Tutorial 7/8

aside: plot $t - \theta_{head} - \theta_{ear}$ & see if $\exists$ manifold, maybe vocalizations/AP on manifold or opto stim "leaves" manifold

maybe rate of vocalizations?

Victorian Era~

General Problem: Given $x_1, \dots, x_n \in \mathbb{R}^d$, $y_1, \dots, y_n \in \mathbb{R}$ want to fit $f(x) = \sum_{i=1}^{p} c_i \varphi_i(x)$ where we choose $\varphi_1, \dots, \varphi_p$. Usually $\varphi_1, \dots, \varphi_p$ are L.I. $\Rightarrow$ is $\therefore$ a basis, a.k.a. a **Dictionary**

$\longrightarrow$ "Fit" := **Loss Function**, a func. of $c_1, \dots, c_p$ e.g. **Mean Square Error (MSE)** is $L(c_1, \dots, c_p) := \frac{1}{n} \sum_{i=1}^{n} \big(y_i - \underbrace{\sum_{j=1}^{p} c_j \varphi_j(x_i)}_{= f(x_i)}\big)^2$, getting f is then the **Least Squares Approximation**

$\longrightarrow$ Generalize to $y_i \in \mathbb{R}^{d'}$. Then $L(c_1, \dots, c_n) := \frac{1}{n} \sum_{k=1}^{d'} \sum_{i=1}^{n} \left(y_i^{(k)} - \sum_{j=1}^{p} c_j \varphi_j^{(k)}(x_i)\right)^2$
$= \frac{1}{n} \sum_{i=1}^{p} \| y_i - f(x_i) \|_2^2$

- Norm $\|v\|_2 = \sqrt{\sum_{j=1}^{d} v_j^2}$

$\longrightarrow$ $\therefore$ want to minimize $L(c_1, \dots, c_n)$ so that $y_i \approx f(x_i)$ ⊛

$\longrightarrow$ Return to case $y_i \in \mathbb{R}$. Define $\Phi \in \mathbb{R}^{n \times p}$, $\Phi_{ij} := \varphi_j(x_i)$ w/ $c = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \in \mathbb{R}^{p \times 1}$ &

$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$. Then ⊛ $\Leftrightarrow \Phi \cdot c \approx y$, a.k.a. c minimizes $\| y - \Phi \cdot c \|_2^2$

- If $y_i \in \mathbb{R}^{d'}$, then Φ becomes a tensor

Proposition: $\exists$ minimizer $c_* = \underset{c}{\operatorname{argmin}}\, L(c)$ which satisfies the **Normal Equations** $\Phi^T \Phi \cdot c = \Phi^T y$

$\longrightarrow$ $\Phi^T \Phi$ is invertible $\Leftrightarrow$ col(Φ) are L.I.

Proposition: $\exists\, \underset{\text{use Gram-Schmidt}}{Q} \in \mathbb{R}^{n \times p}$, $R \in \mathbb{R}^{p \times p} \ni \overbrace{Q^T Q = I}^{\text{orthogonal}}$, R is upper $\triangle$, & $\Phi = QR$ where col(Φ) are L.I.

$\longrightarrow$ Then minimizer $c_* = R^{-1} Q^T y$

Singular Value Decomposition (SVD): Most common method to solve least squares

- **Overfitting:** Fit to noise > trend, occurs when $p >> n \Longrightarrow$ soln: $\downarrow p$
- **Generalizability/Stability:** Fails to fit to data beyond what was fitted
- **Coefficients:** c_* can often be too large/small

$\longrightarrow$ Solution is **Regularization**, where we introduce a Lagrange Multiplier λ by optimizing

$L(c_1, \dots, c_n) + \underbrace{\lambda R(c_1, \dots, c_n)}_{\text{regularization term}}$ e.g. 1) $R(c) = \|c\|_2$ (penalizes large coefficients); **Ridge Regression**

[LASSO] 2) $R(c) = \|c\|_1 = \sum |c_i|$ (encourages "sparsity" a.k.a. makes more $c_j = 0$)

- Great for overcomplete dictionary

3) $R(c) = \lambda \|c\|_2^2 + \mu \|c\|_1$ is **Elastic Net**

$\boxed{\text{Tutorial 7/10}}$ Lowk nothing useful today; brief intro to ODEs @ 454 level

- More precisely, dictionary are terms we guess f to make-up e.g. $\{1, x, x^2, \sin x, \cos x\}$
  - For cont. time SINDy, we want $f(x(t)) \approx \sum_{j=1}^{p} \varphi_j(x) c_j$

$$\underbrace{y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\text{data given}}, \quad \underbrace{y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}}_{\text{derivative needs to be approximated}}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}, \quad \Phi(y) = \begin{bmatrix} | & & | \\ \varphi_1(y) & \cdots & \varphi_p(y) \\ | & & | \end{bmatrix}$$

time given for data

sparsity for L_1-norm

Want $y' = \Phi(y)c$, where $c := \underset{c}{\operatorname{argmin}} \|y' - \Phi c\|_2^2 + \lambda \|c\|_1$

  - " optimal $\tilde{c} \ni \tilde{f}(x) = \Phi(x)\tilde{c}$
  - For discrete time SINDy, instead of $x'(t) = f(x(t))$, want $x_{k+1} = F(x_k)$

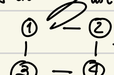
$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}, \text{ want } c \ni \hat{y} = \Phi(y)c$$

same way to find c

  - Get optimal $\tilde{c} \ni \tilde{F}(x) = \Phi(x)\tilde{c}$, can guess future via $x_{k+1} = \tilde{F}(x_k)$

$\boxed{\text{Seminar 7/14}}$

- Halfway into the REU & still no data, what on earth do I do? Just keep reading ig
- Say quadriped has 1 oscillator / limb, governed by



. How could this be modeled?

coupling for front limbs — coupling for left limbs

$$\begin{cases} \dot{x}_1(t) = F(x_1(t)) + G(x_1(t), x_2(t)) + G(x_1(t), x_3(t)) \\ \dot{x}_2(t) = F(x_2(t)) \\ \dot{x}_3(t) = F(x_3(t)) \\ \dot{x}_4(t) = \underbrace{F(x_4(t))}_{\text{same "governance" } F \text{ per limb}} \end{cases}$$

coupling for front / " back / " back

right / left / right

or switch to double dots as $F = ma$, though model gets very complicated

e.g. $\ddot{\varphi}_1 = F(\varphi_1, \dot{\varphi}_1) + G(\varphi_1, \varphi_2, \dot{\varphi}_1, \dot{\varphi}_2) + G(\varphi_1, \varphi_3, \dot{\varphi}_1, \dot{\varphi}_3)$

- Things to consider w/ data-driven modeling:

(1) # of horses / which ones

(2) # of gaits / which ones $\longrightarrow$ Needs flexibility to break symmetries (switch between gaits)

(3) Symmetries assumed of groups of solns $\longrightarrow$ Constrains possibilities explored so that we can find couplings

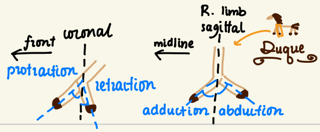
(4) If we want continuous / discrete time models

(5) Autonomous vs. Non-autonomous

no external factors — $\exists$ external factors $\Rightarrow$ proprioceptive feedback

in this case, it's "duty cycle": when the foot hits ground $\Rightarrow$ limits degrees of freedom for locomotion

(6) Hyperparameter Tuning

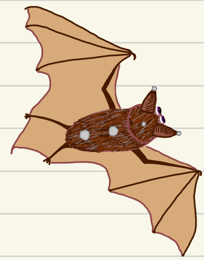


Seminar 7/15

- Apparently taking time to get processed data BUT $\exists$ ugly raw data
- Likely have Fourier basis $\{e^{ik\cdot\varphi}\}$ for the dictionary & try stab @ autonomous ODEs

Github Tutorial 7/17

- Pretty much basics of how git works
- Branch: Make Δ w/o affecting other branches to maintain og code
- Pull Requests (PR): Merge branch & $\therefore$ update og code



- TL:DR; Use ChatGPT for Github help

- Bibtex: Bibliography for LaTeX, a reference manager
  1) Make .bib file
  2) Copy & paste bibtex format of paper into file
  3) Use /citep{key in bibtex}
  4) @ end before /end{document}, have /bibliography{.bib file name}

LinkedIn 7/18

- Soft introduction to digital networking, nothing special tbh

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Neural Network & Machine Learning 7/22

- Work boring & I want lunch >_< like Panda, Takeout, Latino, Sushi, Indian ... except it's been 3 weeks & I haven't been PAID! $$$

- Neuron: $f:\mathbb{R}^{m+1}\to\mathbb{R}$, $f(x)=\sigma(\underset{\text{learnable weight}}{W}X^T)$; $X, W\in\mathbb{R}^{m+1}$, $\sigma:\mathbb{R}\to\mathbb{R}$
  - $\longrightarrow$ σ is an Activation Function e.g. $\sigma=\begin{cases}x & x>0\\ 0 & \text{otherwise}\end{cases}$, $\sigma=\frac{e^x}{1+e^x}$, $\sigma=\tanh(x)$
  - $\longrightarrow$ Neural Network Layer: $f:\mathbb{R}^{m+1}\to\mathbb{R}^n$, $f=(f_1,\dots,f_n)$; neurons combined, so $f(x)=\sigma(WX)$, $W\in\mathbb{R}^{n\times(m+1)}$, $X\in\mathbb{R}^{m+1}$, $\sigma:\mathbb{R}^{m+1}\to\mathbb{R}^{m+1}$
  - $\longrightarrow$ Neural Network: $NN:\mathbb{R}^{m+1}\to\mathbb{R}^n$, $NN(X)=\sigma_L(W_L\sigma_{L-1}(W_{L-1}\sigma_{L-2}(\dots\sigma_1(W_1X)\dots)$

    weights/layer; layer #1, layer #2, layer #L

- Theorem: A Neural Network w/ 1 layer is a universal function appropriator given enough neurons & nonlinear activator functions.

- For loss function: $\mathcal{L}\to\mathbb{R}^{p\times n}\to\mathbb{R}$, then we want gradient $\frac{\partial\mathcal{L}(x,y,\theta)}{\partial\theta_{ij}}$ to go in the direction of (a local???) min

  j^{th} weight @ i^{th} layer

- Say our Neural Network has H layers: $f(x) = l_H \circ l_{H-1} \circ \dots \circ l_1(x)$, w/ loss $\mathcal{L}(x,y,\theta) = \|y - f(x)\|_2^2$
  → W/ 3 layers: $f(x) = W_3\,\sigma_2(W_2\,\sigma_1(W_1 x))$

Forward Pass

x

$z_1 = W_1 x$

$z_2 = \sigma_1(z_1)$

$z_3 = W_2 z_2$

$z_4 = \sigma_2(z_3)$

$z_5 = W_3 z_4$

$\mathcal{L}(x,y,z) = (y - z_5)^4$

Backward Pass

$\mathcal{L}_{z_5} = -2(y - z_5)$

$(z_5)_{z_4} = W_3$

$(z_4)_{z_3} = \sigma_2'(z_3)$

$(z_3)_{z_2} = W_2$

$(z_2)_{z_1} = \sigma_1'(z_2)$

partials

Then $\mathcal{L}_{W_3} = \mathcal{L}_{z_5} \frac{\partial z_5}{\partial W_3}$

$\mathcal{L}_{W_2} = \mathcal{L}_{z_5} \frac{\partial z_5}{\partial z_4} \frac{\partial z_4}{\partial z_3} \frac{\partial z_3}{\partial W_2}$

$\vdots$

Termed **Backpropagation**

- **Gradient Descent:** Find $\mathcal{L}, \mathcal{L}_\theta$, then for **Learning Rate** parameter α, update weights $\theta \to \theta' := \theta - \alpha \mathcal{L}_\theta$, then repeat till $\mathcal{L}_\theta = 0$ (some local min)
  → **Proposition:** If α is sufficiently small, $\mathcal{L}(x,y,\theta') \le \mathcal{L}(x,y,\theta)$

Poster 7/25

- Boring; consider QR code for any R Shiny app
- Maybe be a Methods Poster since ∄ results & it's due in 1 week (definitely finish beforehand as lease expires July 31st)

Idea: Find open source data of quadriped muts. (cat, dog, mice, rats, etc.) & model that

- Ngl I didn't like their example posters @ all; too much text
- No hard requirements; have a reasonable poster; will be @ Al-Marah

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AI Coding 7/29

Poster due Tues. 9pm

- No clue how I'd make a poster in time !!, "make" synthetic data
- All I want to do rn is watch Better Call Saul & play Roblox
- Is convincing me to never use AI for presentations

My Roblox Supermarket !!

← my 2025 Honda Civic

To-Do Today

- Get 8-cell Fitzhugh-Nagumo eqns to "work" (could reproduce trot, walk, pace, nothing else worked)
- ✓ Shop for groceries
- ✓ Ask for poster extension
- ✓ Complete BUMC background check
- ✓ Workout upper body
- ✓ Meet w/ Kevin
- ✓ Attend AI lecture
- ✗ Call back ABA position
- ✓ Ask about JPARCC clinic internship
- ✗ Pay for hotel
- ✗ Pay for secondaries
- ✓ Inquire about stipend
- ✗ Begin to pack-up
- ✗ Begin to make poster

Scientific Writing 8/1

- Will have to write an "extended" abstract w/ minimal math
  → Requirements: (1) Self-contained (4) Audience: Future REU students
  (2) 3-4pgs.
  (3) Intro + Methods + Results + Conclusions + References
  [Discussion?]

To-Do

- Plot oscillatory freqs. over Δ parameters
- 3D plot of gaits
- Pay for hotel
- Pay for secondaries
- Work on poster
- ~~Get github working~~
- Schedule BUMC + Zu Nidito appointments
- Consider RShiny app
- Workout upper body

Final Week to-do

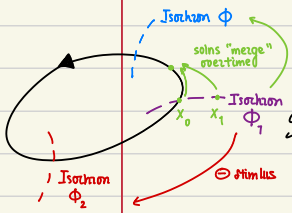
- ~~Get paid...? ;C~~
- Make poster
- Make abstract
- Upload files to git
- Present poster
- Finalize code/qmd
- Attend both BUMCT-DS orientations
- Complete pediatric scavenger hunt
- Figure out parking situation
- Tu Nidito train + shadow
- Submit all secondaries
- JPARCC Interview

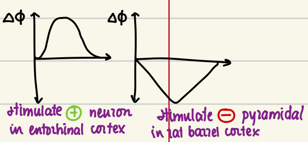
Mathematical Foundations of Neuroscience by Ermentrout

CHAPTER 8: Weak Coupling of Neural Oscillators

- **Binding Problem**: How does the brain integrate seperate processing of an object → unified perception?
  - ↳ Strong evidence for oscillators e.g. $\gamma, \theta, \delta, \alpha, \beta$ waves (orange, chewy 🧋 good memories, smells nice ☺)

- Regularly rhythmic APs can be thought of as limit cycles
- Consider $\dot{x} = F(x)$ in $\mathbb{R}^n$ w/ Γ a T-cycle (recall: $F^T(x^*) = x^*$; $F^n(x^*) \neq x^*$, $\forall n < T$)
  - ↳ Such is Orbitally Asymptotically Stable if I.C. $\to \Gamma$ as $t \to \infty$
  - ↳ Parameterize $\theta \in [0, T)$ along the limit cycle w/ phase $\Theta(x)$ of the oscillator for $x \in \Gamma$
  - ↳ Let $X(t; y)$ be a soln w/ I.C y, a pt. in limit cycle neighborhood w/ x on the limit cycle $\ni$ $\|X(t;x) - X(t;y)\| \to \emptyset$ as $t \to \infty$; then define $\Theta(y) = \Theta(x)$, namely solns are = as $t \to \infty$
  - ◦ Then $N(x) = \{y \mid \Theta(y) = \Theta(x)\}$ is the **Isochron**, which are locally invariant, namely $y \in N(x) \Rightarrow X(T; y) \in N(x)$, where $y \to X(T; y)$ gives the **Poincaré/Return Map**:
  - ◦ Say $\exists$ stimulus to $x \in \Gamma$ on phase $\phi \to \Phi$; map is **Phase Transition Curve** $P(\phi) = \Phi'$
    - ▪ **Type 1 (Weak) Resetting**: $P: [0,T) \to [0,T)$ is invertible
    - ▪ **Type 0 (Strong) Resetting**: " " " isn't "
    - ▪ Ex: Stimulus so strong that AP fires immediately $\Leftrightarrow P(\phi) = 0, \forall \phi \in [0,T)$, so ~~injective~~ $\Rightarrow$ ~~invertible~~ $\Rightarrow$ strong!
  - ◦ **Phase Resetting Curve (PRC)**: $\Delta(\phi) \equiv \underbrace{P(\phi)}_{new} - \underbrace{\phi}_{old}$





- Recall phase of limit cycle has $\frac{d\theta}{dt} = 1$ w/ $P(\phi) = \phi' < \underbrace{T}_{period}$, $\tau = T - \phi' > \emptyset$ (speeds up delay to next spike but doesn't trigger AP), so $\underbrace{T'}_{\text{time of next spike}} = \phi + \tau = \phi + T - \phi' = T - \Delta(\phi)$
  - ↳ **Lemma**: $\Delta(\phi) = T - T'$, so $T' < T \Rightarrow \Delta(\phi) > \emptyset \Rightarrow$ stimulus speeds up cycle
  - ↳ Common to rescale $T \to {[0,1) \atop [0,2\pi)}$ by multiplying by ${1/T \atop 2\pi/T}$ to compare PRCs across different frequencies

