cap f(f^{-1}(U)) = (\cup U_i) \cap U = U$. That proves the statement. $\square$

Problem 1.4.9: Show that $O(n)$ is compact.

Proof Recall $M(n) \cong_D \mathbb{R}^{n^2}$. So a natural norm on $M(n)$ would be $||A|| = \sqrt{\sum_{i,j} A_{ij}^2}$ by using the Euclidean Norm (termed the ***Frobenius Norm***). Recall that in the book, $f : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}, f(A) = AA^T$ has $f^{-1}(I) = O(n)$. Yet f is continuous with the preimage of closed being closed.

Now note that $(AA^T)_{ii} = \sum_{k=1}^n A_{ij}A_{ji}^T = \sum_{k=1}^n A_{ij}A_{ij} = \sum_{j=1}^n A_{ij}^2$. Then $Tr(AA^T) = \sum_{i=1}^n (AA^T)_{ii} = \sum_{i,j} A_{ij}^2$. Thus $||A||^2 = \sum_{i,j} = A_{ij}^2 = Tr(AA^T) = Tr(I) = n \Rightarrow ||A|| = \sqrt{n}$. Thus $O(n)$ is bounded. So by Heine-Borel, $O(n)$ is compact. $\square$

Problem 1.4.11: The ***Special Linear Group*** $SL(n)$ are all $n \times n$ matrices with $det = 1$

(**a**) Prove that $SL(n)$ is a submanifold of $M(n)$ and thus a Lie group.

Proof Let $f : M(n) \cong_D \mathbb{R}^{n^2} \to \mathbb{R}, f(A) = det(A)$. As det is a polynomial, it's smooth. Now parameterize $A(t) = (1+t)A$. That means $\frac{d}{dt}det(A(t)) = n(1+t)^{n-1}detA$. So if $t = 0$, that simplifies to $ndetA \neq 0 \iff detA \neq 0 \iff Df_A$ is surjective. Thus 0 is the only critical value of f (**I don't understand, but that's what the book says**).

Now, recall that f is a submersion at A if $Df_A : T_A(M(n)) \cong \mathbb{R}^{n^2} \to T_{f(A)}(\mathbb{R}) \cong \mathbb{R}$ is surjective iff $\exists B \in T_A(M(n)) \ni Df_A(B) \neq 0$. Try $\gamma(t) := tA, t > 0$. Then $\gamma'(1) = A$, yet $Df_A(\gamma'(1)) = \frac{d}{dt}[det(\gamma(t))]|_{t=1} = \frac{d}{dt}[t^n detA]|_{t=1} = ndetA = n \neq 0$. Thus f is a submersion at A.

Finally, by the Local Submersion Theorem, there's local coordinates $(x_1, \cdots, x_{n^2})$ in a neighborhood of $A \in SL(n) \ni det(x_1, \cdots, x_{n^2}) = x_1$. But if $A \in SL(n)$, then $x_1 = 1 \Rightarrow f^{-1}(1) = \{(1, x_2, \cdots, x_{n^2})\} = \{1\} \times \mathbb{R}^{n^2-1}$. So as $SL(n)$ is locally diffeomorphic to $\mathbb{R}^{n^2-1}$, by definition $SL(n)$ is a codimension-1 submanifold of $M(n)$ $\square$

(**b**) Show that $T_I(SL(n)) = \{A | Tr(A) = 0\}$

Lemma

$det(e^A) = e^{Tr(A)}$

Proof Let $\gamma(t) \in SL(n)$ be smooth with $\gamma(0) = I$ and $f : M(n) \to GL(n), f(A) = exp(A)$. As f is a local diffeomorphism near $0 \to I, \exists \eta(t) \in M(n)$ smooth with $\eta(0) = 0 \ni \gamma(t) = exp[\eta(t)], \forall t \in [0, \epsilon)$. As $\gamma(t) \in SL(n)$, then $det(\gamma(t)) = 1$ and $det(exp(\eta(t))) = exp(Tr(\eta(t))) = 1 \Rightarrow Tr(\eta(t)) = 0, \forall t$. Thus $\frac{d}{dt}Tr(\eta(t))|_{t=0} = Tr(\eta'(0)) = 0$. But then $\gamma'(0) = \frac{d}{dt}exp[\eta(t)]|_{t=0} = \eta'(0) \Rightarrow Tr(\gamma'(0)) = Tr(\eta'(0)) = 0$. Thus the statement follows by Q1.2.12. $\square$

Section 1.5: *Transversality*

Problem 1.5.1: Say $A : \mathbb{R}^k \to \mathbb{R}^n$ is linear with V, W vector subspaces of $\mathbb{R}^n$.

(**a**) Show that $A \pitchfork V$ means $A(\mathbb{R}^k) + V = \mathbb{R}^n$

Proof $A \pitchfork V \iff \forall x \in A^{-1}(V), Im(DA_x) + T_{A(x)}(V) = T_{A(x)}(\mathbb{R}^n) \Rightarrow A(\mathbb{R}^k) + V = \mathbb{R}^n$ $\square$

(**b**) Show that $V \pitchfork W$ means $V + W = \mathbb{R}^n$

Proof Similarly, $T_x(V) + T_x(W) = T_x(\mathbb{R}^n) \Rightarrow V + W = \mathbb{R}^n$ $\square$

Problem 1.5.2: Which of the following linear spaces intersect transversally?

(**a**) xy plane and z-axis in $\mathbb{R}^3$

Solution By definition, $\forall x \in X \cap Z$ for both submanifolds of space Y, then $T_x(X) + T_x(Z) = T_x(Y)$. In this case, they intersect only at the origin. Observe that they're both linear subspaces of $\mathbb{R}^3$ and that the plane is $A = span\{(1,0,0),(0,1,0)\}$ and the z-axis is $B = span\{(0,0,1)\}$. That means $T_x(A) + T_x(B) = A + B = \mathbb{R}^3 = T_x(\mathbb{R}^3)$, thus they're [***Answer***] Transversal

(**b**) xy plane and plane $span\{(3,2,0),(0,4,-1)\}$ in $\mathbb{R}^3$

Solution Observe that $(3,2,0) \in A$, the xy plane, but $(0,4,-1)$ is L.I. from A due to the z component. Denote the second plane as B. Thus, the following is true $\forall x \in \mathbb{R}^3 \supset A \cap B$: $T_x(A) + T_x(B) = \mathbb{R}^3 = T_x(\mathbb{R}^3)$, hence they're [***Answer***] Transversal

(**c**) Plane $span\{(1,0,0),(2,1,0)\}$ and y-axis in $\mathbb{R}^3$

Solution Observe that for plane A and y-axis B, that $(2,1,0) = 2(1,0,0) + 1(0,1,0)$. As $(0,1,0) = -2(1,0,0) + (2,1,0)$, they nontrivially intersect. Thus $\forall x \in A \cap B \neq \emptyset, T_x(A) + T_x(B) = A + B = span\{(1,0,0),(0,1,0)\} = \mathbb{R}^2 \neq \mathbb{R}^3 = T_x(\mathbb{R}^3)$. Thus they're [***Answer***] Not Transversal (**d**) $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$

Solution Case 1: $k+l>n$. Both are subspaces of $\mathbb{R}^n$, so for x in the intersection, of slices A, B respectively, then $T_x(A)+T_x(B)=A+B$ is spanned by the standard basis, thus is $\mathbb{R}^n$, so they're [***Answer***] Transversal

Case 2: $k+l=n$. Then the logic still holds, but they're [***Answer***] Transversal as the condition holds for the only point they intersect: the origin.

Case 3: $k+l<n$. They only intersect at the origin, yet $T_x(A)+T_x(B)=A+B=\mathbb{R}^{k+l}\subset\mathbb{R}^n=T_x(\mathbb{R}^n)$, and thus they're [***Answer***] Not Transversal

(**e**) $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^l \times \{0\}$ in $\mathbb{R}^n$

Solution No matter the case when denoting the slices as A, B respectively, $A+B\subset A$ or B, which is a proper subset of $\mathbb{R}^n$ due to the "cavity" $\{0\}$ that remains. As they have a nontrivial intersection at the origin, follows they're [***Answer***] Not Transversal

(**g**) Symmetric and skew symmetric matrices in $M(n)$

Solution Denote their sets as M_{sym}, M_{skew} respectively. Then $A^T=A=-A\Rightarrow 2A=0\Rightarrow A=0$. As they're both vector subspaces of $M(n)\cong_D\mathbb{R}^{n^2}$, follows that $T_0(A)+T_0(B)=A+B$. But note for arbitrary $C\in M(n), C=\frac{C+C^T}{2}+\frac{C-C^T}{2}$. Observe that $(C+C^T)^T=C+C^T$ and $(C-C^T)^T=-(C-C^T)$. That means C is decomposable into symmetric and skew-symmetric components, meaning $A+B=M(n)=T_0(M(n))$. Thus they're [***Answer***] Transversal

Problem 1.5.3: Let V_1,V_2,V_3 be linear subspaces of $\mathbb{R}^n$. They have a ***Normal Intersection*** if $V_i \pitchfork (V_j\cap V_k)$ for i,j,k not pairwise equal. Prove this holds iff $codim(V_1\cap V_2\cap V_3)=codimV_1+codimV_2+codimV_3$

Proof $V_i \pitchfork (V_j\cap V_k) \iff V_i+(V_j\cap V_k)=\mathbb{R}^n$ as they're linear subspaces, thus $T_x(V_i)=V_i$

Forward: $codim(V_1\cap V_2\cap V_3)=n-dim(V_1\cap V_2\cap V_3)=n-[dim(V_1)+dim(V_2\cap V_3)-dim(V_1+(V_2\cap V_3))]$. Yet as $V_1\pitchfork(V_2\cap V_3)\Rightarrow V_1+(V_2\cap V_3)=\mathbb{R}^n\Rightarrow dim(V_1+(V_2\cap V_3))$, such reduces to $n-[dim(V_1)+dim(V_2\cap V_3)-n]=2n-[dim(V_1)+dim(V_2)+dim(V_3)-dim(V_2+V_3)]=$ (**No idea why**) $3n-dim(V_1)-dim(V_2)-dim(V_3)=codim(V_1)+codim(V_2)+codim(V_3)$

Backward: WLOG, decompose $n-dim(V_1\cap(V_2\cap V_3))=2n-[dim(V_1)+dim(V_2)+dim(V_3)-dim(V_1+(V_2\cap V_3))]=3n-[dim(V_1)+dim(V_2)+dim(V_3)]\Rightarrow n=dim(V_1+(V_2\cap V_3))\Rightarrow\mathbb{R}^n=V_1+(V_2\cap V_3)\Rightarrow V_1\pitchfork(V_2\cap V_3)$. The logic holds when permuting $(123),(132)$, thus follows. $\square$

Note: I'm unable to prove that $dim(V_i+V_j)=n$ for $i\neq j$

Problem 1.5.4: Let X, Z be transversal submanifolds of Y. Prove that $x \in X \cap Z \Rightarrow T_x(X \cap Z) = T_x(X) \cap T_x(Z)$

Proof As $X \pitchfork Z$, then $codim(X \cap Z) = codim(X) + codim(Z) \Rightarrow n - dim(X \cap Z) = n - dim(X) + n - dim(Z) \Rightarrow dim(X \cap Z) = dim(X) + dim(Z) - dim(Y) \Rightarrow dim(T_x(X \cap Z)) = dim(T_x(X)) + dim(T_x(Z)) - n$

Recall that $T_x(X) + T_x(Z) = T_x(Y)$ as $X \pitchfork Z$. Then for subspaces A, B, follows that $dim(A \cap B) = dim(A) + dim(B) - dim(A + B) \Rightarrow dim(T_x(X) \cap T_x(Z)) = dim(T_x(X)) + dim(T_x(Z)) - n$

Thus $dim(T_x(X \cap Z)) = dim(T_x(X) \cap T_x(Z))$. As $X \cap Z \subset X, Z \Rightarrow T_x(X \cap Z) \subset T_x(X) \cap T_x(Z)$, both vector spaces with the same dimension, equality follows. $\square$

Problem 1.5.10: Let $f : X \to X$ have fixed point x. If $+1$ isn't an eigenvalue of $Df_x : T_x(X) \to T_x(X)$, then x is a ***Lefschetz Fixed Point*** of f. f is a ***Lefschetz Map*** if all its fixed points are Lefschetz. Prove that if X is compact and f Lefschetz, then f has only finitely many fixed points.