- Let $\overbrace{x = X_0(\phi) \in \Gamma}^{\text{pt. on limit cycle, I.C.}}$ w/ phase $\overbrace{\phi = \Theta(x) \in [0,T)}^{\text{by definition}}$ & arbitrary perturbation $y \in \mathbb{R}^n$. Then:
  - ↳ **Lemma**: $P(\phi) = \phi' = \Theta(x + y) = \phi + \nabla_x \Theta(x) \cdot y + O(\|y\|^2)$, $\therefore$ for small perturbations, $\Delta(\phi; y) = \nabla_x \Theta(x) \cdot y$
  - ↳ Define $Z(\phi) := \nabla_x \Theta(X_0(\phi))$

- Now let $X_0(t)$ be a T-periodic limit cycle soln w/ $A(t) = D_x F(x)\big|_{X_0(t)}$ the $n \times n$ matrix from linearizing $\dot{x} = F(t)$ around the limit cycle. Then linearized eq. solns satisfy $\dot{y} - A(t)y(t) = (Ly)(t) = \emptyset$

→ Let $\langle u(t), v(t)\rangle = \int_0^T u(t)\cdot v(t)\,dt$ be the inner product on T-periodic functions in $\mathbb{R}^n$

→ For linear operator L, the Adjoint Linear Operator $L^* \ni \langle u, Lv\rangle = \langle L^* u, v\rangle$

- Then: $(L^* y)(t) = -\dot{y} - A(t)^T y(t)$. Note $L^* Z(t) = \emptyset$
- If $X_0(t)$ is a stable limit cycle, then L has a nullspace spanned by scalars of $X_0(t)$
- Adjoint has 1D nullspace $\Rightarrow Z(t) \propto$ eigenfunction
- As $\Theta(X_0(\Phi)) = \Phi \xRightarrow{\text{diff. wrt } \Phi} Z(\Phi)\cdot \frac{dX_0(\Phi)}{d\Phi} = 1$ (normalization)

→ Thus $Z(t)$ is the soln of $L^* Z = \emptyset$ & $Z \cdot \frac{dX_0}{dt} = 1$

Q8.1: Consider $\dot{x} = f(x) > \emptyset$, $x \in S^1 = \mathbb{R}\setminus\mathbb{Z}$

As $x \in S^1$, it has period 1: $f(x) = f(x+1)$. Then the eqn has T-periodic soln $x_0(t)$ w/ period $T = \int_0^T dt = \int_{x_0}^{x_0+1} \frac{dt}{dx}\,dx = \int_0^1 \frac{dx}{f(x)}$. So $Z \cdot \frac{dX_0}{dt} = Z f(x_0(t)) \Rightarrow Z(t) = \frac{1}{f(x_0(t))}$

[Supercritical Hopf Bifurcation]

Q8.2: Consider class of nonlinear oscillators $u' = \lambda(r)u - w(r)v$, $r^2 = u^2 + v^2$, w/ $\lambda(1) = 0$,
$v' = \lambda(r)v + w(r)u$ $\quad w(1) = 1$, & $\lambda'(1) < 0$

W/ stable limit cycle soln $(u,v) = (\cos t, \sin t)$. Calculate the adjoint.

SOLUTION $Z(t) = (u^*(t), v^*(t)) = (a\cos t - \sin t,\ a\sin t + \cos t)$, $a = -\frac{w'(1)}{\lambda'(1)}$ (can't figure out why)

Q8.3: Take the quadratic integrate-&-fire w/ ∞ reset $V' = V^2 + I$, w/ soln $V(t) = -\sqrt{I}\cot(\sqrt{I}t)$

SOLUTION The adjoint is $Z(t)\frac{dV(t)}{dt} = 1 \Rightarrow Z(t) = \frac{1}{V'(t)} = \frac{-1}{\sqrt{I}(-\csc^2(\sqrt{I}t))\sqrt{I}} = \boxed{\frac{\sin^2(\sqrt{I}t)}{I}}$

Class I Excitability: Spiking freq. can be arbitrary low (saddlenode bifurcation)
" II " : $\exists$ cutoff threshold to spiking freq. (Hoph bifurcation)
" III " : Spikes only in response to sudden ↑ current

Dynamic Clamp: ⊕ & ⊖ channels/synapses in real time stimulated by computer Δ current

→ Let $\dot{X} = F(X)$, $X \in \mathbb{R}^n$, w/ $G(t,t_0)$ & onset time t_0; update to $\dot{X} = F(X) + G(t,t_0)$

→ Then $PRC(t_0) = \int_0^{T} Z(t)\cdot G(t,t_0)\,dt$, a time delay/advance due to input G

→ For membrane eqn $X(t)$ & synaptic current $G(t,t_0)$ by the alpha function: $G(t,t_0) = \alpha(t-t_0)(V_{syn} - V(t))$ is the Spike-Time Response Curve (STRC): When neuron spikes due to input & its arrival

Post-Stimulus Time Histogram (PSTH): % of spike @ time t after stimulus onset as $T \times \frac{1}{T} = 1$ spike/period

Proposition: Assume that (I) mean firing rate of neuron is $\frac{1}{T}$ before stimulus + uniform

(II) stimulus is weak enough so that $\nexists$ new spikes

(III) $\longrightarrow$ baseline mean firing rate after 1 cycle

Then (II)+(III) $\Rightarrow$ area $t = T \to 2T$ is 1 since $\nexists$ new spikes w/ 1 spike/period

Say $\exists$ spike @ $t=s$. Then expected next time is $s+T$, but stimulus shifts to s'. Then:

$$s' = \underbrace{s + T}_{\text{expected time w/o stimulus}} - \Delta(\underbrace{T-s}_{\text{time from firing to stimulus}}) := F(s) \Rightarrow F(0)=T,\ F(T)=2T$$

As $F'(s) = 1+\Delta'(T-s) > 0$ as stimulus is weak $\Rightarrow$ invertible $[0,T] \longleftrightarrow [T,2T]$

$$\text{PSTH}: P(\{s'<t\}) = \int_T^t \text{PSTH}(t')\,dt' = P(\{F(s)<t\}) = P(\{s < F^{-1}(t)\}) = \frac{F^{-1}(t)}{T}$$

(distr. of spikes before stim. is uniform) $t\in[T,2T] \to F^{-1}(t) \in [0,T]$

$$\Rightarrow \boxed{F^{-1}(t) = T\int_T^t \text{PSTH}(t')\,dt'}$$

} relates PSTH & PRC

Say oscillator $x(t)$ has PRC $\Delta(t)$, $t\in[0,T)$ for period T, w/ a stimulus applied every T_f time steps. Denote ϕ_n the oscillator phase @ the instant before stimulus arrival

↳ After stimulus, the phase is $P(\phi) := \phi_n + \Delta(\phi_n),\ \phi\in[0,T)$

↳ Between stimuli, oscillator advances by T_f, so: → Phase Transition Curves (PTC)

$$\phi_{n+1} = \phi_n + \Delta(\phi_n) + T_f = P(\phi_n) + T_f := M(\phi_n)$$

Invertible maps on ○ (circle) are characterized completely by their **Rotation Number**: Avg # of O's per stimulus, defined by $\rho := \lim_{n\to\infty} \frac{1}{T}\frac{\phi_n}{n}$; e.g. $\rho = \frac{2}{3} \Rightarrow 2:3$ locking, $\frac{\text{2 oscillator cycles}}{\text{3 stimulus cycles}}$

Denjoy's Theorem: The rotation number is well-defined, e.g. $\exists \rho$ & is independent of the I.C. If $M(\phi)\in C^2[0,T)$, then (1) $\rho\in\mathbb{Q} \Leftrightarrow M(\phi)$ has a periodic orbit of some period, namely $\exists N\in\mathbb{N} \ni \phi_{n+N} \equiv \phi_n \mod T$, (2) $\rho\in\mathbb{I} \Leftrightarrow$ every orbit $\{\phi_n\}$ is dense in ○ (circle), & (3) ρ is continuous of any parameters in function M

Say its 1:1 locking. Then $\phi_{n+1} = \phi_n + T \Rightarrow \Delta(\phi) = T - T_f$

" 2:1 (↳ fires 2x per stimulus) " $\phi_{n+1} = \phi_n + 2T \Rightarrow \Delta(\phi) = 2T - T_f$

" 1:2 (↳ fires once after 2 stimuli) " $\phi_{n+2} = \phi_n + T \Rightarrow \phi_{n+2} = M(\phi_{n+1}) = M^2(\phi)$

- Define normalized PRC $d(\phi) := \dfrac{\Delta(T_0\phi)}{T_0}$ (unitless & 1-periodic) $\rightarrow \delta_1(t) := \sum_{n=-\infty}^{\infty} \delta(t-n)$ (periodized dirac; impulse $\forall n \in \mathbb{Z}$)

  $\longrightarrow$ Say 2 cells are coupled by
$$\theta_1' = \omega_1 + \boxed{\delta_1}(\theta_2)\, d_1(\theta_1)$$
$$\theta_2' = \omega_2 + \boxed{\delta_1}(\theta_1)\, d_2(\theta_2)$$
confused af to continue

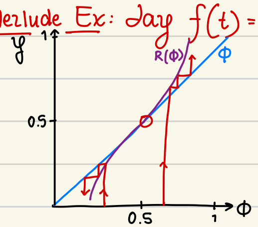
- **Mirollo-Strogatz Theorem**: Say w/o coupling, each oscillator satisfies $x_j(t) = f(t)$, $f(0)=0$, $f(1)=1$, $f'(t)>0$, $f''(t)<0$, where $x_j(t) = 1 \rightarrow$ resets to 0. Each oscillator that fires advances those that didn't fire by $\varepsilon > \emptyset$. Furthermore:
  (I) @ time t^-, if m oscillators fire, others <1 are increased by $m\varepsilon$
  (II) If increment $\rightarrow$ fire, then set to 1 but doesn't advance others
  (III) All oscillators @ 1 are immediately reset to $\emptyset$

  $\therefore$, oscillator(s) can join firing group to induce synchrony, especially if $\exists$ strong coupling. Furthermore, the set of I.C.s in which oscillators aren't completely absorbed has measure zero.

Proof for $N=2$: As f is monotonic, $\exists$ inverse $g(x)$ that gives time. Let Φ be the phase of oscillator B right after A fires. If $\Phi \geq 1$, done, so let $\Phi < 1$. Thus B fires @ $1-\Phi \Rightarrow A$ @ $x_A = f(1-\Phi)$ is kicked by ε. If ≥ 1, done, so let $f(1-\Phi)+\varepsilon < 1$. So phase of A is $h(\Phi) := g(f(1-\Phi)+\varepsilon)$

As oscillators are identical, then $\Phi \rightarrow h^2(\Phi) := R(\Phi)$. Let δ be $\ni f(1-\delta)+\varepsilon = 1$. Then h is defined for $\Phi \in (\delta, 1)$. f & g are monotonically $\uparrow$, so $h'<0$ & $h(\Phi) < \delta \Rightarrow \Phi < h^{-1}(\delta) \Rightarrow R$'s domain is $(\delta, h^{-1}(\delta))$. Note that $h(\delta) = 1 > \delta$ $\Rightarrow$ W/ $\Phi > h^{-1}(\delta)$, $R(\Phi) = 1$. This is "absorption" as phase space is $\bigcirc$, so $0 \equiv 1$; $\Phi < \delta$, $R(\Phi) = 0$

Interlude Ex: Say $f(t) = t(2-t)$. Then $g(x) = 1-\sqrt{1-x}$. Let $\varepsilon = 0.02$. Then graphically:



Method of Cowebs show $\exists$ fixed pt. @ $\Phi = \frac{1}{2}$ that's unstable & will be absorbed to 0 or 1. Thus, WTS $\exists!$ fixed point of $R(\Phi)$ & that it's unstable.

f & g are inverses $\Rightarrow f'(y) = \dfrac{1}{g'(f(y))} \Rightarrow h'(\Phi) = -g'(f(1-\Phi)+\varepsilon) \cdot f'(1-\Phi) = -\dfrac{g'(f(1-\Phi)+\varepsilon)}{g'(f(1-\Phi))}$. Let $u = f(1-\Phi) \Rightarrow h' = -\dfrac{g'(u+\varepsilon)}{g'(u)}$. $f''<0 \Rightarrow g''>0 \Rightarrow g'$ is monotonically $\uparrow \Rightarrow \underbrace{}_{<-1} \Rightarrow$ any fixed pts. is unstable. Let $F(\Phi) = \Phi - h(\Phi)$. As $F(\delta) = \delta - h(\delta) = \delta - 1 < 0$ & $F(h^{-1}(\delta)) = h^{-1}(\delta) - \delta > 0$, $\exists$ root by IVT. As $F'(\Phi) = 1 - h'(\Phi) > 2$, unique! $\square$

- Experiment: Ermentrout & Kopell induced $\gamma = 40$Hz on slices w/ both hippocampal halves
  - → Results: ⊕ cells ⇒ local ⊖ cells fired doublets iff slice had halves synchronized
    - ⊖ cell doesn't fire if ⊕ doesn't fire & feedback ⊖ slows ⊕ cell to 40Hz

- Say $x' = F(x)$ has an asymptotically stable limit cycle. Consider 2 identical coupled oscillators:

$$\frac{dx_j}{dt} = F(x_j(t)) + \varepsilon G_j(x_j(t), x_k(t)),\ j = 1,2 \ \& \ k = 3-j$$

W/ $\theta_j = \Theta(x_j)$, then $\dot{\theta}_j = \nabla_x \Theta(x_j) \cdot \dot{x}_j = 1 + \varepsilon \nabla_x \Theta(x_j) \cdot G_j(x_j, x_k)$

$$\Rightarrow \begin{aligned} \dot{\theta}_1 &= 1 + \varepsilon P_2(\theta_2) d_1(\theta_1) \\ \dot{\theta}_2 &= 1 + \varepsilon P_1(\theta_1) d_2(\theta_2) \end{aligned}$$; $d_j(\theta)$ is PRC for oscillator & $P_j(\theta)$ synaptic input of presynaptic oscillator

$$\Rightarrow \dot{\phi}_j = \varepsilon \nabla\Theta[U(t+\phi_j)] \cdot G_j[U(t+\phi_j), U(t+\phi_k)]$$

$$\Rightarrow y' = \varepsilon M(y,t) \xrightarrow{\text{method of averaging}} \bar{y}' = \frac{\varepsilon}{T}\int_0^T M(\bar{y},t)\,dt$$

Recall that $\nabla_x \Theta(x)$ is soln to adjoint $Z(t)$. Then: $\begin{aligned} \dot{\phi}_1 &= \varepsilon H_1(\phi_2 - \phi_1) \\ \dot{\phi}_2 &= \varepsilon H_2(\phi_1 - \phi_2) \end{aligned}$

w/ $H_j(\phi) = \frac{1}{T}\int_0^T Z(t) \cdot G_j[U(t), U(t+\phi)]\,dt$. (avg interaction w/ the PRC)

Let $\psi = \phi_2 - \phi_1$. Then $\dot{\psi} = \varepsilon[H_2(-\psi) - H_1(\psi)]$

Such generalizes to N coupled oscillators: (*) $\dot{\phi}_j = \varepsilon H_j(\phi_1 - \phi_j, \ldots, \phi_N - \phi_j),\ j = 1,\ldots,N$

→ Phase-Locked State: $\phi_j(t) = \Omega t + \xi_j$; for Ensemble Frequency Ω, $\xi_1 = 0$, & ξ_j are Relative Phases, ∴ solving for N unknowns $\Omega, \xi_2, \ldots, \xi_N$

- Theorem: Let $S = \{\Omega, 0, \xi_2, \ldots, \xi_N\}$ be a phase-locked soln to (*). Let a_{jk} denote the partial of $H_j(\eta_1, \ldots, \eta_N)$ w.r.t. η_k evaluated @ S. Say $a_{jk} \geq 0$ & that $A = (a_{jk})$ is irreducible. Then S is asymptotically stable.

Let $G_j(X,Y) = \overset{\text{intrinsic differences}}{B_j(X)} + \overset{\text{coupling strength}}{g}\,\overset{\text{synaptic current}}{C_j(X,Y)}$ w/ ensemble $\Omega = \varepsilon\left(\omega_j + \sum_k H_{jk}(\xi_k - \xi_j)\right), \xi_1 = 0$

⟶ Chemical synapses : $C_{syn}(X,Y) = -s_Y(V_X - V_{syn})e_V$; e_V := vector of 0's except 1 in voltage component

Electrical synapses : $C_{gap}(X,Y) = (V_Y - V_X)e_V$

$s_Y(t)$:= synaptic response of presynapse

- Let $V(t)$ be the potential & $V^*(t)$ voltage part of adjoint solution
- Then $H_j(\phi) = \omega_j + h(\phi)$, where:

$$h(\phi) = h_{syn}(\phi) = \frac{1}{T}\int_0^T V^*(t)\, s(t+\phi)\left[V_{syn} - V(t)\right]dt$$

$$h_{gap}(\phi) = \frac{1}{T}\int_0^T V^*(t)\left[V(t+\phi) - V(t)\right]dt$$

Experiment: Kelso et al found ↑ speed of alternating finger tapping ⇒ often switch to synchronous tapping, modeled-ish by Wang-Buszaki Model

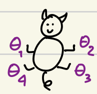
$\boxed{\text{Quadriped Gaits}}$

	L. rear	R. rear	L. front	R. front
PRONK	$x_1(t)$	$x_1(t)$	$x_1(t)$	$x_1(t)$
PACE	"	$x_1(t+\frac12)$	$x_1(t)$	$x_1(t+\frac12)$
BOUND	"	$x_1(t)$	$x_1(t+\frac12)$	$x_1(t+\frac12)$
TROT	"	$x_1(t+\frac12)$	$x_1(t+\frac12)$	$x_1(t)$
JUMP	"	$x_1(t)$	$x_1(t\pm\frac14)$	$x_1(t\pm\frac14)$
WALK	"	$x_1(t\pm\frac12)$	$x_1(t\pm\frac14)$	$x_1(t\pm\frac34)$

Then:

$$\begin{aligned}
\theta_1' &= \overset{\text{L-R coupling}}{H_a}(\theta_2-\theta_1) + \overset{\text{diagonal coupling}}{H_b}(\theta_3-\theta_1) + \overset{\text{same-side coupling}}{H_c}(\theta_4-\theta_1) \\
\theta_2' &= H_a(\theta_1-\theta_2) + H_b(\theta_4-\theta_2) + H_c(\theta_3-\theta_2) \\
\theta_3' &= H_a(\theta_4-\theta_3) + H_b(\theta_1-\theta_3) + H_c(\theta_2-\theta_3) \\
\theta_4' &= H_a(\theta_3-\theta_4) + H_b(\theta_2-\theta_4) + H_c(\theta_1-\theta_4)
\end{aligned}$$

θ_1 : L. front
θ_2 : R. front
θ_3 : R. rear
θ_4 : L. rear



⟶ Phase-locked solutions have form $\theta_j = \omega t + \phi_j$ w/ $\phi_1 = 0$ & ϕ_2, ϕ_3, ϕ_4 the phases of other limbs relative to θ_1

- So walk $W = (0, \pi, \frac{3\pi}{2}, \frac{\pi}{2})$, pace $P = (0,\pi,\pi,0)$, bound $B = (0,0,\pi,\pi)$
- Lemma: $\exists W \Leftrightarrow H_b = H_c$

Say $x=0$ is the soma. Then synaptic current @ soma is $I_{soma}(t) = \int_0^\infty \overset{\text{Green's f(x) for dendrites}}{G(x;s)}\, I(t-s)\,ds$, & the interaction function for synapse @ distance x from soma on a passive dendrite is $H(x;\phi) = \int_0^\infty G(x;s)H(\phi - s)ds$

Elements of Differentiable Dynamics: Ruelle

CHAPTER 1: Differentiable Dynamical Systems

PRELUDE

→ In complete metric space, set of countable intersection of dense open subsets is the **Residual**; a residual set is dense by **Baire's Theorem**

→ **Contraction Mapping Theorem**: If E is complete, then **Contraction** $f: E \to E$, where $\exists \alpha < 1 \ni d(f(x), f(y)) \leq \alpha\, d(x,y)$, has a unique fixed pt. a & $f^i(x) \to a$

→ If E is compact metrizable, then E is a **Cantor Space** if (I) E is **Completely Disconnected**: $\forall x, y \in E, \exists$ disjoint open $X \ni x$, $Y \ni y \ni X \cup Y = E$, & (II) E lacks **Isolated Points**: $x \in E \Rightarrow E \setminus \{x\}$ can't be closed

- **Theorems**: Every 2 Cantor spaces are homeomorphic.
  The topological dimension of a Cantor Space is 0.

→ Say $\mathcal{U}$ open covers $\mathcal{X}$ as **Order at most n** if any intersection of form $\mathcal{U}_{\alpha(0)} \cap \dots \cap \mathcal{U}_{\alpha(n)} = \emptyset$, w/ $\mathcal{X}$ having **Lebesgue Covering / Topological Dimension** $\leq n$ if every open covering $\mathcal{U}$ of $\mathcal{X}$ has a refinement $\mathcal{V}$ of order $\leq n$. If $\dim \mathcal{X} \leq n-1$ is false, then $\dim \mathcal{X} = 1$

→ **Diameter**: $d(X) = \sup\{d(x,y) \mid \forall x, y \in \mathcal{X}\}$

→ W/ compact metric $E \neq 0$ & $r > 0$, say $\exists$ countable covering $\sigma = (\sigma_k)$ w/ diameters $d_k = \operatorname{diam} \sigma_k \leq r$. Then $\forall \alpha \geq 0$, say $m_r^\alpha(E) = \inf_\sigma \sum_k (d_k)^\alpha$.
Then $\lim_{r \to 0} \Rightarrow m_r^\alpha(E) \uparrow \to m^\alpha(E)$, the **Hausdorff Measure** of E in dim α

- Say $\dim_H E = \sup\{\alpha \mid m^\alpha(E) > 0\}$, the **Hausdorff Dimension** of E

→ Norm is $\|\cdot\|: E \to \mathbb{R} \ni \|0\| = 0$, $\|x\| > 0$ if $x \neq 0$, $\|\alpha x\| = |\alpha|\|x\|$, $\forall \alpha \in \mathbb{R}$ or $\mathbb{C}$, & $\|x+y\| \leq \|x\| + \|y\|$

- If E is complete w/ metric $d(x,y) = \|y-x\|$, E is a **Banach Space**
- Real E can be extend to $E_{\mathbb{C}} = \{x + iy \mid x, y \in E\}$, the **Complexification** of E, a $\mathbb{C}$ Banach Space, via $\|x+iy\| = \sqrt{\|x\|^2 + \|y\|^2}$

→ Real Banach Space E is a **Hilbert Space** if $\exists$ bilinear map $x, y \to (x,y)$ $E \times E \to \mathbb{R}$ (scalar product) $\ni (x,y) = (y,x)$ & $(x,x) = \|x\|^2$

- **Theorem**: (Banach $\Rightarrow$ Hilbert) $\Leftrightarrow \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

PRELUDE

- If E, F (e.g. $\mathbb{R}^n$) are real Banach Spaces, the **Derivative** of $f: E \to F$ is $f': E \to \mathcal{L}(E,F) \ni \lim_{\|\varepsilon\| \to 0} \frac{\|f(x+\varepsilon) - f(x) - f'(x)\varepsilon\|}{\|\varepsilon\|} = 0$; $f'(x) = D_x f$
  - Define $D_x^0 f = f(x)$; $f^{(k)}(x) = D_x^k f = D_x(D_x^{k-1} f) \in \mathcal{L}(E^k, F)$
  - $g: \mathcal{U} \to F$ is **Hölder Continuous** of exponent $\alpha \in (0,1]$ if $\forall x \in \mathcal{U}$ has neighborhood V

    $$\ni |g|_{V,\alpha} = \sup\left\{ \frac{\|g(y) - g(z)\|}{\|y - z\|^{\alpha}} \,\middle|\, y \neq z \in V \right\} < \infty$$

    ↳ If $\alpha = 1$, then g is **Lipschitz** & $|g|_{V,1}$ is the **Lipschitz Constant** of g in V

    ↳ If $f^{(r)}: \mathcal{U} \to \mathcal{L}^r(E,F)$ is Hölder Cont. of exponent α, say $f \in C^{(r,\alpha)}$

$f \in C^{\infty}(\mathcal{U})$ if $\forall x \in \mathcal{U}$, $\exists$ neighborhoods V where all derivatives exists & is bounded. → Additionally, if $\forall a \in \mathcal{U}, \exists A, R > 0 \ni \|f^{(k)}(a)\| \leq \frac{k!A}{R^k}$, then $f \in C^{\omega}$, a.k.a. is **Real Analytic**, e.g. there's a convergent Taylor Expansion $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D_a^k f(x-a, \dots, x-a)$

  - $\tilde{f}: \tilde{\mathcal{U}} \subset \tilde{E} \to \tilde{F}$ (open; $\mathbb{C}$-Banach Spaces) is **Holomorphic** if it's locally given by convergent Taylor Expansion
  - Continuous $f: \mathcal{U} \to F$ is **Compact** if for bounded $B \subset \mathcal{U}$, $f(B)$ is compact in F

    ↳ **Schauder Fixed-Point Theorem**: If B is closed, convex, bounded, f compact, & $f(B) \subset B$, then f has a fixed point in B

- Let $\mathcal{U}$ be a bounded open subset of E. Say $f: \bar{\mathcal{U}} \to E$ differs from identity by a compact map & $a \in E \setminus f(\partial \mathcal{U})$. The **Leray-Schauder Degree** $\deg(f, a, \mathcal{U}) \in \mathbb{Z}$: (I) depends continuously on f. And (II) if $f(x) = a$ has a finite # of solns $x_i \in \mathcal{U}$, & if f is differentiable @ x_i w/ $D_{x_i} f$ invertible & $D_{x_i} f = \mathbb{1} +$ compact operator, then $\deg(f, a, \mathcal{U}) = \sum_i \deg(D_{x_i} f)$

  ↳ If E is finite-dimensional, then $\deg(f, a, \mathcal{U})$ is the **Brower Degree**

- **Inverse Function Theorem (InFT)**: Let E, F be Banach Spaces, $\mathcal{U} \subset E$ open, & $a \in \mathcal{U}$. Let $f \in C^r(\mathcal{U}, F)$, $r \in \mathbb{N}$, & $\exists (D_a f)^{-1} \in \mathcal{L}(F, E)$. Then $\exists$ open $a \in \tilde{\mathcal{U}} \subset \mathcal{U}$ & $f|_{\tilde{\mathcal{U}}}$ has unique inverse $g: f(\tilde{\mathcal{U}}) \to \tilde{\mathcal{U}} \ni g \in C^r$ & $D_{f(y)} g = (D_y f)^{-1}$; $y \in \tilde{\mathcal{U}}$

- **Implicit Function Theorem (ImFT)**: Let E, F be Banach Spaces, $\mathcal{U} \subset E$, $V \subset F$ open, $a \in \mathcal{U}$, $b \in V$. Let $f \in C^r(\mathcal{U} \times V, E)$, & $D^1_{(a,b)} f$ w.r.t. 1st arg invertible in $\mathcal{L}(E)$. Then $\exists$ open $a \in \tilde{\mathcal{U}} \subset \mathcal{U}$, $b \in \tilde{V} \subset V$ & $\exists! g: \tilde{V} \to \tilde{\mathcal{U}} \ni f(g(y), y) = f(a,b)$ for $y \in \tilde{V} \ni g \in C^r(\tilde{V})$ &

  $$D_y g = -\left(D^1_{(g(y),y)} f\right)^{-1} D^2_{(g(y),y)} f$$

  derivate w.r.t. 2nd arg

- Let $\mathcal{U} \subset E$, $V \subset F$ (open; Banach Spaces). If $f \in C^r(\mathcal{U}, F)$ has $f: \mathcal{U} \to V$ w/ smooth $f^{-1}: V \to \mathcal{U}$, f is a **C^r Diffeomorphism**

PRELUDE

(See next pgs.)

- Let $\underset{\text{open}}{\mathcal{U}} \subset \underset{\text{Banach space}}{E}$. $\mathcal{X} \subset \mathcal{U}$ is a C^r **Submanifold** if $\forall x \in \mathcal{X}$, $\exists$ open $x \in \tilde{\mathcal{U}} \subset \mathcal{U}$ & C^r diffeomorphism $f: \tilde{\mathcal{U}} \to \underset{\text{open}}{V} \times \underset{\text{open}}{W} \subset \underset{\text{Banach Spaces}}{\mathcal{F} \times G} \ni f(\mathcal{X} \cap \tilde{\mathcal{U}}) = V \times \{0\}$
  - → $\operatorname{codim}_x \mathcal{X} := \dim G$; if independent of x, it's the **Codimension** of $\mathcal{X}$
  - → **Tangent Space** $T_x X$ to $\mathcal{X}$ @ x is $(D_x f)^{-1}(V)$
- Let $a \in \mathcal{U} \subset E$, $b \in \underset{\text{open}}{V} \subset \mathcal{F}$ (Banach spaces) & $f(a) = b$ where $f \in C^r(\mathcal{U}, \mathcal{F})$. If $D_a f$ is injective & if $(D_a f)E$ is a closed subspace of $\mathcal{F}$ which has a closed complement, then f is an **Immersion** @ a
  - → **Lemma**: $\exists$ open $a \in \tilde{\mathcal{U}} \ni f(\tilde{\mathcal{U}})$ is a C^r submanifold of V tangent to $(D_a f)E$ @ b
  - → Let $b \in Y$ be a submanifold of V. If $(D_a f)E + T_b Y = \mathcal{F}$ & $(D_a f)^{-1} T_b Y$ has a closed complement in E, then f is **Tranversal** to Y @ a
    - **Lemma**: $\exists$ open $a \in \tilde{\mathcal{U}} \ni \tilde{\mathcal{U}} \cap f^{-1}(Y)$ is a C^r submanifold of $\mathcal{U}$ tangent to $(D_a f)^{-1} T_b Y$ @ a
  - → If $D_a f$ is surjective & if $(D_a f)^{-1}\{0\}$ has a closed complement in E, f is **Submersion** @ a
    - **Lemma**: $\exists \begin{matrix} a \in \tilde{U} \\ f(\tilde{U}) \subset \underset{\text{open}}{\tilde{V}} \end{matrix}$ & diffeomorphisms $\begin{matrix} \varphi: \tilde{U} \to U_1 \times U_2 \subset E_1 \times E_2 \\ \psi: \tilde{V} \to V_1 \subset E_1 \end{matrix} \ni \psi \circ f \circ \psi^{-1}(x_1, x_2) = x_1$ @ a
- Let M be Hausdorff. A **Chart** of M is $(\mathcal{U}, \varphi, E)$ ($\mathcal{U}$: open subset of M; φ: homeomorphism of $\mathcal{U} \to \underset{\text{open}}{V} \subset E$; E: Banach space). Let $r \in [0, 1]$, then a **C^r Atlas** is set of charts $(\mathcal{U}_\alpha, \varphi_\alpha, E_\alpha)$ ($\mathcal{U}_\alpha$ open covers M; $\varphi_\beta \circ \varphi_\alpha^{-1} \in C^r$); 2 atlases are **$C^r$ Equivalent** if union is C^r Atlas
  - → Set M w/ C^r equivalence classes of atlases is a **C^r Banach Manifold**
  - → If $x \in \mathcal{U}$, it's a **Chart** @ x
  - → If $r \geq 1$, M is a **Differentiable / Smooth Manifold**
  - → If C^r manifold M has atlas $\{(\mathcal{U}_\alpha, \varphi_\alpha, E_\alpha)\}$ w/ all E_α are equal to a Banach Space E, then M is **Type E** & $\dim M = \dim E$
    - A manifold of type $\mathbb{R}^m$ is **Finite Dimensional**
  - → Open $\mathcal{U} \subset E$ (Banach space) is a **C^∞** manifold w/ atlas of the only chart $(\mathcal{U}, \varphi, E)$, where $\varphi: \mathcal{U} \to E$ is the canonical map
- Open Cover $\{\mathcal{U}_\alpha\}$ of $\mathcal{X}$ is **Locally Finite** if $\forall x \in \mathcal{X}$, $\exists$ neighborhood $x \in \mathcal{U}_x \ni \mathcal{U}_x$ intersects only finitely many elements of the open cover
  - → $\{V_\beta\}$ is a **Refinement** of $\{\mathcal{U}_\alpha\}$ if such is an open cover & $\forall j, \exists i \ni V_j \subset \mathcal{U}_i$
  - → $(\mathcal{X}, \tau)$ is **Paracompact** if all open covers have a locally finite refinement
- **Theorem**: For manifold topological space (M, τ), then: seperable $\Leftrightarrow$ paracompact $\Leftrightarrow$ homeomorphic $\cong$ complete metrizable space $\Leftrightarrow$ metrizable

PRELUDE

- For M, N be C^r manifolds w/ $s \leq r$. Say $f: M \to N$ is of **Class C^s** if $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is of Class C^s for charts $(U_\alpha, \varphi_\alpha, E_\alpha)$ & $(V_\beta, \psi_\beta, F_\beta)$ of C^r atlases of M, N, denoted **$C^s(M,N)$**
  - $\longrightarrow$ If $s \geq 1$, f is **Differentiable / Smooth**, & if f has a smooth inverse, then f is a **C^s Diffeomorphism**
  - $\longrightarrow$ **Proposition**: If M is compact, all diffeomorphisms form an open subset in $C^s(M,N)$

- Banach Space E has the **C^r Extension Property**, $r \leq \infty$, if $\exists\, \varphi \in C^r(E, \mathbb{R})$ vanishing outside unit ball $E_0(1)$ & $=1$ in $E_0(\delta)$, $\delta > 0$
- **Proposition**: Let M be seperable C^r manifold of Type E, where E has the C^r extension property. $\forall$ locally finite open covering $\{X_\alpha\}$ of M, $\exists$ family $\{\psi_\alpha\}$ of C^r functions $M \to \mathbb{R} \ni \psi_\alpha(x) \geq 0$, $\psi_\alpha(x) = 0$ for $x \notin X_\alpha$ & $\sum_\alpha \psi_\alpha = 1$ on M. The family is a **C^r Partition of Unity**
- If M, N are manifolds w/ C^r atlases $\{(U_\alpha, \varphi_\alpha, E_\alpha)\}, \{(V_\beta, \psi_\beta, F_\beta)\}$, the **Product** $M \times N$ w/ atlas $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta, E_\alpha \times F_\beta)\}$ is a C^r manifold
  - $\longrightarrow$ $X \subset M$ is a **C^r Submanifold** if $\forall$ charts (U, φ, E) of a C^r atlas of M, $\varphi(X \cap U)$ is a C^r submanifold of $\varphi(U)$
  - $\longrightarrow$ If codimension of $\varphi(X \cap U)$ is constant when $X \cap U \neq \emptyset$, that's the **Codimension** of $X \neq \emptyset$
  - $\longrightarrow$ Say $r \geq 1$. $f \in C^r(M,N)$ is an **Immersion** if it's an immersion $\forall x \in M$
    Transversal to a C^r submanifold Y of N if it's tranversal $\forall x \in f^{-1}(Y)$
    - 2 submanifolds X, Y of M are **Transversal** @ $x \in M \cap N$ if inclusion $X \to M$ is transversal to Y @ x
- **Proposition**: If $r \geq 1$ & $C^r(M,N)$ is an immersion & $f: M \to f(M)$ is a homeomorphism, then $C^r(M)$ is a C^r submanifold of N & f is a C^r diffeomorphism
- **Proposition**: If $r \geq 1$ & $f \in C^r(M,N)$ is transversal to C^r submanifold Y of N, then $f^{-1}(Y)$ is a C^r submanifold of M
- **Whitney's Theorem**: If M is an m-dimensional C^r manifold, $r \geq 1$, $\exists\, C^r$ embedding $f: M \to \mathbb{R}^{2m+1}$ $\ni f(M)$ is a closed C^ω submanifold of $\mathbb{R}^{2m+1}$
- **Thom's Transversality Lemma**: Let M, N, Y be C^r manifolds, where M is compact & Y a closed submanifold of N. Then the set of maps $M \to N$ that're transversal to Y is open & dense in $C^r(M,N)$ when $r \geq 1$