Proof Denote diagonal $\Delta(X)$ and let $W_f(X) := \{(x, f(x)) | x \in X\}$. I claim $\forall x \in X, T_{(x,f(x))}(W_f(X)) = W_{Df_x}(T_x(X))$. PROOF: Define $F : X \to W_f(X), F(x) = (x, f(x))$, which is clearly smooth. As $F^{-1}(x, f(x)) = x$ is smooth as a projection, follows that F is a diffeomorphism. Hence $DF_x : T_x(X) \to T_{(x,f(x))}(W_f(X))$ is an isomorphism. $\blacksquare$

That means $dim W_f(X) = dim(X), \Delta(X) = W_{id}(X)$. Now, I claim $\Delta(X) \pitchfork W_f(X)$. PROOF: Say $(x, x) \in \Delta(X) \cap W_f(X)$. Then $T_{(x,x)}\Delta(X) = \Delta(T_x(X))$ and $T_{(x,x)}W_f(X) = W_{Df_x}(T_x(X))$ by the 1st claim. As $+1$ isn't an eigenvalue of Df_x, then $\Delta(T_x(X)) \pitchfork W_{Df_x}(T_x(X))$ in $T_x(X) \times T_x(X)$, thus proves the statement. $\blacksquare$

Now: $codim(\Delta(X) \cap W_f(X)) = codim\Delta(X) + codim W_f(X) = 2dim X = dim X \times X \Rightarrow dim(\Delta(X) \cap W_f(X)) = 0$. As X is compact, so is $X \times X$. As $W_f(X)$ is closed with $x \to (x, x)$ also continuous, thus is compact. So with $f = id$, then $\Delta(X)$ is compact too. Thus $\Delta(X) \cap W_f(X) \subset X \times X$ is closed and compact. As compact 0-dimensional manifolds are finite singletons, follows. $\square$

Problem 1.5.11: If $C \subset \mathbb{R}^k$ is closed, then $\exists$ submanifold $X \subset \mathbb{R}^{k+1} \ni X \cap \mathbb{R}^k = C$. Consider $\mathbb{R}^k$ as a submanifold of $\mathbb{R}^{k+1}$ by the canonical inclusion.

Theorem

Every closed subset of $\mathbb{R}^k$ is the zero set of some smooth $f : \mathbb{R}^k \to \mathbb{R}$

$\boxed{\underline{\textbf{\textit{Proof}}}}$ Say $g : \mathbb{R}^k \to \mathbb{R}$ is smooth where $g^{-1}(0) = C$. So set $X = \{(x_1, \cdots, x_k, g(x_1, \cdots, x_k))\} \subset \mathbb{R}^{k+1}$ and define $\Phi : \mathbb{R}^k \to \mathbb{R}^{k+1}, \Phi(x) = (x, g(x))$. Such is clearly smooth. Note that $D\Phi_x : \mathbb{R}^k \to \mathbb{R}^{k+1}, D\Phi_x(v) = (v, Dg_x(v))$. But then the 1st k components forces injectivity, thus Φ is an immersion. As $\Phi^{-1}(x, g(x)) = x$ is continuous, the preimage of compact is thus compact, making Φ an embedding. Thus $Im(\Phi) = X$ is a k-dimensional embedded submanifold of $\mathbb{R}^{k+1}$. But then $X \cap \mathbb{R}^k = \{(x, 0)|g(x) = 0\} = \{(x, 0)|x \in C\} = C$ $\square$

<u>Addendum</u>: As C is ***any!!*** closed set, transversality is key for "nice" intersections!

Section 1.6: *Homotopy and Stability*

Problem 1.6.1: Say $f_0, f_1 : X \to Y$ are homotopic. Show $\exists$ homotopy $\tilde{F} : X \times I \to Y, \tilde{F}(x,t) = f_0(x), \forall t \in [0, \frac{1}{4}]$ and $\tilde{F}(x,t) = f_1(x), \forall t \in [\frac{3}{4}, 1]$

[**SCRATCH**] Say $f_0 \sim f_1$ by homotopy F. Define $\tilde{F}(x,t) = \begin{cases} f_0(x) & t \in [0, 1/4] \\ F(x, 2t - 1/2) & t \in (1/4, 3/4) \\ f_1(x) & t \in [3/4, 1] \end{cases}$. Such is continuous, including at $\tilde{F}(x, 1/4^-) = \tilde{F}(x, 1/4^+) = f_0(x), \tilde{F}(x, 3/4^-) = \tilde{F}(x, 3/4^+) = f_1(x)$. But then is it smooth at $t = 1/4$? Observe that $f_0(x)_t = 0, [F(x,s)_s \frac{d}{dt}(4t-1)]_{t=1/4} = 4[F(x,0)]_s$, which it isn't known if $[F(x,0)]_s = 0$. So then we could take F and force a sigmoid change, e.g. F is "how" it changes, but "when" it changes occurs by viewing slowly away from $f_0(t)$ and slowly approaching $f_1(t)$, which would force a 0 derivative. Hence the following.

Proof Define $\tilde{F}(x,t) = \begin{cases} f_0(x) & t \in [0, 1/4] \\ F(x, g(2t - 1/2)) & t \in (1/4, 3/4) \\ f_1(x) & t \in [3/4, 1] \end{cases}, f(t) = e^{-1/t}, g(t) = \frac{f(t)}{f(t) + f(1-t)}$. Then $g : (0,1) \to \mathbb{R}$ with $g(0^+) = 0, g(1^-) = \frac{1/e}{1/e + 0} = 1$. Note that an online calculator has $g'(t) = \frac{(2t^2 - 2t + 1) exp[1/t + 1/(1-t)]}{[(t-1)t[exp(1/t) + exp(1/(1-t))]]^2}$ with $g'(0) = g'(1) = 0$ using Desmos.

Now at $t = 1/4$, they're still smooth w.r.t x. For $t = \frac{1}{4}^-, f_0(x)_t = 0$. For $t = \frac{1}{4}^+, [F(x,s)]_s[g'(2t - \frac{1}{2})] * 2|_{t=1/4} = 0$. Thus extends to $t = 3/4$, thus $\tilde{F}$ is a valid homotopy for the statement. $\square$

Problem 1.6.2: Show that homotopy is an equivalence relation.

Proof Clearly $f \sim f$ by the trivial map turned homotopy. If $f \sim g$ by homotopy $F(x,t)$, clearly $g \sim f$ by $F(x, 1-t)$. Lastly, let $f \sim g, g \sim h$ by homotopies F and G. Using from Q1.6.1, obtain $\tilde{F}, \tilde{G}$. Define $H(x,t) = \begin{cases} \tilde{F}(x, 2t) & t \in [0, 1/2] \\ \tilde{G}(x, 2t-1) & t \in [1/2, 1] \end{cases}$, which in terms of t, is like a dual sigmoid curve. Smooth is preserved by function composition, thus what remains to check that H behaves at $t = 1/2$. Observe that $H(x, 1/2^-) = F(x, 1^-) = g_0(x), H(x, 1/2^+) = G(x, 0^+) = g_0(x)$. Then one can similarly (and annoyingly) show that $[H(x,t)_t]|_{t=1/2} = 0$ via the left-right limits converging. $\square$

Problem 1.6.3: Show that every connected manifold X is ***Arcwise Connected***: $\forall x_0, x_1 \in X, \exists$ smooth curve $f : I \to X, f(0) = x_0, f(1) = x_1$

Proof I claim that $x_0 \sim x_1$, where they have a smooth arc, is an equivalence relation. PROOF: Trivial that $x_0 \sim x_0$ as the constant map is smooth. If $x_0 \sim x_1$ by arc f, then $f(1-t)$ is also smooth by function composition, thus $x_1 \sim x_0$. Now for $x_0 \sim x_1, x_1 \sim x_2$ by f, g. We can view them as homotopies $\{x\} \times I \to X$ where $f_t(x) = f(t), g_t(x) = g(t)$, so by transitivity from Q1.6.2, there's a smooth map $h : \{x\} \times I \to X, h_0(x) = x_0, h_1(x) = x_2$, hence $x_0 \sim x_2$. ■

As manifolds are locally diffeomorphic, there's an open $x \in B \subset X \ni B \cong_D \mathbb{R}^k$ which arcwise connected is preserved. BWOC, say $[x_0] \neq X$. Then $[x_0]^C$ is also open. But X is connected and $x_0 \in [x_0] \neq \emptyset$, thus X is arcwise connected. □

Problem 1.6.4: Manifold X is ***Contractible*** if it's identity map is homotopic to some constant map $X \to \{x\}, x \in X$. Show that X is contractible iff all maps of an arbitrary manifold $Y \to X$ are homotopic.

Proof Forward: By definition, there's homotopy $F : X \times I \to X, F(x, 0) = x, F(x, 1) = x_0$ for some fixed $x_0 \in X$. Let $f_0, f_1 : Y \to X$ be smooth. Define constant map $c_{x_0} : Y \to X, c_{x_0}(y) = x$. Then $f_0 \sim c_{x_0}$ by homotopy $F \circ f_0$ and similarly $f_1 \sim c_{x_0}$. Thus $f_0 \sim f_1$ due to Q1.6.2.

Backward: Y is arbitrary, so let $Y = X$. Consider $id_X : X \to X$ and constant map c_{x_0}. So by assumption, $id_X \sim c_{x_0}$, namely X is contractible. □

Problem 1.6.5: Show that $\mathbb{R}^k$ is contractible.

Proof Simply take $H : \mathbb{R}^k \times [0, 1] \to \mathbb{R}^k, H(x, t) = (1 - t)x + tx_0$. Done! □

Problem 1.6.8: Prove that diffeomorphisms of compact manifolds constitute a stable class.

Proof Let $f_0 : X \to Y$ be a diffeomorphism with f_t homotopies of f_0. WLOG, say X and Y are connected as they're both compact, so they have finitely many connected components. As local diffeomorphisms and embeddings are stable by the Stability Theorem, $\exists \epsilon > 0 \ni \forall t \in [0, \epsilon), f_t$ are too. Fix t. $f_t(X)$ is open in Y being a local diffeomorphism and closed as the continuous image of compact is compact. Yet Y is connected, so clopen means $f_t(X) = Y$ □

Problem 1.6.9: Let smooth $\rho : \mathbb{R} \to \mathbb{R}$ have $\rho(s) = 1$ for $|s| < 1$ and $\rho(s) = 0$ for $|s| > 2$. Define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = x\rho(tx)$. Verify this is a counterexample to all 6 parts of the Stability Theorem as $\mathbb{R}$ isn't compact.

Proof Observe that $f_0(x) = x\rho(0) = x$. The identity map has all 6 properties. Now let $t > 0$.

- **Diffeomorphism**: If $x_1 \neq x_2$ are large, then $f_t(x_1) = f_t(x_2) = 0$
- **Local Diffeomorphism**: $f_t'(x) = \rho(tx) + xt\rho'(tx)$. For large x, then $\rho(tx) = \rho'(tx) = 0$, thus the derivative can't be an isomorphism
- **Immersion and Submersion**: As $f_t'(x) = 0, f$ can't be injective nor surjective
- **Embedding**: That's a proper injective immersion, but it's not an immersion
- **Transversal**: Consider submanifold $Z = \{0\} \subset \mathbb{R}$. Recall f is transversal to Z if $\forall x \in f^{-1}(Z), Im(Df_x) + T_0(Z) + T_0(\mathbb{R})$. Yet $f_t^{-1}(0) = 0$ when $x > 2/t$. Out there, f is constant, thus $Im(Df_x) = 0$. But then $0 = \mathbb{R}$, a contradiction

□

Problem 1.6.10: A ***Deformation*** of submanifold Z in Y is a homotopy $i_t : Z \to Y$ where $i_0 : Z \to Y$ is the inclusion map and each i_t is an embedding. Thus $Z_t := i_t(Z)$ is a smooth varying submanifold of Y with $Z_0 = Z$. Show that if Z is compact, then any homotopy i_t of its inclusion map is a deformation for small t.

Proof i_0 is an embedding and Z is compact, thus by the Stability Theorem, $\exists \epsilon > 0 \ni \forall t \in [0, \epsilon), i_t$ is also an embedding homotopy. Thus i_t is also a deformation of Z. □

Problem 1.6.11: Say $f_s : X \to Y$ is a family of smooth maps between manifolds with X compact, indexed by parameter s over S, subset in some Euclidean space. Then $\{f_s\}$ is a ***Smooth Family*** of mappings if $F : X \times S \to Y, F(x, s) = f_s(x)$ is smooth. Check that the Stability Theorem generalizes: if f_0 belongs to any class listed, then $\exists \epsilon > 0 \ni ||s_0 - s|| < \epsilon \Rightarrow$ f_s belongs to the same class.