START

- m-Sphere is a submanifold of $\mathbb{R}^{m+1}$: $S^m = \{x \in \mathbb{R}^{m+1} \mid \sum x_i^2 = 1\}$
- m-Torus is $T^m = \bigtimes_{i=1}^{m} S^1$
- Semigroup: Set S has associative b.o.
- W/ $\dot{x} = \mathcal{X}(x)$, say soln is $(f^t)_{t \geq 0}$, a family s.t. $f^t(x)$ maps I.C. x after time t, e.g. $f^0(x) = x$
  - → Then (f^t) forms a semigroup & forms a Semiflow (associative + identity) on differentiable manifold M & a Continuous Time Dynamical System (CTDS), else Discrete for $t \in \mathbb{Z}$ (which $\exists -t \Rightarrow$ forms a group!) (DTDS)
  - → If $f^t \in C^r(M)$, then (f^t) is a Differentiable Dynamical System (DDS)

- Dynamical System
  - Discrete — Defined for only
    - $\oplus$ times — Determined by smooth $f = f^1$
    - $\oplus$ & $\ominus$ times — " " diffeomorphism
      - ↳ If $f \in C^r \Rightarrow f^{-1} \in C^r$ by InFT, so ∴ f is a C^r diffeomorphism
  - Continuous
    - $\oplus$ times — semiflow
    - $\oplus$ & $\ominus$ times — Flow: Family of $(f^t)_{t \in \mathbb{R}}$ diffeomorphisms + is a group, $f^{-t} = (f^t)^{-1}$

- Group of Transformations G of M is a Group of Symmetries for (f^t) if $f^t \circ g = g \circ f^t$ $\forall g \in G, t$, a.k.a. (f^t) is G-Equivariant

- Hamilton's Equations: $\dot{q}_i = \mathcal{H}_{p_i}$, $\dot{p}_i = -\mathcal{H}_{q_i}$, $i = 1, \dots, n$, in $2n$ dim. space (derivative $/dt$; partial w.r.t. p_i)
  - → Hamilton Function $\mathcal{H}$ depends smoothly on $2n$ coordinates but not on time
  - → n is the Degrees of Freedom of system
  - → $\mathcal{H}$ is constant of motion/energy, invariant under time, so phase space decomposed into family of subsets where $\mathcal{H} = c \in \mathbb{R}$ termed Energy Surfaces
    - Volume element $dq_1 \dots dq_n dp_1 \dots dp_n$ is also preserved across time by Liouville's Theorem, & is ∴ a Conservative, not Dissipative, System

- Let D be the space of differentiable dynamical systems on manifold M
  - → Say M is obtained by gluing pieces $\mathcal{U}_i$ or $\mathcal{U}_i'$ (open) of Banach Spaces which $f_0 : \mathcal{U}_i \to f_0(\mathcal{U}_i)$, $f_0(\mathcal{U}_i) \subset \mathcal{U}_i'$. Then $f_0 : M \to M$ maps are replaced by a collection of Banach space maps $f_0|_{\mathcal{U}_i}$
    - If N_i are C^r neighborhoods of " , define $N = \{f \in D \mid f(\mathcal{U}_i) \subset \mathcal{U}_i' \ \& \ f|_{\mathcal{U}_i} \in N_i, \forall i\}$
    - So neighborhoods of f_0 are all subsets of D that contain N as above when the family $\{\mathcal{U}_i\}$ vary over a class of open covers of M

→ If $\mathcal{M}$ is compact, all finite open cover $\{\mathcal{U}_i\}$ of $\mathcal{M}$ yields the same topology termed the C^r Topology : uniform convergence of maps $M \to M$ & their derivatives up to order $r \in \mathbb{Z}$, the space termed $C^r(M, M)$

- $\mathrm{Diff}^r(\mathcal{M})$: Space of C^r diffeomorphisms of compact manifold M w/ the C^r topology; it's an open subset of $C^r(M, M)$

→ For CTDS, consider $\mathcal{M} \times [0,1] \to \mathcal{M}$ (semiflows)
or $\mathcal{M} \times [-1,1] \to \mathcal{M}$ (flows)

- If $\mathcal{M}$ is compact, define $\mathcal{F}^r(\mathcal{M})$: Topology of uniform convergence on $\mathcal{M} \times [-1,1]$, or equivalently compacts of $M \times \mathbb{R} : (x,t) \longrightarrow f^t x$, & its derivatives up to order r

- Diffusion Equation: $x_t = \chi \Delta x$, where $\xi \to x(\xi)$ is a real function as subset on $\mathbb{R}^3$ & $\chi > 0$ is a Diffusion Coefficient

- Theorem: Let $\dot{x} = \mathcal{X}(x)$ be defined on open $\mathcal{U} \subset E$ (Banach Space) & where $\mathcal{X} : \mathcal{U} \to E$ has $\mathcal{X} \in C^s(\mathcal{U})$, $s \geq 1$. Then $\exists$ local existence & uniqueness of solns, e.g. for $a \in \mathcal{U}$, $\exists \varepsilon, T > 0 \ni$ has a unique soln $t \to f^t x$ for $t \in [-T, T]$ & I.C. $x \in B_\varepsilon(a)$; $f^t(x) \in C^s(\mathcal{U})$

- Let m-dimensional manifold $\mathcal{M}$ be viewed as a submanifold of $\mathbb{R}^n$ (take $n = 2m+1$ by Whitney's Theorem). Then $\forall x \in \mathcal{M}$. Then $\exists$ Tangent Space $T_x M$, an m-dimensional linear space

as $\boxed{T_x M \cong \mathbb{R}^m}$ for any m-dim manifold M, for chart $\varphi : x \in U \subset M \to \mathbb{R}^m$; define basis of $T_x M$
$$\frac{\partial}{\partial x^i} := \frac{d}{dt}\left(\varphi^{-1}(\varphi(x) + t e_i)\right)\Big|_{t=0}$$
from pullback of standard basis

$\mathbb{R}$ define as $\underset{\text{iso-morphic}}{\cong} E$ banach space

→ Vector Field $\mathcal{X}$ on $\mathcal{M}$ is a function $x \in \mathcal{M}$ a Tangent Vector $\mathcal{X}(x) \in T_x(\mathcal{M}), \forall x$

→ $\exists$ open $\mathcal{U}_\alpha \subset M$, $V_\alpha \subset \mathbb{R}^m$ & maps $\psi_\alpha : V_\alpha \to \mathbb{R}^n \ni \psi_\alpha(\mathcal{U}_\alpha) = \mathcal{V}_\alpha$ (by def. of M being a manifold)

→ So for vector field $\mathcal{X}$ on $\mathcal{M}$, $\exists\, Y_\alpha : V_\alpha \to \mathbb{R}^m \ni \mathcal{X}(\psi_\alpha(y)) = (D_y \psi_\alpha)\, Y_\alpha(y)$
$$\Rightarrow Y_\beta(\psi_\beta^{-1} \circ \psi_\alpha(y)) = (D_y(\psi_\beta^{-1} \circ \psi_\alpha))\, Y_\alpha(y) \text{ for } \psi_\alpha y \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$$

→ Similar to coordinate Δ $y \to \psi_\beta^{-1} \circ \psi_\alpha(y)$; $\mathcal{X}$ is of class C^s if Y_α are C^s & $\mathcal{M}$ is at least of class C^{s+1}

→ Tangent Bundle $TM = \bigcup_{x \in M} T_x M$, a 2m-dimensional manifold of class C^{r-1} if M is C^r

- If $\mathcal{X}$ is a smooth vector field on $\mathcal{M}$, a soln of $\dot{x} = \mathcal{X}(x)$ on $\mathcal{M}$ is a func. defined on $(T^-, T^+) \subset \mathbb{R}$, where on subintervals it's the form $t \to \psi_\alpha(y_\alpha(t)) \ni \dot{y}_\alpha = Y_\alpha(y_\alpha)$ is satisfied in V_α; the soln is an Integral Curve of $\mathcal{X}$

- For manifold $\mathcal{M}$ w/ open $\mathcal{U} \subset E$ (Banach Space), take $T_x\mathcal{M}$ as copy of E, $\forall x \in \mathcal{U}$. Then TM is identified w/ $\mathcal{U} \times E$ & a C^r function $\mathcal{U} \to E$ corresponding to a C^r vector field

→ Integral Curve of $\mathcal{X}$ is everywhere tangent to $\mathcal{X}$ & $\mathcal{X}(f^t x)$ is the velocity vector of $t \to f^t(x)$. By local existence & uniqueness, each I.C. $x \in \mathcal{M}$ determines a soln $t \to f^t(x)$ on a maximal $(T^-(x), T^+(x)) \ni 0$ w/ soln unique

↳ $\Gamma = \{(x,t) \in M \times \mathbb{R} \mid t \in (T^-(x), T^+(x))\}$ is open in $M \times \mathbb{R}$ & that $(x,t) \to f^t(x)$ is C^s on Γ if X is C^s. (f^t) is thus termed a Local Flow

↳ Proposition: If M is a compact manifold, the vector field X is bounded & $f^t(x) \not\to \infty$, $\therefore \exists f^t(x), \forall t \in \mathbb{R}$, thus $\Gamma = M \times \mathbb{R}$ & (f^t) is a C^s flow

Say we now have $\dot{x} = X(x,t,\lambda)$ for $t \in J$ & $\lambda \in \Lambda$ manifold. Then can reduce to $\dot{\tilde{x}} = \tilde{X}(\tilde{x})$ on $M \times J \times \Lambda$ w/ $\tilde{X}(x,t,\lambda) = (X(x,t,\lambda), 1, 0)$

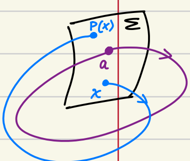
Let (f^t) be a dynamical system on manifold w/ $(x,t) \to f^t(x)$ continuous. The Orbit of $x \in M$ is $\{f^t(x)\}$ for t in some set e.g. $\mathbb{R}$ or $\mathbb{Z}$, etc. The Forward Orbit has $t \geq 0$, Backward Orbit " $t \leq 0$

↳ a is a Fixed Point if $orb(a) = \{a\}$ or $f(a) = a$ |||

↳ a is a Critical Point if $f(a) = 0$

↳ a is a Periodic Point if $\exists t > 0 \ni f^t(a) = a$; inf t is the Period of a denoted $T(a) \geq 0$. If a is also fixed, then in cont.: $T(a)=0$, discrete: $T(a)=1$, & $\{f^t(a) \mid t \in [0, T(a)]\}$ is the Periodic/Closed Orbit

↳ Let Σ be a smooth codimension-1 submanifold w/ a that's transversal to the orbit of a, a.k.a. is a Local Cross-Section of the flow. If x is close to a, choose t near $T(a) \ni f^t(x) \in \Sigma$. Define $P: x \to f^t(x)$ as the Poincaré/Return Map from a neighborhood of a in $\Sigma \to \Sigma$, where P is of class C^r w/ a fixed



Let (U, φ, E) & (U', φ', E') be C^r charts of manifold M around x. Then these Δ of coordinates hold: $(T_x\varphi')(T_x\varphi)^{-1} = (\varphi'\varphi^{-1}, D_{\varphi x}(\varphi'\varphi^{-1}))$. Now let $f: M \to N$ be a C^r map to C^r manifold N. Let (V, ψ, F) be a C^r chart of N around $f(x)$. Then the Tangent Map to f @ x is the linear map $T_x f: T_x M \to T_{f(x)} N$,

$$(T_{f(x)}\psi)(T_x f)(T_x\varphi)^{-1} = (\psi f \varphi^{-1}, D_{\varphi x}(\psi f \varphi^{-1}))$$

↳ So if $g: N \to O$ is another C^r map, Chain Rule: $T_x(g \circ f) = (T_{f(x)} g)(T_x f)$

⊛ Confused? A C^r submanifold of M is a subset $\Sigma \ni \forall x \in \Sigma$, $\exists C^r$ chart $(U, \varphi, F \times G)$ (Banach spaces) of M around $x \ni \varphi(x) = (0,0)$ & $\varphi(\Sigma \cap U) = (F \times \{0\}) \cap \varphi(U)$ (a.k.a. locally looks like $F \cap \varphi(U)$ as subspace of $F \times G$)

↳ Tangent space to Σ @ x is $T_x\Sigma = (T_x\varphi)^{-1}(F \times \{0\})$, a subspace of $T_x M$

↳ Codimension of Σ = dim G

- W/ $(x,t)\to f^t(x)$ smooth $\Rightarrow \exists\ \dot{f}^t(a)$ @ $t=T(a)$, defining vector $\mathcal{X}$. If a isn't fixed, then $\mathcal{X}\neq 0 \Rightarrow$ by Thm 7T, piece of orbit $\mathcal{O}=\{f^t(a)\,\big|\,|t-T(a)|<\varepsilon\}$ for some $\varepsilon>0$ is a C^r submanifold of $\mathcal{M}$ of dim = 1, tangent to a @ $\dot{f}^t(a)\big|_{t=T(a)}=\mathcal{X}\neq 0$
  - ↳ Recall submanifolds Σ & $\mathcal{O}$ of $\mathcal{M}$ are Transversal @ $a\in\Sigma\cap\mathcal{O}$ if $T_a\Sigma + T_a\mathcal{O} = T_a\mathcal{M}$
  - ↳ As codim $\Sigma = 1$, dim $\mathcal{O}=1$, & span $\mathcal{X} = T_x\mathcal{O}$ in our case, transversality means $\mathcal{X}\notin T_a\Sigma$. More math gets....

- **Poincaré Return Proposition**: Let (f^t) be a semiflow on C^r manifold $\mathcal{M}$ & a periodic pt. of period $T(a)>0$. Say $(x,t)\to f^t(x)$ is continuous & that in neighborhood of $(a,T(a))$, that is C^r, $r\geq 1$. Let Σ be a C^r submanifold of codim $\Sigma=1$ transversal to a to orb(a). For $x\in\mathcal{M}$ near a, $\exists!\,\tau(x)\in\mathbb{R}$ near $T(a)\ni f^{\tau(x)}(x)\in\Sigma$. τ is C^r & $P: x\to f^{\tau(x)}(x)$ is C^r from a neighborhood of a in $\Sigma\to\Sigma$

- Let $\tau: N\to\mathbb{R}$ be a C^r function that's $\oplus$, bounded, & bounded away from 0. Consider $\{(y,l)\in N\times\mathbb{R}\mid 0\leq l\leq\tau(x)\}$ & glue $(y,\tau(y))$ to $(g(y),0)$ to get C^r manifold $\mathcal{M}$
  - ↳ Define flow on $\mathcal{M}$ by $((y,l),t)\to f^t(y,l)=(y,l+t)$, $0\leq l+t\leq\tau(y)$, which the " is C^r & corresponds to vector field $\mathcal{X}=(0,1)$
  - ↳ $\mathcal{M}$ is a Suspension of N by g, (f^t) is the Special Flow over g w/ Roof Func. τ

- Say a fixed point a of smooth $f:\mathcal{M}\to\mathcal{M}$ is Hyperbolic if $T_af: T_a\mathcal{M}\to T_a\mathcal{M}$ is too, e.g. spectrum of T_af is disjoint from unit circle
  - ↳ a is Attracting if spectrum of T_af lies in $\{z\mid |z|<1\}$, else Repelling
  - ↳ To study a fixed point is having $(\mathcal{U},\varphi,E)$ be a C^r chart of $\mathcal{M}$ around a w/ $r\geq 1$. Say $\varphi(a)$ is the origin 0_E. So $\tilde{f}=\varphi^{-1}\circ f\circ\varphi$ is defined in a neighborhood of 0_E
  - ↳ Spectrum of bounded linear operator T on Banach Space $\mathcal{X}$ over scalar field $\mathbb{K}$ is all $\lambda\in\mathbb{K}\ni\lambda I-T$ lacks an inverse that's also a bounded linear operator
  - ↳ Hyperbolicity of linear operator $A=D_0\tilde{f}$ decomposes into spectrum that's $\frac{\text{in}}{\text{out}}$ unit O, e.g. $\exists$ closed subspaces E^-, E^+ of $E\ni$ (I) $(E^-)^c=E^+$, (II) $AE^-\subset E^-$ & the spectrum of restriction A^- of A to E^- lies in , & (III) $AE^+\subset E^+$ & the " " " A^+ of A to E^+ " " unit circles

- Operators A_- & $(A_+)^{-1}$ have spectral radii <1. Thus, $\exists \rho < 1 \ni \lim_{n\to\infty} \|(A_-)^n\|^{\frac{1}{n}} < \rho$ & $\lim_{n\to\infty} \|(A_+)^{-n}\|^{\frac{1}{n}} < \rho$
  - $\hookrightarrow$ $\exists$ norms $\|\cdot\|_\pm$ on $E^\pm \ni A_-$ & $(A_+)^{-1}$ are contractions
  - $\hookrightarrow$ If $\|X\|_- = \sum_{n=0}^{\infty} \rho^{-n} \|(A_-)^n x\|$, then $\|A_-\|_- < \rho$
    $\|Y\|_+ = \sum_{n=0}^{\infty} \rho^{-n} \|(A_+)^{-n} Y\|$ $\quad \|(A_+)^{-1}\|_+ < \rho$
  - $\hookrightarrow$ Define $\|\cdot\|$ on E by $\|x+Y\| = \max\{\|X\|_-, \|Y\|_+\}$, $X \in E^-$, $Y \in E^+$
  - $\hookrightarrow$ Decompose $A = D_0\tilde{f}$ into linear contraction A_- on E^- & invertible operator A_+ on E^+ w/ a contracting inverse, where $E^- = \{X \mid A^n x \to 0\}$, the Contracting & Expanding $E^+ = \{Y_0 \mid \exists (Y_n)_{n\geq 0}, AY_{n+1} = Y_n$ & $Y_n \to 0\}$ subspaces of E
  - $\hookrightarrow$ Also say $V_a^- = (T\varphi^{-1})E^-$ are Contracting subspaces of T_aM for f @ a, & if f is a
    $V_a^+ = (T\varphi^{-1})E^+$ " Expanding $\quad$ diffeomorphism, f^{-1} inverts contract $\updownarrow$ expand
  - $\hookrightarrow$ As f is smooth, $x \to D_x\tilde{f}$ is continuous & say $\|D_x\tilde{f}\| < \beta < 1$ for $\|x\| < \varepsilon$ w/ $\varepsilon > 0$
  - $\hookrightarrow$ If $x \in B_\varepsilon(0)$, then $\|\tilde{f}(x)\| \leq \beta\|x\|$, so $\tilde{f}^n(x) \to 0$ as $n \to \infty$, $\therefore$ $f^n(x) \to a$ for x in neighborhood $\varphi^{-1}(B_\varepsilon(0))$ of a; attraction is Exponentially Fast
    - If a repels, then $E^- = \{0\}$ & f is locally invertible near a by the InFT, a.k.a. f is a Local Diffeomorphism, & a is attracting for f^{-1}

- **Grobman-Hartman Theorem**: For Banach Space E, neighborhood $\tilde{\mathcal{U}}$ of 0 in E, & C^1 map $\tilde{f}: \tilde{\mathcal{U}} \to E$, if 0 is a hyperbolic fixed point & if $D_0\tilde{f}$ is invertible, then $\tilde{f}$ is topologically conjugate to $D_0\tilde{f}$ near 0.

  a.k.a. $\exists$ open $0 \in V$ & homeomorphism h of V
  $\longrightarrow$ neighborhood of 0 in $\tilde{U} \ni h(0) = 0$ & $h(D_0\tilde{f}) = \tilde{f}(h)$

Horse REU: Linear Regression

1.4 Pseudo-Inverse, Least Squares, & Regression

- For constraint A & vector b known & x unknown, then $Ax=b$
  - ⟶ If A is square invertible $\Rightarrow \exists! x$ per b; A square or rectangle $\Rightarrow$ there's 0, 1, or ∞ solns x
  - ⟶ Say $A \in \mathbb{C}^{n \times m}$, $n \ll m$ (short-fat matrix), then $\mathrm{col}(A)$ likely to span $\mathbb{R}^n$, so likely ∞ solns, Under-Determined
  - ⟶ " " , $n \gg m$ (tall skinny) " " can't span $\mathbb{R}^n$, so only $\exists$ soln if $b \in \mathrm{col}(A)$, Over-Determined
  - ⟶ Solution space of $Ax=b$ determined by $A = \tilde{U}\tilde{\Sigma}\tilde{V}^*$

$\mathrm{col}(A) \oplus \mathrm{Ker}(A^*) = \mathbb{R}^n$ ||| $\mathrm{col}(A) = \mathrm{span}(\text{colums of } A) = \mathrm{col}(\tilde{U})$ ||| $\mathrm{row}(A) = \mathrm{span}(\text{rows of } A) = \mathrm{col}(A^*)$

$\mathrm{col}(A^*) \oplus \mathrm{Ker}(A) = \mathbb{R}^n$ ||| $[\mathrm{col}(A)]^{\perp} = \mathrm{Ker}(A^*) = \mathrm{col}(\hat{U}^{\perp})$ ||| $\mathrm{Ker}(A) = [\mathrm{row}(A)]^{\perp} = \mathrm{col}(\hat{V}^{\perp})$

- **Proposition**: $b \in \mathrm{col}(A)$, $\dim(\mathrm{Ker}(A)) \neq 0 \Rightarrow \exists \infty$ solns x

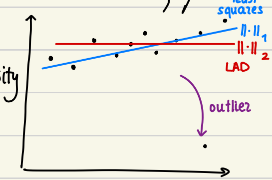
- In over-determined case w/o solns, still want soln x that minimizes sum-squared error $\|Ax-b\|_2^2$, $x :=$ Least-Squares Solution; SVD $A = \tilde{U}\tilde{\Sigma}\tilde{V}^*$ has each right matrix invertible
  - ⟶ Moore-Penrose Left Pseudo-Inverse $A^\dagger$ of A: $A^\dagger \triangleq \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^* \Rightarrow A^\dagger A = \tilde{V}\tilde{V}^*$
    - $A^\dagger A\tilde{x} = A^\dagger b \Rightarrow \tilde{x} = \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^* b \Rightarrow A\tilde{x} = \tilde{U}\tilde{U}^* b$; $\tilde{x}$ only exact soln if $b \in \mathrm{col}(\tilde{U})$

- Condition Number of matrix A: How sensitive matrix & $^{-1}$ are to input errors, $\uparrow \propto$ worse performance, $\kappa(A) = \frac{\sigma_{max}(A)}{\sigma_{min}(A)}$
  - ⟶ If x has error $A(x+\epsilon_x) = b + \epsilon_b \underset{\substack{\text{w/ singular vector} \\ \text{corresponding to } \sigma_{max}}}{\overset{\text{if } \epsilon_x \text{ aligned}}{=}} \underbrace{\sigma_{min} x}_{b} + \underbrace{\sigma_{max}\epsilon_x}_{\epsilon_b}$, where $\underset{\text{signal-to-noise}}{\text{output = input}}$ $\frac{\|b\|}{\|\epsilon_b\|} = \frac{\|x\|}{\|\epsilon_x\|} \cdot \kappa(A)$
  - ⟶ $x + \epsilon_x \approx A^\dagger(b+\epsilon_b) = \frac{b}{\sigma_{max}} + \frac{\epsilon_b}{\sigma_{min}}$ when ϵ_b aligned w/ singular vector corresponding to σ_{min}
    b " " " " " σ_{max}

- $\begin{bmatrix} | \\ b \\ | \end{bmatrix} = \begin{bmatrix} | \\ a \\ | \end{bmatrix} x = \tilde{U}\tilde{\Sigma}\tilde{V}^* x \Rightarrow x = \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^* b$, where $\tilde{\Sigma} = \|a\|_2$, $\tilde{V} = 1$, $\tilde{U} = \frac{a}{\|a\|_2} \Rightarrow x = \frac{a^* b}{\|a\|_2^2}$ for linear regression

3.5 Sparse Regression

- l_1-norm used to regularize regression to (1) penalize outliers & (2) promote parsimonious models, w/ procedure termed Least Absolute Deviation (LAD)
  - ⟶ Least Absolute Shrinkage & Selection Operator (LASSO): l_1 used, sparsity
  - ⟶ Parsimony / Occam's Razor: Simplest correct model is likely the true one



- W/ $Ax=b$; least-squares regression usually has x all non-zero coefficients, suggesting *all* of A is used to predict b
  - ⟶ LASSO adds l_1 penalty to regularize, e.g. stop overfitting, $x = \underset{x'}{\mathrm{argmin}} \|Ax' - b\|_2 + \lambda\|x\|_1$
  - ⟶ Usually λ varied through values & fit is validated w/ holdout data (often 80% training vs. 20% testing)
  - ⟶ Kitchen-Sink Approach: All predictive info thrown in $\xrightarrow{\text{sifted, sieved}}$ truly relevant predictors

4.3 Regression & $Ax=b$

Overdetermined Too many data for linear soln, so want to minimize least squares l_2 error E_2 by $\hat{x} = \underset{x}{\text{argmin}} \|Ax-b\|_2$

$\longrightarrow \hat{x} = \underset{x}{\text{argmin}} \|Ax-b\|_2 + \lambda_1 \|x\|_1 + \lambda_2 \|x\|_2$

regularization for ℓ_1 & the ℓ_2 norms $\Longrightarrow$ constrain soln, often prefer l_1 for sparsity

Underdetermined Too little data where it's $\min \|x\|_p$ subject to $Ax=b$, can also be $\min(\lambda_1\|x\|_1 + \lambda_2\|x\|_2)$ for some sparsity

7.2 DMD

- DMD finds spatiotemporal patterns from ↑dim. data, where each DMD mode $\lambda = \underbrace{a}_{\text{growth/decay rate}} + \underbrace{b}_{\text{oscillation}} i$ (eigenvalue)

$\longrightarrow$ Snapshots $\{(x(t_k), x(t'_k))\}_{k=1}^{m}$ where $t'_k = t_k + \Delta t = t_{k+1}$

$$X = \begin{bmatrix} | & | & & | \\ x(t_1) & x(t_2) & \dots & x(t_m) \\ | & | & & | \end{bmatrix} ; \quad X' = \begin{bmatrix} | & | & & | \\ x(t'_1) & x(t'_2) & \dots & x(t'_m) \\ | & | & & | \end{bmatrix}$$

  - Uniform sampling has $x_k = x(k\Delta t)$; algorithm seeks leading spectral decomposition (eigen value / vector) of best-fit linear operator $\mathcal{A} \Rightarrow x_{k+1} \approx Ax_k$; $A = \underset{A}{\text{argmin}} \|X' - AX\|_F$ $= X'X^{\dagger}$ (pseudo-inverse); for high-dim $x \in \mathbb{R}^n$, then $A \in \mathbb{R}^{n^2}$ & ... words

    Frobenius Norm: $\|A\|_F = \sqrt{\sum_{i,j} (A_{ij})^2}$

$\longrightarrow$ Step #1: SVD of $X \approx \tilde{U}\tilde{\Sigma}\tilde{V}^*$ ($\mathbb{C}^{n\times r}$, $\mathbb{C}^{n\times r}$, $\mathbb{C}^{m\times r}$) where r is chosen dim to reduce to; columns of $\tilde{U}$ are Proper Orthogonal Decomposition (POD) Modes w/ $\tilde{U}^*\tilde{U} = I$

$\longrightarrow$ Step #2: $A = X'\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^* \Rightarrow \tilde{A} = \tilde{U}^*A\tilde{U} = \tilde{U}^*X'\tilde{V}\tilde{\Sigma}^{-1}$ (reduced matrix w/ same λ as A) w/ $\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k$

$\longrightarrow$ Step #3: Spectrally decompose $\tilde{A}$ as $\tilde{A}W = W\Lambda$ (columns of W = eigenvectors of A; Λ diagonal has DMD λ's = λ's of A)

$\longrightarrow$ Step #4: High-dim DMD nodes reconstructed as $\Phi = X'\tilde{V}\tilde{\Sigma}^{-1}W$. Then $A\Phi = \Phi\Lambda$

- DMD is very sensitive to noise, a soln is to compute forward/backwards avg; $X' \approx A_1X$ & $X \approx A_2X'$ where $\underbrace{A_2^{-1}}_{\text{backwards}} \approx \underbrace{A_1}_{\text{forwards}}$ (time), then $A = \frac{A_1 + A_2^{-1}}{2} = \underset{A}{\text{argmin}} \frac{1}{2}(\|X' - AX\|_F + \|X - A^{-1}X'\|_F)$ &

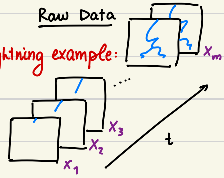
  estimate modal amplitudes as $b = \underset{b}{\text{argmin}} (\|X - \Phi \operatorname{diag}(b) T(\omega)\|_F + \gamma\|b\|_1)$

- DMD works ✓: linear, periodic, quasi- " ✗: many fixed pts., unstable periodic, chaos (nonlinear)

- What? **Dynamic Mode Decomposition (DMD)**:

Raw Data

Lightning example:



$\Rightarrow$ $X = \begin{bmatrix} | & & | \\ x_1 & \dots & x_{m-1} \\ | & & | \end{bmatrix}$ staggered $+\Delta t$ $X' = \begin{bmatrix} | & & | \\ x_2 & \dots & x_m \\ | & & | \end{bmatrix}$

Finds best linear operator $A \ni X' \approx AX$; b/c insane # of data bloats A's size too much to compute, find leading eigen values/vectors (reduce high-dim) (find spatio-temporal patterns)

$\longrightarrow$ Essentially PCA + Fourier Transform

$\boxed{7.3}$ SINDy

- W/ $\dot{x} = f(x)$ & want to approximate $f(x) \approx \sum_{k=1}^{p} \theta_k(x)\xi_k = \Theta(x)\xi$

- **Sparse Identification of Nonlinear Dynamics (SINDy)**: W/ $X = \overbrace{\begin{bmatrix} x(t_1) \\ \vdots \\ x(t_m) \end{bmatrix}}^{\text{time-series data}}$, $\dot{X} = \overbrace{\begin{bmatrix} \dot{x}(t_1) \\ \vdots \\ \dot{x}(t_m) \end{bmatrix}}^{\text{derivatives @ time}}$

  candidate dictionary $\Theta(X) = \overbrace{\begin{bmatrix} \varphi_1(X) & \varphi_2(X) & \cdots \end{bmatrix}}^{\text{as much as we want!}}$, & want $\dot{X} = \Theta(X)\Xi$, where each column ξ_k in Ξ is vector of coefficients in k^{th} row has $\xi_k = \underset{\xi_k'}{\operatorname{argmin}} \|\dot{X}_k - \Theta(X)\xi_k'\|_2 + \underbrace{\lambda\|\xi_k'\|_1}_{\text{for sparsity/parsimonious models}}$

  $\dot{x}_k = \Theta(X)\xi_k$

- Extension of SINDy to PDEs is **PDE Functional Identification of Nonlinear Dynamics (PDE-FIND)**

  spatial time-series data $\mathcal{U} \in \mathbb{C}^{mn \times 1}$ (time pts. m, spatial locals n) + additional inputs of knowns $Q \in \mathbb{C}^{mn\times 1}$ $\longrightarrow$ dictionary of candidate terms of soln $\Theta(\mathcal{U}, Q) \in \mathbb{C}^{mn \times D}$ (D: # of candidates)

  combine as $\mathcal{U}_t = \Theta(\mathcal{U}, Q)\underbrace{\xi}_{\substack{\text{coefficient for}\\ \text{term in }\Theta}}$, sparse via SINDy w/ L_1 norm, via $\hat{\xi} = \underset{\xi}{\operatorname{argmin}} \|\Theta(\mathcal{U}, Q)\xi - \mathcal{U}_t\|_2^2 + \epsilon\underbrace{\kappa(\Theta(\mathcal{U}, Q))}_{\text{condition number}}\|\xi\|_0$

  $\|x\|_p = \sqrt[p]{\sum_i x_i^p} \Rightarrow \|x\|_0 = \{\#\text{ of nonzero terms}\}$

  $0^0 := 0$, $\sqrt[0]{0} = 0$, $a \neq 0$ has $\sqrt[0]{a} = a^0 = 1$

  $\therefore$ discourages overfitting

  $\longrightarrow$ Needs lots of data e.g. $u_t = cu_x$ (linear advection), $u_t = 6uu_x + u_{xxx}$ (Korteweg-de Vries)

- What if system has rational nonlinearities? $\dot{x}_k = \dfrac{f_N(x)}{f_D(x)} \Rightarrow f_N(x) - f_D(x)\dot{x}_k = 0$

  $\longrightarrow$ Try implicit $\Theta(X, \dot{x}_k(t)) = \begin{bmatrix} \underbrace{\Theta_N(X)}_{\text{dictionary for numerator}} & \underbrace{\operatorname{diag}(\dot{x}_k(t))\Theta_D(X)}_{= [\dot{x}_k(t) \;\; (\dot{x}_k x_k)(t) \;\; (\dot{x}_k x_k^2)(t) \;\; \ldots]} \end{bmatrix}$ (tradition for $\Theta_N(X) = \Theta_D(X)$)

  - Then $\Theta(X, \dot{x}_k(t))\xi_k = 0$ — sparse SINDy produces $\xi_k = 0$, so instead get sparsest vector in null(Θ)

Applications of Lie Groups to Differential Equations by Olver

Chapter 1: Introduction to Lie Groups

1.1 Manifolds

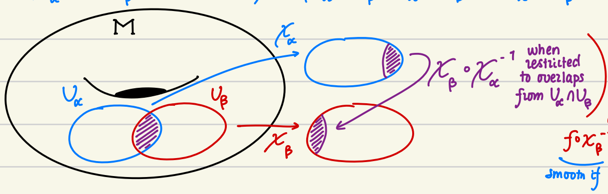
- We want a coordinate-free definition, so an **m-Dimensional Manifold** M is a countable collection $\{U_\alpha\} \subset M$, termed **Coordinate Charts**, & 1-to-1 $\chi_\alpha : U_\alpha \to V_\alpha$ (connected open $\subset \mathbb{R}^m$), termed **Local Coordinate Maps** $\ni$
  - open cover: (a) $\bigcup_\alpha U_\alpha = M$, (b) $\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \to \chi_\beta(U_\alpha \cap U_\beta)$ is smooth (local diffeomorphism)
  - Hausdorff: (c) If $x \in U_\alpha$ & $\tilde{x} \in U_\beta$ are distinct, $\exists$ open W of $\chi_\alpha(x)$ in V_α $\ni \chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\tilde{W}) = \Phi$, $\tilde{W}$ of $\chi_\beta(\tilde{x})$ in V_β
  - $\longrightarrow$ M is endowed w/ a topology $\ni \forall$ open $W \subset V_\alpha \subset \mathbb{R}^m$ that $\chi_\alpha^{-1}(W)$ is open in M; such forms a $\boxed{\text{basis}}$ & $\therefore$ $U \subset M$ is open $\Leftrightarrow \forall x \in U$, $\exists$ neighborhood $x \in \chi_\alpha^{-1}(W) \subset U$
  - $\longrightarrow$ As $\chi_\beta \circ \chi_\alpha^{-1}$ is smooth $\Rightarrow$ M is a **Smooth Manifold** of class C^∞, else if analytic or is up to k-differentiable, then M is an **Analytic** / **C^k-Manifold**, respectively

- Ex: Show that $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is a 2D Manifold.