Addendum: So smooth families generalize homotopies. Say S is a smooth manifold in $\mathbb{R}^k$

Proof **Too hard!** $> \_ <$ □

Section 1.7: *Sard's Theorem and Morse Functions*

Problem 1.7.1: Show that $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$ when $k < l$

Proof Let $\epsilon > 0$. Consider $\mathbb{Z}^k \subset \mathbb{R}^k$. Such is countable, so enumerate $\mathbb{Z}^k = \{x_1, \cdots\}$. Then for each $x_n \in \mathbb{Z}^k$, construct open cubes each of length 2 around x_n. As $k < l \Rightarrow 0 < l - k$, shrink the last $l - k$ lengths of R_x down to $\frac{\sqrt[l-k]{\epsilon}}{2^{l-k+1}}$. Thus $Vol(R_{x_n}) = 2^k(\frac{\sqrt[l-k]{\epsilon}}{2^{l-k+n+1}})^{l-k} = \frac{\epsilon}{2^{n+1}}$. Then $\mathbb{R}^k \subset \cup R_{x_n}, \sum_{n=1}^{\infty} Vol(R_{x_n}) = \frac{\epsilon}{2}$. Thus $\mathbb{R}^k$ has measure zero in $\mathbb{R}^l$ $\square$

Problem 1.7.3: Say Z is a submanifold of X with $dimZ < dimX$. Prove that Z has measure zero in X without using Sard's Theorem.

Proof As manifolds are second countable and locally diffeomorphic, there's countably many coordinate charts $\{(U_i, \phi_i)\}, \phi_i : U_i \to V_i \subset \mathbb{R}^n \ni \phi_i(Z \cap U_i) \subset \mathbb{R}^{dimZ} \times \{0\}^{dimX - dimZ}$. By Q1.7.1, $\mathbb{R}^{dimZ}$, thus $\cup\phi_i(Z \cap U_i) \subset \mathbb{R}^{dimZ}$, has measure zero in $\mathbb{R}^{dimX}$. As diffeomorphisms preserve measure, follows that $Z \cap U_i$, thus $Z = \cup_i Z \cap U_i$ being a countable union, preserves measure. Thus Z has measure zero. $\square$

Problem 1.7.4: Prove that $\mathbb{Q}^n$ has measure zero in $\mathbb{R}^n$

Proof Note that $\mathbb{Q}^n = \{x_1, \cdots\}$ is countable. Let $\epsilon > 0$. For each $x_k \in \mathbb{Q}^n$, construct open cubes with each side of length $\sqrt[k]{\frac{\epsilon}{2^{k+1}}} \Rightarrow Vol(R_{x_k}) = \frac{\epsilon}{2^{k+1}}$. Thus $\mathbb{Q}^n \subset \cup R_{x_k}, \sum_{k=1}^{\infty} Vol(R_{x_k}) = \frac{\epsilon}{2}$, thus $\mathbb{Q}^n$ has measure zero in $\mathbb{R}^n$ $\square$

Problem 1.7.6: Prove that S^k is ***Simply Connected*** in $k > 1$, namely every smooth map of $S^1 \to S^k$ is homotopic to a constant.

Proof Let $f : S^1 \to S^k$ be smooth. As $dim(S^1) = 1 < k = dim(S^k)$, follows that every value is critical as the derivative can't be surjective anywhere. Yet by Sard's Theorem, the set of critical values has measure 0, thus $f(S^1) \subset S^k$ has measure 0. So choose any $p \in S^k \setminus f(S^1)$ and send $S^k \setminus \{p\} \cong_D \mathbb{R}^k$ via stereographic projection. As $f(S^1)$ is mapped onto a contractible space, f can shrink to a point. $\square$

Problem 1.7.8: Analyze the critical behavior of the following at the origin e.g. if it's degenerate, isolated, max, or min.

(**a**) $f(x,y) = x^2 + 4y^3$

Solution $Hf_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & 24y \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Such is singular with $det(Hf_{(0,0)}) = 0$, thus is [**Answer**] Degenerate

(**b**) $f(x,y) = x^2 - 2xy + y^2$

Solution $Hf_{(0,0)} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ with determinant 0, and thus is [**Answer**] Degenerate

(**d**) $f(x,y) = x^2 + 11xy + y^2/2 + x^6$

Solution $Hf_{(0,0)} = \begin{bmatrix} 2+30x^4 & 11 \\ 11 & 1 \end{bmatrix}|_{(0,0)} = \begin{bmatrix} 2 & 11 \\ 11 & 1 \end{bmatrix}$ with determinant $2 - 11^2 < 0$ and thus a [**Answer**] Saddle

(**e**) $f(x,y) = 10xy + y^2 + 75y^3$

Solution $Hf_{(0,0)} = \begin{bmatrix} 0 & 10 \\ 10 & 2 \end{bmatrix}$ with determinant -100, and is thus a [**Answer**] Saddle

Note: Recall in MATH454 where if x is a critical value, then $detHf_x < 0$ means saddle, inconclusive if $detHf_x = 0$, and $detHf_x > 0$ has local max if double partial (w.r.t. any variable) of f at x is (+) and min if vice versa.

Problem 1.7.9: Prove the Morse Lemma in $\mathbb{R}$

Proposition

For any smooth f on $\mathbb{R}$ and any $a \in \mathbb{R}$, $\exists$ smooth $g \ni f(x) = f(a) + (x-a)f'(a) + (x-a)^2 g(x)$

Proof Let $a \in \mathbb{R}$ be a nondegenerate critical point of f. The Morse Lemma says there's a diffeomorphism ϕ around a s.t. $f \circ \phi(x) = f(a) + f''(a)x^2$. As a is critical, $f'(a) = 0$. Then $\exists g \ni f(x) = f(a) + (x-a)^2 g(x)$. Yet $f''(x) = 2g(x) + 4(x-a)g'(x) + (x-a)^2 g''(x) \Rightarrow f''(a) = 2g(a) \neq 0$ as a is nondegenerate, thus g is nonvanishing near a as g is continuous.

We want to transform the coordinates s.t. $(x-a)^2 g(x) = f''(a)u^2 \Rightarrow u = \phi(x) := (x-a)\sqrt{\frac{g(x)}{f''(a)}}$. Via an online calculator, $\phi'(x) = \frac{(x-a)g'(x)}{2f''(a)\sqrt{g(x)/f''(a)}} + \sqrt{\frac{g(x)}{f''(a)}}$. As $2g(a) = f''(a) \Rightarrow \sqrt{\frac{g(a)}{f''(a)}} \neq 0$, follows $\phi'(a) \neq 0$. So via InFT, ϕ is indeed a local diffeomorphism around a. But then $f \circ \phi(u) = f(a) + f''(a)u^2$ is valid near $x = a$ $\square$

Problem 1.7.11: Prove that if a is a nondegenerate critical point of f, then there's a local coordinate system $(x_1, \cdots, x_n)$ around a $\ni f = f(a) + \sum_{i=1}^n \epsilon_i x_i^2, \epsilon_l = \pm 1$

Proof As a is nondegenerate with f smooth, then Hf_a is symmetric and nonsingular, thus diagonalizable, say $Hf_a = A^T DA$ for diagonal D full of eigenvalues. So by Morse's Lemma, $f = f(a) + x^T Hf_a x = f(a) + x^T A^T DAx$. Let $y = Ax \Rightarrow y^T = x^T A^T$. Then $f = f(a) + y^T Dy = f(a) + \sum_{i=1}^n \lambda_i y_i^2$. Now normalize via $z_i = \sqrt{|\lambda_i|} y_i$. Thus $f(x) = f(a) + \sum_{i=1}^n sgn(\lambda_i) z_i^2$ $\square$

Problem 1.7.12: Prove that f in Q1.7.11 has a max at a if $\epsilon_i = 1, \forall i$, min if all is -1, else neither if neither is the case.