  SOLUTION: Have 2 coordinate charts $U_1 = \{(\theta,\rho) \mid \theta,\rho \in \underbrace{(0,2\pi)}_{=A}\}$, $U_2 = \{(\theta,\rho) \mid \theta,\rho \in \underbrace{(\pi,3\pi)}_{=B}\}$ w/
$$\chi_1 \circ \chi_2^{-1}(\theta,\rho) = \begin{cases} (\theta,\rho) & \theta,\rho \in A \\ (\theta-2\pi,\rho) & \theta \in B, \rho \in A \\ (\theta,\rho-2\pi) & \theta \in A, \rho \in B \\ (\theta-2\pi,\rho-2\pi) & \theta,\rho \in B \end{cases}$$

- **Lemma**: If M, N are smooth manifolds of dimension m & n, so too is $M \times N$ w/ dimension m+n.

  Proof: For coordinate charts χ_α & $\tilde{\chi}_\beta$ on M & N, then $\chi_\alpha \times \tilde{\chi}_\beta : U_\alpha \times \tilde{U}_\beta \to V_\alpha \times \tilde{V}_\beta \subset \mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}$ ∎



a.k.a. smoothness is well-defined if chart independent, a.k.a. transition maps are smooth e.g. $\underbrace{f \circ \chi_\beta^{-1}}_{\text{smooth if}} = \underbrace{(f \circ \chi_\alpha^{-1})}_{\text{smooth \&}} \circ \underbrace{(\chi_\alpha \circ \chi_\beta^{-1})}_{\text{smooth}}$

- $\therefore$ one can adjoin coordinate chart $\chi : U \to V \subset \mathbb{R}^m$ as long as it's **Compatible**, e.g. $\forall \alpha$, $\chi \circ \chi_\alpha^{-1}$ is smooth on $\chi_\alpha(U \cap U_\alpha)$, & if χ is a diffeomorphism, then it's a **Change of Coordinates**
  - $\longrightarrow$ $\therefore$ any "property" of M we want to be "stable" $\Leftrightarrow$ coordinate free/independent
  - $\longrightarrow$ **Atlas**: Maximal collection of charts on M (satisfies manifold def. except being countable) (a.k.a. any extra chart breaks compatibility w/ existing charts)

- "Let $x = (x^1, \dots, x^m)$ be local coordinates on M" $\Leftrightarrow$ $\exists$ local coordinate chart $\chi_\alpha : U_\alpha \subset M \to V_\alpha \subset \mathbb{R}^m$ (open) $\ni \forall p \in U_\alpha$ has local coordinates $x = \chi_\alpha(p)$. As χ_α is injective, each p has a distinct x
  - $\longrightarrow$ $y = (y^1, \dots, y^m)$ are also local coordinates $\Leftrightarrow$ $\exists$ diffeomorphism $y = \Psi(x)$ in $\mathbb{R}^m$

Let M, N be smooth manifolds, then $F: M \to N$ is **smooth** if its local coordinate expression is a smooth map in every (coordinate chart), (e.g. $\forall\, x_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^m$ on M, $\tilde{x}_\beta : \tilde{U}_\beta \to \tilde{V}_\beta \subset \mathbb{R}^n$ on N, then $\tilde{x}_\beta \circ F \circ x_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth on $x_\alpha(U_\alpha \cap F^{-1}(\tilde{U}_\beta))$)

↳ Ex1.1: Let $f: \mathbb{R} \to S^1$, $f(t) = (\cos t, \sin t)$ W/ angular coordinate θ on S^1, f is linear & ∴ smooth

↳ Ex1.2: Let $f: \mathbb{T}^2 \to \mathbb{R}^3$, $f(\theta, \rho) = ((\sqrt{2} + \cos\rho)\cos\theta, (\sqrt{2} + \cos\rho)\sin\theta, \sin\rho)$

F is clearly smooth in θ & ρ & injective. Such embeds as a torus in $\mathbb{R}^3$ via $x^2 + y^2 + z^2 + 1 = 2\sqrt{2}(x^2 + y^2)$

Let $F: M \left(\substack{\text{m-dim} \\ \text{manifold}}\right) \to N \left(\substack{\text{n-dim} \\ \text{manifold}}\right)$ be smooth. The **Rank** of F @ $x \in M$ is the rank (max # of L.I. rows/columns of matrix) of JF_x

↳ F is of **Maximal Rank** on $S \subset M$ is $\max\{\text{rank}(Jf_x) \mid x \in S\} \leq \min\{m, n\}$

↳ Ex1.3: F, irregardless of manifolds $M \subset \mathbb{R}^2$ or $N \subset \mathbb{R}$, w/ $F(x,y) = xy$ has rank 1, $\forall x \in \mathbb{R}^2 \setminus \{(0,0)\}$

Theorem: Let $F: M \to N$ be of maximal rank @ $x_0 \in M$. Then $\exists$ local coordinates $x = (x_1, \ldots, x_m)$ near x_0 & $y = (y_1, \ldots, y_n)$ near $y_0 = F(x_0)$ $\ni$ in these coordinates F has form $y = (x_1, \ldots, x_m, 0, \ldots, 0)$ $n > m$

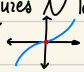
$y = (x_1, \ldots, x_n)$ $n \leq m$

Proof: Obvious corollary of Implicit Function Theorem. □

For smooth manifold M, a **Submanifold** $N \subset M$ has a smooth, bijective ($\Phi(\tilde{N}) = N$) $\Phi: \tilde{N} \to N \subset M$ that satisfies the maximal rank condition everywhere, where **Parameter Space** $\tilde{N}$ is a manifold

↳ Φ is an **Immersion** (∴ "submanifold" in book = immersed submanifold)

↳ Ensures N lacks singularities e.g. $\Phi(t) = (t^2, t^3)$ is smooth w/ $\text{Im}\,\Phi = \{(x,y) \mid y^2 = x^3\}$, but



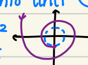
$J\Phi(t) = (2t, 3t^2)$ isn't of maximal rank @ $t = 0$

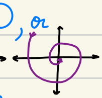
Ex: Let $\tilde{N} = \mathbb{R}$, $\Phi: \mathbb{R} \to M$ parametrizing 1D submanifold $N = \Phi(\mathbb{R})$ of M

↳ Ex1.4: $M = \mathbb{R}^3$, $\Phi(t) = (\cos t, \sin t, t)$

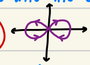
where Φ is injective & $\dot{\Phi} = (-\sin t, \cos t, 1)$ never vanishes, thus maximal rank condition holds & ∴ Φ is an immersion

↳ Ex1.5: $M = \mathbb{R}^2$, $\Phi(t) = \left((1 + \frac{1}{e^t})\cos t, (1 + \frac{1}{e^t})\sin t\right)$ As $t \to \infty$, then N spirals into unit ○, or $\tilde{\Phi}(t) = (e^{-t}\cos t, e^{-t}\sin t)$ spirals into the origin; both submanifolds of $\mathbb{R}^2$.

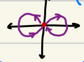




↳ Ex1.6: $M = \mathbb{R}^2$, $\hat{\Phi}(t) = (\sin t, 2\sin(2t))$



where $\hat{\Phi}(k\pi) = (0,0)$, $\forall k \in \mathbb{Z}$. Modify as $\Phi(t) = (\sin(2\arctan t), 2\sin(4\arctan t))$ gets



where it crosses origin once & as $t \to \infty$ / $t \to -\infty$ has $\arctan \to -\pi/2 \cdot 4 = 2\pi$; $\sin \to 0^-$ / $\arctan \to \pi/2 \cdot 2 = \pi$; $\sin \to 0^+$, ∴ Φ is injective. As max rank condition holds, Φ is an immersion

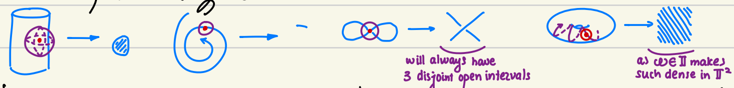
↳ Ex1.7: $M = \mathbb{T}^2$ w/ angular coordinates (θ, ρ). Let $\Phi: \mathbb{R} \to \mathbb{T}^2$, $\Phi(t) = (t \bmod 2\pi, \omega t \bmod 2\pi)$ for $\omega \in \mathbb{R}$

W/ $\dot{\Phi} = (1, \omega)$, max rank condition is satisfied. But if $\omega \in \mathbb{Q}$, then Φ isn't injective as $\Phi(t + 2\pi q) = \Phi(t)$, $\omega = p/q$, so $\text{Im}\,\Phi$ is just a (closed curve) on $\mathbb{T}^2$. Or has 1D manifold $\tilde{N} = S^1$ via $\tilde{\Phi}(\theta) = (q\theta, \omega q\theta)$; $\theta \in S^1$ & p, q coprime

⊖ if $\omega \in \mathbb{I}$, then $\Phi(\mathbb{R})$ is a dense submanifold of $\mathbb{T}^2$

Regular Submanifold $\mathcal{N}$ of $\mathcal{M}$ is parameterized by $\phi: \tilde{\mathcal{N}} \to \mathcal{M} \ni \forall x \in \mathcal{N}$, $\exists$ open neighborhood $x \in \mathcal{U} \subset \mathcal{M} \ni \phi^{-1}(\mathcal{U} \cap \mathcal{N})$ is a connected open subset of $\tilde{\mathcal{N}}$

⟶ Ex1.8:



✓ regular ✗ regular

Theorem: n-dimensional manifold $\mathcal{N} \subset \mathcal{M}$ is regular $\Leftrightarrow \forall x_0 \in \mathcal{N}$, $\exists$ local coordinates $x = (x_1, \ldots, x_m)$ on neighborhood $x_0 \in \mathcal{U} \ni \mathcal{N} \cap \mathcal{U} = \{x \mid x^{n+1} = \ldots = x^m = 0\}$, a **Flat Coordinate Chart** on $\mathcal{M}$

⟶ a.k.a. for regular submanifolds, we can toss out $\tilde{\mathcal{N}}$ & have $x = (x_1, \ldots, x_m)$ on $\mathcal{U} \subset \mathcal{M}$ induce local coordinates $\tilde{x} = (x_1, \ldots, x_n)$ on $\mathcal{U} \cap \mathcal{N}$, ∴ the parameterization is replaced by $\mathcal{N} \subset \mathcal{M}$

Instead of defining a surface S in $\mathbb{R}^3$ parametrically, try implicitly via $S = \{(x,y,z) \mid F(x,y,z) = 0\}$ for if ∇F never vanishes on S, then by the ImFT, $\forall (x_0, y_0, z_0) \in S$, 1 of the variables can be expressed by the other 2

⟶ Thus if $F(x_0, y_0, z_0) \neq 0$, $\exists$ open neighborhood U_α of $(x_0, y_0, z_0) \ni$ in U_α, S is given as graph $z = f(x,y)$ of some smooth $f: \tilde{V}_\alpha \subset \mathbb{R}^2$, ∴ define local coordinate chart on S by projecting along the z-axis

- e.g. set $\tilde{U}_\alpha = S \cap V_\alpha$ w/ $\chi_\alpha: \tilde{U}_\alpha \to \tilde{V}_\alpha$, $\chi_\alpha(x,y,z) = (x,y)$, similarly if F_x or $F_y \neq 0$
- On $\tilde{U}_\alpha \cap \tilde{U}_\beta$, if $\tilde{U}_\alpha$ is given by $z = f(x,y)$, then $\tilde{U}_\alpha$ is given by $\chi_\alpha(x,y,z) = (x,y)$
$\tilde{U}_\beta$ " " $y = h(x,z) \to \chi_\beta(x,y,z) = (x,z)$

$$\chi_\beta \circ \chi_\alpha^{-1}(x,y) = \chi_\beta(x, y, f(x,y)) = (x, f(x,y))$$ (has smooth & $\chi_\alpha \circ \chi_\beta^{-1}$ smooth, ∴ S is a 2D submanifold of $\mathbb{R}^3$)

Theorem: Let $\mathcal{M}$ be a smooth m-dimensional manifold & $F: \mathcal{M} \to \mathbb{R}^n$, $n \leq m$, be smooth. If F is of maximal rank on $\mathcal{N} = \{x \mid F(x) = 0\}$, then $\mathcal{N}$ is a regular $m-n$ dimensional submanifold of $\mathcal{M}$.

Proof: Via ImFT & ⊛, can choose local coordinates $x = (x_1, \ldots, x_m) \in \mathcal{M}$ near each $x_0 \in \mathcal{N} \ni F(x) = (x_1, \ldots, x_n)$. ∴ we have $\mathcal{N} = \{x_1 = \ldots = x_n = 0\}$, e.g. x's give flat local coordinates for $\mathcal{N}$ near x_0. Moreover, last $m-n$ components $(x_{n+1}, \ldots, x_m)$ provide local coordinates on $\mathcal{N}$, ∴ $\mathcal{N}$ is regular. □

⟶ F only needs to be maximal rank on $\mathcal{N}$, but if everywhere, then every level set $\{(x,y,z) \mid F(x,y,z) = c\}$ is a regular $m-n$ dimensional submanifold, $\forall c \in \mathbb{R}$

⟶ Ex1.9: $F(x,y,z) = x^2 + y^2 + z^2 - 2\sqrt{2(x^2+y^2)}$ Maximal rank everywhere except on z-axis & circle $\{x^2 + y^2 = 2, z = 0\}$. If $c \in (-2, \sqrt{2} - 2)$, level sets are tori, if $c \geq \sqrt{2} - 2$, spheres indented @ the z-axis

- **Curve** C on smooth manifold $\mathcal{M}$ is parameterized by smooth $\Phi: \underset{\text{interval}}{\mathcal{I}} \subset \mathbb{R} \to \mathcal{M}$ (all valid curves)
  - $\longrightarrow$ C is defined by m functions $x = \phi(t) = (\Phi^1(t), \dots, \Phi^m(t))$ & is **Degenerate** if $\Phi(t) = x_0 \in \mathcal{M}, \forall t \in I$
  - $\longrightarrow$ C is **Closed** if $\Phi(a) = \Phi(b)$, $I = [a,b]$
  - $\longrightarrow$ $(\mathcal{X}, \tau)$ is **Connected** if $\nexists A, B \in \tau \ni A \sqcup B = \mathcal{X}$ & **Pathwise-Connected** if $\forall x, y \in \mathcal{X}, \exists$ smooth curve $x \to y$

Book Assumption All manifolds (includes Lie Groups) are connected.

- Manifold $\mathcal{M}$ is **Simply Connected** if $\forall$ closed curve $C \subset \mathcal{M}$, C can be **Continuously Deformed** to a point, e.g. $\exists$ continuous $\underset{\text{homotopy}}{\mathcal{H}}: [0,1] \times [0,1] \to \mathcal{M} \ni \mathcal{H}(t, \underset{\text{deformation}}{0}) = x_0$ & $\mathcal{H}(\underset{\text{parameterization}}{t}, 1)$ parameterizes C, $\forall t \in [0,1]$
  - $\longrightarrow$ Ex1.10: $\mathbb{R}^m \setminus \{0\}$, but not $\mathbb{R}^2 \setminus \{0\}$, is simply connected.

- **Theorem**: $\forall$ manifolds $\mathcal{M}$, $\exists$ simply connected **Covering Manifold** $\pi: \tilde{\mathcal{M}} \to \mathcal{M}$, where the **Covering Map** is onto & a local diffeomorphism
  - $\longrightarrow$ Ex1.11: $\Phi: \tilde{\mathcal{M}} \to \mathbb{S}^1$ has $\pi: \mathbb{R} \to \mathbb{S}^1$, $\pi(t) = (\cos t, \sin t)$
    - ✓ surjective
    - ✓ $\mathbb{R}$ is simply connected
    - ✓ local diffeomorphism

1.2 Lie Groups

- **r-Parameter Lie Group** is group G that's an r-dimensional smooth manifold $\ni$ multiplication $m: G \times G \to G, m(g,h) = gh$ & inversion $i: G \to G, i(g) = g^{-1}$ are smooth maps between manifolds
  - $\longrightarrow$ Ex1.12: $\mathbb{R}^r$ has b.o. $(+)$ as "multiplication" & $(-)$ as "inverse", both clearly smooth
  - $\longrightarrow$ Ex1.13: $SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in [0, 2\pi) \right\}$ Observe the group can be identified w/ $\mathbb{S}^1 \subset \mathbb{R}^2$, which defines the manifold structure; group + reflections = **Orthogonal Group** $O(2) = \{\mathcal{X} \in GL(2) \mid \mathcal{X}^T\mathcal{X} = I\}$, which $O(n)$ can be reimagined lying in $\mathbb{R}^{n^2}$ via $\mathcal{X}^T\mathcal{X} - I = 0$
    - $O(n)$ has exactly $\frac{n(n+1)}{2}$ eqns are independent & satisfy maximal rank everywhere on $O(n)$. $\therefore$ by a prior theorem, $O(n)$ is a regular submanifold of dim $\frac{n(n-1)}{2}$ in $GL(n)$
    - $O(n)$ has matrix $\cdot$ & $^{-1}$ still smooth & $\therefore$ is still a Lie Group
  - $\longrightarrow$ Ex1.14: **Upper Triangular Matrices** $T(n)$ w/ all 1's on diagonal has (length n, & each $\triangledown$ is size $\frac{n^2-n}{2} = \frac{n(n-1)}{2}$) $\therefore T(n) \subset \mathbb{R}^{n(n-1)/2}$ but isn't isomorphic e.g. in $T(3)$, $(x,y,z) \cdot (x',y',z') \overset{?}{=} (x+x', y+y', z+z'+xy')$ isn't commutative, even though it's a $\frac{n(n-1)}{2}$-parameter Lie Group

- **Lemma**: If G & $\mathcal{H}$ are r & s-parameter Lie Groups, then $G \times H$ is an $r+s$ parameter Lie Group w/ b.o. $(g,h) \cdot (\tilde{g}, \tilde{h}) = (g \cdot \tilde{g}, h \cdot \tilde{h})$
  - $\longrightarrow$ Ex1.15: $\mathbb{T}^n \cong \overset{n}{\underset{i=1}{\times}} \mathbb{S}^1 \cong \overset{n}{\underset{i=1}{\times}} SO(2)$ is an r-parameter Lie Group w/ b.o. componentwise $(+) \bmod 2\pi$ & is the only connected, compact, Abelian " " up to isomorphism

Ex1.16: $GL(n)$ Has 2 connected components: $GL^+(n) = \{X \mid \det X > 0\}$ ✓ lie group
$GL^-(n) = \{X \mid \det X < 0\}$ ✗

→ **Lemma**: The connected component of the identity of a Lie Group is too a Lie Group.

Lie Subgroup H of Lie Group G is a submanifold given by a lie group homomorphism $\Phi : \underset{\text{lie group}}{\tilde{H}} \to H \subset G \ni \Phi(\tilde{H}) = H$, $\Phi(h \cdot \tilde{h}) = \Phi(h)\cdot\Phi(\tilde{h})$

→ Ex1.17: $\forall \omega \in \mathbb{R}$, $H_\omega = \{(t, \omega t) \bmod 2\pi \mid t \in \mathbb{R}\} \subset \mathcal{T}^2$ is a 1-parameter Lie Group

$\omega \in \mathbb{Q} \Rightarrow H_\omega \cong SO(2)$ & is a closed regular Lie subgroup

$\omega \in \mathbb{I} \Rightarrow H_\omega \cong \mathbb{R}$

Theorem: For Lie Group G, H is a closed (topologically closed) subgroup $\iff$ H is a regular submanifold of G & ∴ a Lie Group.

→ e.g. if $H = \{g \mid F_i(g) = 0, i \in \{1, \dots, n\}\}$ where F_i's are all continuous real-valued funcs., then clearly H as the kernel is a Lie subgroup of G w/o even checking maximal rank!

→ Ex1.18: $O(n)$ As A is given as the kernel of n^2 eqns $A^TA - I = 0$, such is a Lie Group

$SL(n) = \{A \in GL(n) \mid \det A = 1\}$ Is n^2-1 dim Lie Group as it's the kernel of $\det A - 1$

r-Parameter Local Lie Group V is made of connected open subsets $V_0 \subset V \subset \mathbb{R}^r \ni 0 \in V$ (origin = identity), multiplication & inversion smooth, where additionally multiplication has Associativity: if $x, y, z \in V$ & $m(x,y), m(y,z) \in V$, then $m(x, m(y,z)) = m(m(x,y), z)$, Identity: $\forall x \in V$, $m(0,x) = x = m(x,0)$, & Inverses: $\forall x \in V_0$, $m(x, i(x)) = 0 = m(i(x), x)$

→ Q: Why not state it's a group? Local = b.o. doesn't have to be closed e.g. $\boxed{B_1(0) \subset \mathbb{R}^3}$ is a 3-parameter local Lie Group under $(+)$ but clearly $(+)$ isn't closed. Yet all holds, so only where defined e.g. for $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = x = y = z$, sure $x+y, y+z, x, y, z \in B_1(0)$ yet $x+y+z \notin B_1(0)$

Ex1.19: Let $V = \{x \mid |x| < 1\} \subset \mathbb{R}$ w/ $m(x,y) = \frac{2xy - x - y}{xy - 1}$; $x, y \in V$

✓ associativity & identity, though $i(x) = \frac{x}{2x-1}$ is defined only for $x \in V_0 = \{x \mid |x| < \frac{1}{2}\}$, ∴ V is a local 1-parameter Lie Group under m. Note that by def., the domain for $i \subset$ domain for m to work

Theorem: Let $V_0 \subset V \subset \mathbb{R}^r$ be a local Lie Group w/ multiplication m & inversion i. Then $\exists$ global Lie Group G & coordinate chart $\chi : U^* \to V^* \subset V_0 \ni e_G \in U$, $\chi(e_G) = 0$, & $\chi(g \cdot h) = m(\chi(g), \chi(h))$ & $\chi(g^{-1}) = i(\chi(g))$ (e.g. local comes from coordinate chart around identity in global Lie Group)

Moreover, $\exists!$ connected & simply-connected Lie Group G^* w/ the above properties. If G is any other such Lie Group, $\exists$ covering map $\pi : G^* \to G$ that's a group homomorphism w/ G^* & G locally isomorphic as Lie Groups. G^* is then the **Simply-Connected Covering Group of G**

Ex 1.20: The only connected & simply connected 1-parameter Lie Group is $\mathbb{R}$.

Let $\chi: U^{*} \to V^{*} \subset \mathbb{R}$, $\chi(t) = \frac{t}{t-1}$, $t \in U^{*} = \{|t|<1\}$, then:

$$\chi(t+s) = m(\chi(t), \chi(s)) = \frac{2\chi(t)\chi(s) - \chi(t) - \chi(s)}{\chi(t)\chi(s) - 1} \quad \& \quad \chi(-t) = i(\chi(t)) = \frac{\chi(t)}{2\chi(t)-1}$$

$\therefore$ Ex 1.19 is validated via χ & the Theorem.

Proposition: Let G be a connected Lie Group & $U \subset G$ a neighborhood of the identity. Let $U^k := \{g_1 \cdot g_2 \cdot \ldots \cdot g_k \mid g_i \in U\}$. Then $G = \bigcup_{k=1}^{\infty} U^k$ (a.k.a. $\forall g \in G, \exists k \in \mathbb{N}, g_i \in U \ni g = \prod_{i=1}^{k} g_i$)

- Usually Lie Groups arise as a group of transformations on some manifold M e.g. $SO(2)$ rotates $M = \mathbb{R}^2$
- Let M be a smooth manifold. Then a **Local Group of Transformations** acting on M is given by a local Lie Group G, open $U \ni \{e\} \times M \subset U \subset G \times M$ the domain of group action, & smooth $\Psi: U \to M \ni$ (just treat as multiplication b/c notation is too convoluted)
  - (a) If $(h,x), (g, \Psi(h,x)), (g \cdot h, x) \in U$, then $\Psi(g, \Psi(h,x)) = \Psi(g \cdot h, x)$
  - (b) $\forall x \in M, \Psi(e,x) = x$, & (c) $(g,x) \in U \Rightarrow (g^{-1}, \Psi(g,x)) \in U$ & $\Psi(g^{-1}, \Psi(g,x)) = x$
  - ⟶ Then each group transformation is a diffeomorphism
  - ⟶ $\forall x \in M$, then $G_x = \{g \in G \mid (g,x) \in U\}$ is a local Lie Group
  - ⟶ $\forall g \in G$, $\exists$ open submanifold $M_g = \{x \in M \mid (g,x) \in U\} \subset M$
  - ⟶ Such is a **Global Group of Transformations** if $U = G \times M$
  - ⟶ " " **Connected** if (I) G is a connected Lie Group, (II) U is connected, & M " " manifold (III) $\forall x \in M, G_x$ is connected

★ **Book Assumption** All local groups of transformations are connected.

- **Orbit** of a local transformation group is a minimal, nonempty, group-invariant subset of M, a.k.a. (I) $x \in O$, $g \in G$, & $g \cdot x$ is defined $\Rightarrow g \cdot x \in O$ & (II) $\tilde{O} \subset O$ & $\tilde{O}$ satisfies (I), then $\tilde{O} = O$ or $\tilde{O} = \phi$
  - most general definition, same as these definitions
  - ⟶ If global, then **Orbit through x** is $O_x = \{g \cdot x \mid g \in G\}$
  - ⟶ If local, as G may not be closed, have $O_x = \{g_1 \cdot \ldots \cdot g_k \cdot x \mid k \geq 1, g_i \in G, \& \ g_1 \cdot \ldots \cdot g_k \cdot x \text{ is defined}\}$
  - both are submanifolds of M

- Let G be a local group of transformations acting on M. Then G acts...
  - ⟶ **Semi-Regularly** if all orbits O are the same dimension as submanifolds of M
  - ⟶ **Regularly** if action is semi-regular & $\forall x \in M, \exists$ neighborhood U of $x \ni$ each orbit of G intersects U in a pathwise connected subset
  - ⟶ **Proposition**: If G acts regularly on M, then each orbit of G ($\forall x \in M, O_x$) is a regular submanifold of M.
  - ⟶ Group Action is **Transitive** if $\exists!$ orbit, which would be M

• **Ex1.21**: Let $0\neq a\in\mathbb{R}^m$, $G=\mathbb{R}$. Let the **Group of Translations** be $\Psi(\varepsilon,x)=x+\varepsilon a$. O_x are lines $/\!/\ a$

• **Ex1.22**: Let $\alpha_1,\dots,\alpha_m\in\mathbb{R}$ be not all zero & $G=(\mathbb{R}^+,\cdot)$. Then $\mathbb{R}^+$ acts on $\mathbb{R}^m$ as the **Group of Scale Transformations** via $\Psi(\overset{\in\mathbb{R}^+}{\lambda},x)=(\lambda^{\alpha_1}x_1,\dots,\lambda^{\alpha_m}x_m)$. Save for the origin, O_x are all 1D regular submanifolds of $\mathbb{R}^m$ e.g. for $\mathbb{R}^2$, $\Psi(\lambda,(x,y))=(\lambda x,\lambda^2 y)$ has nontrivial orbits to half of axis & $\frac{1}{2}$ of $y=\kappa x^2$ for some $\kappa\in\mathbb{R}^+$

• **Ex1.23**: Let $G=\mathbb{R}$, $M=\mathbb{R}^2$, $\Psi(\varepsilon,(x,y))=\left(\frac{x}{1-\varepsilon x},\frac{y}{1-\varepsilon y}\right)$ on $U=\left\{(\varepsilon,(x,y))\ \middle|\ \begin{matrix}\varepsilon<\frac{1}{x} \text{ for } x>0\\ \varepsilon>\frac{1}{x} \text{ for } x<0\end{matrix}\right\}$

Orbits are rays emanating from the origin & action is regular on $\mathbb{R}^2\setminus\{0\}$

• **Ex1.24**: **Irrational Flow** on $\mathbb{T}^2=M$, $\mathbb{R}=G$, $\omega\in\mathbb{R}$. Define $\Psi(\varepsilon,(\theta,\rho))=(\theta+\varepsilon,\rho+\omega\varepsilon)$ mod 2π. O_x are all 1D submanifolds & ∴ G acts semi-regularly. If $\omega\in\mathbb{Q}$, each curve is closed & action is regular. Yet $\omega\in\mathbb{I}$ has action irregular as  ∄ path $a\to b$ in $O_x\cap U$

1.3 Vector Fields

• Say C is a smooth curve on manifold M via parameterization $\phi: \underset{\text{subinterval of } \mathbb{R}}{I}\to M$. In local coordinates $x=(x_1,\dots,x_m)$, then C is given by m smooth funcs. $\phi(\varepsilon)=(\phi_1(\varepsilon),\dots,\phi_m(\varepsilon))$, $\varepsilon\in\mathbb{R}$. Then each $x=\phi(\varepsilon)$ of C, such has a **Tangent Vector** being $\dot\phi(\varepsilon):=\frac{d\phi}{d\varepsilon}:=(\dot\phi_1(\varepsilon),\dots,\dot\phi_m(\varepsilon))$

⟶ Notation: $v_x:=\dot\phi(\varepsilon):=\dot\phi_1(\varepsilon)\frac{\partial}{\partial x_1}+\dots+\dot\phi_m(\varepsilon)\boxed{\frac{\partial}{\partial x_m}}$ think as unit vector $\hat{x}_m$ but for $T_x(M)$!

⟶ 2 curves $C=\{\phi(\varepsilon)\}$ & $\tilde C=\{\tilde\phi(\varepsilon)\}$ through $x=\phi(\varepsilon^*)=\tilde\phi(\theta^*)$ have the same tangent vectors $\Leftrightarrow \dot\phi(\varepsilon^*)=\dot\phi(\theta^*)$

⟶ This independent of local coordinates near x; if $x=\phi(\varepsilon)$ & $y=\Psi(x)$ is any diffeomorphism, then $y=\Psi(\phi(\varepsilon))$ is the local coordinate formula for C in terms of y

∘ Observe $v_y=\sum_{j=1}^{m}\dot\psi_j(\phi(\varepsilon))\frac{\partial}{\partial y_j}=\sum_{j=1}^{m}\sum_{k=1}^{m}\underbrace{\frac{\partial\psi_j}{\partial x_k}}_{\text{invertible}}(\phi(\varepsilon))\frac{d\phi_k}{d\varepsilon}\frac{\partial}{\partial y_j}$

• **Tangent Space** to M @ x is $T_x(M)=\{\text{all tangent vectors @ } x\}$

⟶ **Proposition**: If M is m-dimensional, then $T_x(M)$ is too w/ basis $\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\right\}$ for a V.S.

⟶ **Tangent Bundle** of M is $T(M)=\bigcup_{x\in M}T_x(M)$; **Proposition**: $T(M)$ is a 2m-D smooth manifold

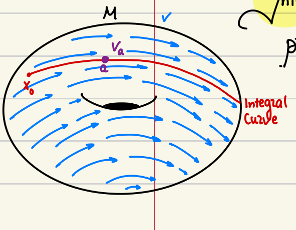
• **Vector Field** v on M assigns $v_x\in T_x(M)$ to $x\in M$ w/ v_x varying smoothly w.r.t. x. Such has the form $v_x=\xi_1\frac{\partial}{\partial x_1}+\dots+\xi_m(x)\frac{\partial}{\partial x_m}$ where each $\xi_i(x)$ is smooth

⟶ **Integral Curve** of v is a smooth parameterized curve $x=\phi(\varepsilon)$ whose tangent vector @ any pt. coincides w/ v, e.g. $\dot\phi(\varepsilon)=v_{\phi(\varepsilon)}$, $\forall\varepsilon\in\mathbb{R}$

∘ Namely $x=\phi(\varepsilon)$ must be a soln to the autonomous ODE $\frac{dx_i}{d\varepsilon}=\xi_i(x)$, $i=1,\dots,m$

∘ W/ I.C. $\phi(0)=x_0$, ∃! soln, ∴ implies ∃! maximal integral curve through $x_0=\phi(0)\in M$

∄ longer integral curve w/ same I.C.



Proposition: If v is a smooth vector field on manifold M & $f(x)$ is any smooth, real-valued function defined for $x \in M$, then $f \cdot v$ is a smooth vector field w/ $(f\cdot v)_x = f(x) v_x$ where $v = \sum \xi_i(x) \frac{\partial}{\partial x_i}$ means $f \cdot v = \sum f(x) \xi_i(x) \frac{\partial}{\partial x_i}$

→ If f near vanishes, integral curve of $f \cdot v$ & v coincide

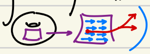
→ e.g. $2v$ traversed $2\times$ speed as v, yet same curve

Let v be a vector field & denote the parameterized maximal integral curve through x in M by $\Psi(\varepsilon, x)$, where Ψ is the **Flow** generated by v, where $\varepsilon \in I_x$ (some interval containing 0)

→ **Properties:** $\Psi(\varepsilon, \Psi(\delta, x)) = \Psi(\delta + \varepsilon, x)$, $\forall x \in M$; $\delta, \varepsilon \in \mathbb{R}$, $\Psi(0,x) = x$, & $\frac{d}{d\varepsilon}\Psi(\varepsilon, x) = v_{\Psi(\varepsilon,x)}$; follows by definition of Integral Curve

- If v is for steady state fluid flow, then integral curves of v are **Streamlines**
- See flow is same as a local group action of Lie Group $\mathbb{R}$ on M & $\therefore$ is a 1-parameter group of transformations (fancy for time ($\mathbb{R}$) "acts" on particle via mvt. along stream)

→ v is the **Infinitesimal Generator** of the action as by Taylor's Theorem: $\Psi(\varepsilon, x) = x + \varepsilon \xi(x) + O(x^2)$, where ξ are coefficients of v

- Then orbits O_x based on group action is the maximal integral curve of v through x
- Conversely for any 1-parameter group of transformations acting on M, then can "rescue vector field" via $v_x = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Psi(\varepsilon, x)$ (tangent vector @ initial condition)
- **Proposition:** There's a bijection between local 1-parameter groups of transformations & their infinitessimal generators up to a relevant domain. (e.g. )

Flow obtained via **Exponentiation** of vector field v, where $e^{\varepsilon v} x := \Psi(\varepsilon, x)$ } soln to ODE $\frac{d}{d\varepsilon}\Psi(\varepsilon,x) = v(\Psi(\varepsilon,x))$, $\Psi(0,x) = 0$

→ Then $e^{(\delta+\varepsilon)v}x = e^{\delta v} e^{\varepsilon v} x$, $e^{0 \cdot v} x = x$, & $\frac{d}{d\varepsilon} e^{\varepsilon v} x = v_{e^{\varepsilon v} x}$

→ **Ex1.25:** Let $M = \mathbb{R}$, $v = \frac{\partial}{\partial x} := \partial_x$. Then $e^{\varepsilon v} x = e^{\varepsilon \partial_x} x = x + \varepsilon$

For vector field $x\partial_x$, then $e^{\varepsilon x \partial_x} x = e^{\varepsilon} x$ ← $= x \sum \frac{\varepsilon^n}{n!} = e^{\varepsilon} x$

$(x\delta_x)[x\delta_x x] = (x\delta_x)(x) = x$, $\therefore e^{\varepsilon x \delta_x} = \sum \frac{\varepsilon^n}{n!}(x\delta_x)^n(x)$

→ **Ex1.26:** Let $M = \mathbb{R}^m$, $v_a = \sum a_i \partial_{x_i}$, $a \in \mathbb{R}^m$. Then $e^{\varepsilon v_a} x = x + \varepsilon a$