Proof Case 1: All is $+1$. Then $f(x) \geq f(a), \forall x$ (Note that this is global when considering solely the quadratic, else "local" to the original f if considering the local coordinate changes.). The logic extends to Case 2 when all is -1

Case 3: WLOG, say $\epsilon_1 = +1, \epsilon_2 = -1$. Along $x_1 = t, x_2 = x_3 = \cdots = 0, f(x) = f(a) + t^2 > f(a)$. Yet for $x_1 = x_3 = \cdots = 0, x_2 = t$, then $f(x) = f(a) - t^2 < f(a)$. Hence f acts like a saddle. $\square$

Problem 1.7.15: Let X be a submanifold of $\mathbb{R}^N$. Prove that there's a linear map $l : \mathbb{R}^N \to \mathbb{R}$ whose restriction to X is a Morse function.

Proof Let $l_a(x) = a \cdot x, a \in \mathbb{R}^N$. Recall that $x' \in X$ is a critical point iff $D(l_a)_{x'}(v) = l_a|_{x=x'}(v) = a \cdot v = 0 \Rightarrow a \in (T_x(X))^\perp$. Thus $l_a|_X$ is Morse iff x is nondegenerate $\forall x \in \{x | a \in (T_x(X))^\perp\}$. Yet in the book, recall that if $f : X \to \mathbb{R}$, for almost every $a \in \mathbb{R}^n, f_a := f + a \cdot x$ is Morse on X. Thus for $f = 0$, follows for almost every $a \in \mathbb{R}^N, l_a|_X$ is Morse. As $\mathbb{R}^N$ is clearly not of measure zero, the statement follows. $\square$

Problem 1.7.16: Let f be smooth on open $U \subset \mathbb{R}^k$. Prove that f is Morse $\iff det(Hf_x)^2 + \sum_{i=1}^k (f_{x_i})^2 > 0, \forall x \in U$

Proof Forward: Let $a \in U$ be a critical point of f. By nondegeneracy, $det(Hf_x) \neq 0 \Rightarrow det(Hf_a)^2 > 0$. As $(f_{x_i})^2 \geq 0$, the statement follows for all critical points in U. Now let $a \in U$ not be a critical point of f. But then $\nabla f(x) \neq 0 \Rightarrow \sum_{i=1}^k (f_{x_i})^2 > 0$. As $det(Hf_a)^2 \geq 0$, follows.

Backward: This is essentially proving $det(Hf_a) \neq 0, \forall$ critical points $a \in U$ of f. But $\nabla f(a) = 0 \Rightarrow \sum_{i=1}^k (f_{x_i})^2 = 0 \Rightarrow det(Hf_a)^2 > 0 \Rightarrow det(Hf_a) \neq 0$, thus f is Morse. $\square$

Problem 1.7.17: Say f_t is a homotopic family on $\mathbb{R}^k$. Show that if f_0 is Morse in some neighborhood of compact K, then so is every f_t for small t.

Proof As f_0 is Morse in some neighborhood V of compact K, $\exists$ open $U \ni K \subset U \subset V$. As the proof of Q1.7.16 doesn't require U open as long f is smooth on the relevant subset, restrict to K. Define $\Phi : K \times [0,1] \to \mathbb{R}, \Phi(x,t) = det(H(f_t)_x)^2 + \sum_{i=1}^{k}(f_t)_{x_i})^2$. Then $\forall x \in K, \Phi(x,0) > 0$ by Q1.7.16. The determinant is a polynomial and f is smooth, thus Φ is continuous on $K \times [0,1]$. Continuous of compact is uniformly continuous. So say $\Phi(x,0) \geq \epsilon > 0$. Then $\exists \delta > 0 \ni \forall t \in [0,\delta), \Phi(x,t) > \frac{\epsilon}{2} > 0$. Thus f_t remains Morse in a neighborhood of K after a small perturbation. $\square$

Problem 1.7.18: Let f be Morse on a compact manifold X with a homotopic family $f_t \ni f_0 = f$. Show that Morse functions constitute a stability class.

Proof Let $\{U_1, \cdots, U_N\}$ be a finite open cover of compact X. Then by combining the logic of Q1.7.16-17, say $\Phi : X \times [0,1] \to \mathbb{R}, \Phi(x,0) \geq \epsilon > 0, \forall x \in X$, where by uniform continuity, $\exists \delta_1, \cdots, \delta_N > 0$. Set $\delta := min\{\delta_1, \cdots, \delta_N\}$. Then $\forall t \in [0,\delta), \Phi(x,t) > \frac{\epsilon}{2} > 0, \forall x \in X$, meaning that Morse functions are stable. $\square$

Section 1.8: *Embedding Manifolds in* $\mathbb{R}^n$

Problem 1.8.5: Prove that $p : T(X) \to X, p(x, v) = x$ is a submersion.

Proof Observe $Dp_{(x,v)} : T_{(x,v)}(T(X)) \to T_x(X)$. If $dimX = n$, then recall $T(X)$ is also a manifold of dimension $2n$. Tangent spaces at a point have the same dimension as the manifold, thus there's a local chart s.t. $T_{(x,v)}(T(X)) \cong \mathbb{R}^{2n}, T_x(X) \cong \mathbb{R}^n$. Let $(a, b) \in T(X)$. Then there's a local diffeomorphism around $(a, b), p(a, b) \ni \alpha, \beta$ onto $\mathbb{R}^{2n}, \mathbb{R}^n$, respectively. Let ρ be the canonical projection $\mathbb{R}^{2n} \to \mathbb{R}^n$. Then $p(a, b) = \alpha \circ \rho \circ \beta^{-1}(a, b)$, which by the chain rule with the derivative of diffeomorphisms invertible and projection surjective, that means Dp is surjective as well. $\square$

Problem 1.8.6: ***Vector Field*** on manifold $X \subset \mathbb{R}^N$ is a smooth map $\vec{v} : X \to \mathbb{R}^N \ni \vec{v}(x)$ is always tangent to X at x. Prove the following definition is equivalent: $\vec{v}$ on X is a ***Cross Section*** of $T(X) \ni p \circ \vec{v} = id_X$ (p from Q1.8.5).

Proof Forward: $\vec{v}(x)$ instead of mapping to $\mathbb{R}^N$ can be thought of as mapping to $(x, \vec{v}(x)) \in T(X)$, a tangent vector at x. But the projection map p only takes $x \in T(X)$, which means $p \circ \vec{v} = id_X$

Backward: $p \circ \vec{v}(x) = x \Rightarrow \vec{v}(x) = p^{-1}(x) = (x, v)$ for some $v \in T_x(X)$. But then $\vec{v}(x)$ thus gets some v based on the derivative at x, a.k.a. a tangent vector at x. $\square$

Problem 1.8.7: $x \in X$ is a ***Zero*** of $\vec{v}$ if $\vec{v}(x) = 0$. If k is odd, show $\exists \vec{v}$ on S^k lacking zeros.

Proof Tangency means the vector at x is orthogonal. If $\vec{v}(x) = 0$, then if $\vec{v}(x)$ merely permutes the coordinates like the symmetric group S^{k-1}, that'd force all the components to be 0, contradicting $x \in S^k$. So try $\vec{v}(x) = (-x_2, x_1, \cdots, -x_k, x_{k-1})$. Observe $x \cdot \vec{v}(x) = 0 \Rightarrow x \perp \vec{v}(x)$ and $\vec{v}(x) = 0 \Rightarrow x = 0$, yet $||x|| = 0 \neq 1$, a contradiction. $\square$

Problem 1.8.10: Prove the ***Whitney Immersion Theorem***, that every k-dimensional manifold X can be immersed in $\mathbb{R}^{2k}$

Proof By Whitney's Embedding Theorem, $\exists$ embedding $f : M \to \mathbb{R}^{2k+1}$. Define $g : T(M) \to \mathbb{R}^m, g(x, v) = Df_x(v), m > 2k$. Then $dimT(M) = 2k$ means x is a critical point of g, $\forall x \in T(M)$. By Sard's Theorem, $\exists a \in \mathbb{R}^m \ni 0 \neq a \notin g(T(M))$. Let π project $\mathbb{R}^m$ onto the orthogonal complement of a, H_a. Then $\pi \circ f : M \to H_a$ is an immersion if $D(\pi \circ f)$ is injective.

BWOC, say $\exists 0 \neq v \in T_x(M) \ni D(\pi \circ f)_x(v) = 0$. By the chain rule, $\pi \circ Df_x(v) = 0$. The projection of $Df_x(v)$ into H_a is only zero if such is a scalar multiple of a, namely $Df_x(v) = ta$ for some $t \in \mathbb{R}$. Note that $t = 0$ contradicts $Df_x(v) = 0 \notin g(T(M))$, so either $g(x, \frac{1}{t}) = a$ or $Df_x(\frac{1}{t}) = a$, a contradiction as a can't lie in $Im(g)$. Thus $D(\pi \circ f)$ is injective. But $\pi \circ f$ immerses into an $m - 1$ dimensional subspace of $\mathbb{R}^m$. Repeat inductively until $m = 2k$ $\square$

Problem 1.8.11: Show that if X is a compact k-dimensional manifold, $\exists f : X \to \mathbb{R}^{2k-1}$ that's an immersion except at finitely many points of X.

Proof Let $f : X \to \mathbb{R}^{2k}$ by Q1.8.10 be an immersion and define $F : T(X) \to \mathbb{R}^{2k}, F(x,v) = Df_x(v)$. I claim that if a is a regular value of F, then $F^{-1}(a)$ is finite. PROOF: By the Preimage Theorem, $F^{-1}(a)$ is a submanifold of dimension $dim T(X) - dim \mathbb{R}^{2k} = 0$. BWOC, say $F^{-1}(a)$ is infinite. Observe that $Df_x(\lambda v) = \lambda Df_x(v) = \lambda a$, so normalize each term in $F^{-1}(a)$ via $\frac{1}{||v||}F(x, \frac{v}{||v||}), w := \frac{v}{||v||}$. So note that $S := \{(x,v) \in T(X) | ||v|| = 1\} = X \times S^{k-1}$ is compact. So for any sequence $(x_i, w_i) \in F^{-1}(a) \subset S$, then as X and S are compact, such is sequentially compact, namely $x_{i_l} \to x, w_{i_l} \to w \in S^{k-1}$. Then $Df_{x_{i_l}}(v_{i_l}) = ||v_{i_l}|| Df_{x_{i_l}}(w_{i_l}) = a \neq 0 \Rightarrow Df_{x_{i_l}}(w_{i_l}) = \frac{a}{||v_{i_l}||}$. Take the limit as $l \to \infty$ obtains $\frac{a}{||v_{i_l}||} \to Df_x(w) \neq 0$ as f is injective and w with a nonzero norm. Thus $Df_x(\lambda w)$ for some $\lambda > 0$. Then $(x, \lambda w) \in F^{-1}(a)$ is an accumulation point, contradicting $F^{-1}(a)$ being 0-dimensional. ■

By Sard's Theorem, the critical values of F has measure 0, so choose any regular value a of F. Let π be the ***Orthogonal Projection*** perpendicular to a, namely $\pi(v) = v - proj_a v = v - (\frac{a \cdot v}{||a||^2})a$. Namely, the portion of v going in a is subtracted. But $\pi(v) = 0 \Rightarrow v = proj_{av} \Rightarrow$ v is already in the direction a $\Rightarrow Ker(\pi) = span(a)$. Thus define the hyperplane orthogonal to a as $Im(\pi) = H_a \cong_I \mathbb{R}^{2k-1}$. I claim that $g := \pi \circ f$ is an immersion except on $F^{-1}(a)$. PROOF: Observe that Dg_x is injective $\iff Ker(\pi \circ Df_x) = \{0\}$. Thus Dg_x isn't injective $\iff \exists v \neq 0 \ni Df_x(v) \in Ker(\pi) = span(a) \iff Df_x(v) = \lambda a$ for some $\lambda \neq 0$. Rescale to get $Df_x(\frac{v}{\lambda}) = a$. But as $F^{-1}(a)$ is finite, there's thus only finitely many $x \in X \ni g$ isn't an immersion. ■

But then g is the desired immersion of $X \to H_a \cong_I \mathbb{R}^{2k-1}$ and thus the statement follows. □

Note: $F^{-1}(a)$ are termed ***Cross Caps***

Problem 1.8.12: Whitney showed that for maps of 2-manifolds into $\mathbb{R}^3$, a typical cross cap looks like $(x,y) \to (x, xy, y^2)$. Check that this an immersion except at the origin. What does its image look like?

Proof Observe that $Jf = \begin{bmatrix} 1 & 0 \\ y & x \\ 0 & 2y \end{bmatrix}$. Such is injective iff there's no $(a,b) \in \mathbb{R}^2 \ni Jf_{(x,y)}(a,b) = 0 \iff$ the column vectors are L.I., which $0 = \lambda, x = \lambda y, 2y = \lambda * 0 \Rightarrow \lambda = x = y = 0$. Thus, f is an immersion everywhere except at the origin. Now rewrite $X = x, Y = xy, Z = y^2 \Rightarrow Y = Xy \Rightarrow y = \frac{Y}{X}$ for $X \neq 0 \Rightarrow Z = (\frac{Y}{X})^2 \Rightarrow Y^2 = X^2 Z$. As $Z = y^2$, it's [***Answer***] $\boxed{Y^2 = X^2 Z, Z \geq 0}$ □

Note: The shape is the ***Whitney Umbrella***, a 3D parabola that intersects at the origin.

Problem 1.8.13: An open cover $\{V_\alpha\}$ of manifold X is ***Locally Finite*** if $\forall x \in X, \exists$ neighborhood U_x that intersects only finitely many V_α. Show that any open cover $\{U_\alpha\}$ admits a locally finite refinement $\{V_\alpha\}$

Proof For open cover $\{U_\alpha\}$ of manifold X, such admits a Partition of Unity of smooth functions $\{\theta_i\}$, which recall $V_i = \{x|\theta_i(x) > 0\} \subset U_{\alpha'}$ for some α'. As θ_i is continuous and $(0, \infty)$ is open, so is V_i. Note $\forall x \in X$, each has a neighborhood where only finitely many functions are nonzero. As $\forall x \in X, \sum_i \theta_i(x) = 1$, erasing all the irrelevant i's means the sum is finite. Then there's finitely many i' s.t. $\theta_{i'}(x) > 0$ and thus $x \in V_{i'}$ for finitely many i'. So $\{V_i\}$ indeed open covers X and is a locally finite refinement of $\{U_\alpha\}$ $\square$

Note: Not fully correct as it's more like $(0, 1]$, but maybe I expand the codomain of θ_i?

Problem 1.8.14: Generalize Q1.3.10 to noncompact submanifolds. Thus the ***Generalized Inverse Function Theorem*** states that if $f : X \to Y$ is smooth and injective on submanifold Z of X and $\forall x \in Z, Df_x$ is an isomorphism, then f has $Z \cong_D f(Z)$

Proof The InFT states each x_i has a local diffeomorphism, say $f : V_x \to U_x$. Let g_x be the local inverse. As the proof of Q1.8.13 didn't rely on X being a manifold, yet $f(Z) \subset \cup_{x \in Z} U_x$, such admits a locally finite refinement $\{U_i\}$ of $f(Z)$. Note that each $U_i \subset U_{x'}$ for some x' and thus $g_{x'}$ is a valid inverse on U_i as it's constrained from domain $U_{x'}$

Now define $W := \{y \in \cup_i U_i | \forall i, j \ni y \in U_i \cap U_j, g_i(y) = g_j(y)\}$, namely is the set where all local inverses agree. As f is injective on Z, it must be that $g_i(y) = g_j(y), \forall y \in f(Z) \cap U_i \cap U_j$. Namely $f(Z) \subset W$. Thus define $g(y) := g_i(y)$ for any $i \ni y \in U_i$, which is well-defined. As each g_i is smooth, g is is smooth. With $x \in Z, f(x) \in f(Z) \subset W \Rightarrow g \circ f(x) = x$, and as f is injective on Z, then $f \circ g(x) = x$. That makes f a diffeomorphism from Z to $f(Z)$ $\square$

Note: So f being injective on Z **doesn't** imply that $g_i(y) = g_j(y)$ as there's a more fundamental problem that W may not even open cover $f(Z)$. So the U_i's would have to be defined more carefully, but StackExchange makes it look really complicated so I'm calling it quits.

Problem 1.8.15: Prove the ***Smooth Urysohn Theorem***: if A, B are disjoint, closed subsets of manifold X, then $\exists$ smooth function ϕ on X $\ni 0 \leq \phi \leq 1$ where $\phi(A) = \{0\}, \phi(B) = \{1\}$

Proof By Q1.8.12, smooth manifolds are ***Paracompact***: every open cover admits a locally finite refinement. As paracompact Hausdorff spaces are normal, $\exists$ open $U, V \subset X \ni A \subset U, B \subset V \ni U \cap V = \emptyset$. As X is open, then $X \setminus (A \cup B)$ is open. Thus $\mathcal{U} := \{U, V, X \setminus (A \cup B)\}$ open covers X. So there's a Partition of Unity $\{\theta_i\}$. Recall for each i, θ_i is 0 outside some closed set $C_i \subset U, V$, or $X \setminus (A \cup B)$. As $U \cap V = \emptyset$ and B lies in V but not $X \setminus (A \cup B)$, follows $\exists x \in V \ni \sum_i \theta_i = 1$, thus there's some $\theta_i \ni C_i \subset V$. Thus $I_V := \{i | C_i \subset V\} \neq \emptyset$. Define $\phi := \sum_{i \in I_V} \theta_i$

Such is the sum of smooth functions and thus is smooth. If $x \in A \Rightarrow x \notin V \Rightarrow \phi(x) = 0$, as $\theta_i(x)$ is zero outside of $C_i \subset V$, which means it's zero on U, thus A. If $x \in B \Rightarrow x \in V \Rightarrow \phi(x) = 1$ by construction. Lastly, as $\sum_i \theta_i(x) = 1, \forall x \in X$, and $0 \leq \theta_i(x), \forall x \in X$ and $\forall i$, follows that reducing the i's that summed over only means $0 \leq \phi(x) \leq 1, \forall x \in X$. Thus the statement follows. $\square$