For $v_A = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} a_{ij} x_j \right) \partial_{x_i}$, $A = (a_{ij})$ has flow $e^{\varepsilon v_A} = e^{\varepsilon A} x$

where $e^{\varepsilon A} = 1 + \varepsilon A + \frac{\varepsilon^2 A^2}{2} + \dots$

→ **Ex1.27:** Consider group of rotations in plane $\Psi(\varepsilon, (x,y)) = (x \cos\varepsilon - y \sin\varepsilon, x \sin\varepsilon + y\cos\varepsilon)$

"Recover" $v = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ where: $\xi(x,y) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(x\cos\varepsilon - y\sin\varepsilon) = -y$

$= \boxed{-y\partial_x + x\partial_y}$ $\qquad \eta(x,y) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(x\sin\varepsilon + y\cos\varepsilon) = x$

→ such solves $\frac{dx}{d\varepsilon} = -y$, $\frac{dy}{d\varepsilon} = x$

→ **Ex1.28:** Consider local group action $\mathbb{R}^2 \circlearrowleft \mathbb{R}$ via $\Psi(\varepsilon, (x,y)) = \left(\frac{x}{1-\varepsilon x}, \frac{y}{1-\varepsilon x} \right)$

$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(\frac{x}{1-\varepsilon x} \right) = \frac{-x(-x)}{(1-\varepsilon x)^2}\Big|_{\varepsilon=0} = x^2 \Rightarrow \boxed{v = x^2\partial_x + xy\partial_y}$

$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left(\frac{y}{1-\varepsilon x} \right) = \frac{-y(-x)}{(1-\varepsilon x)^2}\Big|_{\varepsilon=0} = xy$

- Effect of Δ of coordinates $y=\Psi(x)$ on vector field v is determined on effect on each v_x, $\forall x\in M$
  → So if $v=\sum \xi_i(x)\partial_{x_i}$, then $v=\sum_{j=1}^{m}\sum_{i=1}^{m}\xi_i(\Psi^{-1}(y))(\partial\Psi_j)_{x_i}(\Psi^{-1}(y))\partial_{y_i}$

- **Proposition**: Say v is a vector field $\ni v_{x_0}\neq 0$. $\exists$ local coordinate chart $y=(y_1,\dots,y_m)$ @ $x_0 \ni v=\partial_{y_1}$
  Proof: Linearly Δ coordinates $\ni x_0=0$ & $v_{x_0}=\partial_{x_1}$. As $\xi_1(x)$ of ∂_{x_1} is continuous & ⊕ in a neighborhood of x_0, the integral curves of v cross the hyperplane $\{(0,x^2,\dots,x^m)\}$ transversally, & ∴ in a neighborhood of $x_0=0$, each $x=(x_1,\dots,x_m)$ is uniquely the flow of some pt. $(0,y^2,\dots,y^m)$ on the hyperplane. ∴ $x=e^{y_1 v}(0,y_2,\dots,y_m)$ for y_1 near 0 gives a diffeomorphism $x\longrightarrow y$.
  ∴ for small ε: $e^{\varepsilon v}(y_1,\dots,y_m)=(y_1+\varepsilon,y_2,\dots,y_m)$, ∴ flow is just translation in direction of y_1 □

- Let v be a vector field on M w/ $f:M\to\mathbb{R}$ smooth. How does f Δ from the flow?
  → If $v=\sum\xi_i(x)\partial_{x_i}$, then $\frac{d}{d\varepsilon}f(e^{\varepsilon v}x)=\sum_{i=1}^{m}\xi_i(e^{\varepsilon v}x)f_{x_i}(e^{\varepsilon v}x)=v(f)[e^{\varepsilon v}x]$
  → Then v acts as a 1st order partial differential operator on real-valued f on M
  - As $f(e^{\varepsilon v}x)=f(x)+\varepsilon v(f)(x)+O(\varepsilon^2)$, then $v(f)$ gives the infinitesimal Δ of f under the flow generated by v; if Taylor Series converge in ε, then the **Lie Series** $f(e^{\varepsilon v}x)=\sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}v^k(f)(x)$
  - So if f is just coordinate funcs. x, then $f\longrightarrow\xi$ e.g. $e^{\varepsilon v}x=x+\varepsilon\xi(x)+\frac{\varepsilon^2}{2}v(\xi)(x)+\dots$

  → Then v_x applied to f gives a real number $v(f)(x)$, such determined by Linearity: $v(f+g)=v(f)+v(g)$, & Leibniz' Rule: $v(f\cdot g)=v(f)\cdot g+f\cdot v(g)$, both eqns evaluated @ x
  - Say v_x @ x defines a **Derivation** on space of smooth, real-valued funcs. f defined near x
  - **Proposition**: Every derivation on the " funcs. @ x is a tangent vector

- Let M,N be smooth manifolds & $F:M\to N$ smooth. Each parameterized curve $C=\{\phi(\varepsilon)\mid\varepsilon\in I\}$ on M is another curve $\tilde{C}=F(C)=\{\tilde\phi(\varepsilon)=F(\phi(\varepsilon))\mid\varepsilon\in I\}$ on N
  → ∴ F also induces tangent vector map $\frac{d\phi}{d\varepsilon}$ @ $x=\phi(\varepsilon)\longrightarrow\frac{d\tilde\phi}{d\varepsilon}$ @ $F(x)=F(\phi(\varepsilon))$, termed the **Differential** of F, $dF:T_x(M)\to T_{F(x)}(M)$, $dF(\phi(\varepsilon)):=\frac{d}{d\varepsilon}(F(\phi(\varepsilon)))$
  - ∴ if $v_x=\sum\xi_i\partial_{x_i}\Rightarrow dF(v_x)=\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\xi_i(\partial F_i)_{x_i}(x)\right)\partial_{y_i}=\sum_{j=1}^{n}v(F_j(x))\partial_{y_j}=JF_x$: jacobian of F @ x lol

  → Ex1.29: Let $M=\mathbb{R}^2$, $N=\mathbb{R}$ w/ coordinates (x,y) & (s), respectively. Let $F:\mathbb{R}^2\to\mathbb{R}$, $\underbrace{F(x,y)=s}_{\text{arbitrary}}$
  W/ $v=a\partial_x+b\partial_y$. Then $dF(v_{(x,y)})=\{aF_x(x,y)+bF_y(x,y)\}\left.\frac{d}{ds}\right|_{F(x,y)}$
  So if $F(x,y)=\alpha x+\beta y$, then " $=(a\alpha+b\beta)\left.\frac{d}{ds}\right|_{s=\alpha x+\beta y}$

- **Lemma**: If $\underbrace{M\overset{\text{smooth maps}}{\xrightarrow{F}N\xrightarrow{H}}P}_{\text{manifolds}}$, then $T_x(M)\xrightarrow{dF}T_{F(x)}(N)\xrightarrow{dH}T_{H\circ F(x)}(P)$

  $d(H\circ F)=dH\circ dF$

Caution! If $f: \mathcal{M} \to \mathcal{N}$ is a smooth map of manifolds & v is a vector field on $\mathcal{M}$, then $dF(v)$ may **not** be well-defined on $\mathcal{N}$ e.g. $v = y\partial_x + \partial_y$, $s = F(x,y) = \alpha x + \beta y$ has $dF(v_{(x,y)})$ " " " on $\mathbb{R}$ as a vector field unless $\alpha = 0$

⟶ **Proposition**: If F is a diffeomorphism, then $dF(v)$ is a vector field on $\mathcal{N}$.

⟶ Vector fields v on $\mathcal{M}$ & w on $\mathcal{N}$ are **F-Related** if $dF(v_x) = w_{F(x)}$, $\forall x \in \mathcal{M} \Rightarrow dF(v) = w$, thus they have the "same" integral curves: $F(e^{\varepsilon v}x) = e^{\varepsilon dF(v)} F(x)$

· If v & w are vector fields on $\mathcal{M}$, then the **Lie Bracket** is the unique vector field $[v,w](f) = v(w(f)) - w(v(f))$, $\forall$ smooth $f: \mathcal{M} \to \mathbb{R}$

Proof: $v = \sum \xi_i(x)\partial_{x_i}$, $w = \sum \eta_i(x)\partial_{x_i}$

$$[v,w] = \sum_{i=1}^{m} \{v(\eta_i) - w(\xi_i)\}\partial_{x_i} = \sum_{i=1}^{m}\sum_{j=1}^{m} \{\xi_j(\partial\eta_i)_{x_j} - \eta_j(\partial\xi_i)_{x_j}\}\partial_{x_i}$$

⟶ Ex 1.30: $v = y\partial_x$, $w = x^2\partial_x + xy\partial_y$ $[v,w] = v(x^2)\partial_x + v(xy)\partial_y - w(y)\partial_x =$

$v(w) \longrightarrow v(x^2) = y\partial_x(x^2) = 2xy \longrightarrow 2xy\partial_x + y^2\partial_y$

$v(xy) = y\partial_x(xy) = y^2$

$w(v) \longrightarrow w(y) = x^2\partial_x(y) + xy\partial_y(y) = xy \to xy\partial_x$

$\longmapsto [v,w] = v(w) - w(v) = \boxed{xy\partial_x + y^2\partial_y}$

· **Properties**: Bilinearity: For constants c & c', then $[cv + c'v', w] = c[v,w] + c'[v',w]$ (includes 2nd component for $[v, cw + c'w']$)

Skew-Symmetry: $[v,w] = -[w,v]$

Jacobi-Identity: $[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$

↪ if reduce to $(uvw) = (123)$, then says $(123) + (312) + (231) = 0$, e.g. $\exists$ cyclic symmetry

⟶ Lemma: If $F: \mathcal{M} \to \mathcal{N}$ is smooth & v, w are vector fields on $\mathcal{M} \ni dF(v), dF(w)$ are F-related to well-defined vector fields on $\mathcal{N}$, then $dF([v,w]) = [dF(v), dF(w)]$ is too " "

Proof: Let $f: \mathcal{N} \to \mathbb{R}$, $y = F(x) \in \mathcal{N}$, then $dF([v,w])f(y) = [v,w]\{f(F(x))\} = v(w\{f(F(x))\}) - w(v\{f(F(x))\}) = v\{dF(w)f(F(x))\} - w\{dF(v)f(F(x))\} = dF(v)dF(w)f(y) - dF(w)dF(v)f(y) = [dF(v), dF(w)]f(y)$ ☐

⟶ $[v,w]$ can be thought as the "infinitesimal commutator" of 2 1-parameter groups $e^{\varepsilon v}$ & $e^{\varepsilon w}$

· **Theorem**: Let v, w be smooth vector fields on manifold $\mathcal{M}$. Then $\forall x \in \mathcal{M}$, the commutator ($[g,h] = g^{-1}h^{-1}gh$) $\Psi(\varepsilon, x) = [e^{\sqrt{\varepsilon}w}, e^{\sqrt{\varepsilon}v}] = e^{-\sqrt{\varepsilon}w}e^{-\sqrt{\varepsilon}v}e^{\sqrt{\varepsilon}w}e^{\sqrt{\varepsilon}v}x$ defines a smooth curve for small $\varepsilon \geq 0$. Then $[v,w]_x$ is the tangent vector to this curve @ $\Psi(0,x) = x$, e.g. $[v,w]_x = \frac{d}{d\varepsilon}\Big|_{\varepsilon = 0^+}\Psi(\varepsilon, x)$

Proof: $v=\sum \xi_i(x)\partial_{x_i}$, $w=\sum \eta_i(x)\partial_{x_i}$. Let $y=e^{\sqrt{\varepsilon}v}x$, $z=e^{\sqrt{\varepsilon}w}y$, $u=e^{-\sqrt{\varepsilon}v}z$. Then $\Psi(\varepsilon,x)=e^{-\sqrt{\varepsilon}w}u$. Then $\Psi(\varepsilon,x)=u-\sqrt{\varepsilon}\,\eta(u)+\frac{\varepsilon}{2}w(\eta)(u)+O(\varepsilon^{3/2})$
$=$ (math I don't care about) $= x+\varepsilon\big(v(\eta)(x)-w(\xi)(x)\big)+O(\varepsilon^{3/2})$. $\therefore \frac{d}{d\varepsilon}\Big|_{\varepsilon=0^+}\Psi(\varepsilon,x)$
$=\big(v(\eta)-w(\xi)\big)(x)$ □

Theorem: Let v,w be vector fields on M. Then $e^{\varepsilon v}e^{\theta w}x=e^{\theta w}e^{\varepsilon v}x$, (flows are commutative)
$\forall \varepsilon,\theta\in\mathbb{R}, x\in M$, where both sides are defined $\Leftrightarrow [v,w]=0$ (vector fields are commutative w/ $v(w)=w(v)$)

Proposition: Let $F:M\to\mathbb{R}^n$, $n\leq m$, be of maximal rank on $N=\operatorname{Ker}F$, thus N is a regular $(m-n)$-dimensional submanifold. Given $y\in N$, $T_y(N)=\operatorname{Ker}(dF_y)$ $=\{v\in T_y(M) \mid dF(v)=0\}$

Proof: If $\phi(\varepsilon)$ parameterizes smooth curve $C\subset N$ through $y=\phi(\varepsilon_0)$,
then $F(\phi(\varepsilon))=0, \forall\varepsilon$. $\therefore 0=\frac{d}{d\varepsilon}F(\phi(\varepsilon))=dF(\dot\phi(\varepsilon))$, $\therefore \dot\phi(\varepsilon)\in\operatorname{Ker}(dF)$
Converse follows (curve lies in N, the kernel of F) as $\operatorname{rank}(dF_y)=n$, $\therefore$ use Rank-Nullity Theorem. □

→ **Ex1.31**: Consider unit sphere $S^2\subset\mathbb{R}^3$ $\forall p\in S^2$, then $T_pS^2=\operatorname{Ker}F$ for $F:\mathbb{R}^3\to\mathbb{R}$,
$F(x,y,z)=x^2+y^2+z^2-1$. $\therefore T_pS^2=\{a\partial_x+b\partial_y+c\partial_z \mid 2ax+2by+2cz=0\}$
Then T_pS^2 consists of all vectors v_p in $\mathbb{R}^3$ that's $\perp$ to radial vector p
- More generally, same argument extends to $S=\operatorname{Ker}F$ where dF corresponds to ∇F

what? 1-form is $dF=F_x\,dx+F_y\,dy+F_z=2x\,dx+2y\,dy+2z\,dz=\nabla F$. $\therefore$ a vector $v=a\partial_x+b\partial_y+c\partial_z\in T_p(S^2)$ has $v\perp\nabla F(p)\Leftrightarrow \nabla F(v)\cdot p=\begin{pmatrix}2a\\2b\\2c\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0$

Let N be a submanifold on M. If v is a vector field on M, then v is a vector field on $N\Leftrightarrow$
$v_y\in T_y(N), \forall y\in N$. Then $\exists$ vector field $\tilde v$ on parameterization space $\tilde N\ni d\phi(\tilde v)=v$ on N

Lemma: If v,w are tangent to a submanifold N, then so is $[v,w]$

Proof: Let $\tilde v,\tilde w$ be corresponding vector fields on $\tilde N$. Then $d\phi[\tilde v,\tilde w]=[d\phi(\tilde v),d\phi(\tilde w)]=[v,w]$,
$\forall y\in N$. Then $[v,w]_y\in T_y(N)=d\phi(T_{\tilde y}(\tilde N)), \forall y\in N$. □

Let $v_1,\dots,v_r$ be vector fields on smooth manifold M. Then an **Integral Submanifold** of $\{v_1,\dots,v_r\}$ is a submanifold $N\subset M\ni\forall y\in N$, $T_y(N)=\operatorname{span}\{v_{1y},\dots,v_{ry}\}$. (vector field v_2 @ y)

→ $\{v_1,\dots,v_r\}$ is **Integrable** if $\forall x_0\in M$, there $\exists$ an integral submanifold

→ **Ex1.32**: Let $M=\mathbb{R}^3$ & define $v_1=\partial_x+y\partial_z$, $v_2=\partial_y+x\partial_z$. Intuitively, N is 2D, so say $z=f(x,y)$. Then
$r(x,y)=(x,y,f(x,y))$ has $\partial_x r=\partial_x+f_x\partial_z$ & $\partial_y=\partial_y+f_y\partial_z$
$v_1=\partial_x+y\partial_z$ & $v_2=\partial_y+x\partial_z$ $\Rightarrow \begin{matrix}f_y=x\\ f_x=y\end{matrix}\Rightarrow f(x,y)=xy+C\xrightarrow[\text{let } C=0]{\text{WLOG}}$ $\boxed{N=\left\{\frac{(x,y,z)}{z=xy}\right\}}$

- Vector fields $\{v_1, \dots, v_r\}$ on $\mathcal{M}$ is in **Involution** if $\exists$ smooth real-valued funcs. $c_{ij}^k(x), x \in \mathcal{M}; i,j,k = 1, \dots, r \ni$
$\forall i,j, [v_i, v_j] = \sum_{k=1}^{r} c_{ij}^k \cdot v_k$ (a.k.a. any Lie Bracket is a L.C. of vector fields)

- **Frobenius' Theorem**: Let $v_1, \dots, v_r$ be smooth vector fields on $\mathcal{M}$. Then it's integrable $\Leftrightarrow$ it's in involution.
(must be finite)
&
  - $\longrightarrow$ Let $\mathcal{H}$ be any collection of vector fields that form a vector space. Then $\mathcal{H}$ is in **Involution** if $[v,w] \in \mathcal{H}$. Let $\mathcal{H}_x \subset T_x(\mathcal{M})$ where $\mathcal{H}_x = \text{span}(v_x \mid \forall v \in \mathcal{H})$. An **Integral Submanifold** of $\mathcal{H}$ is a submanifold $\mathcal{N} \subset \mathcal{M} \ni T_y(\mathcal{N}) = \mathcal{H}_y, \forall y \in \mathcal{N}$
  - $\longrightarrow$ $\mathcal{H}$ is **Rank-Invariant** if $\forall v \in \mathcal{H}$, $\dim \mathcal{H}_{e^{\varepsilon v}x}$ is constant w.r.t ε varying
  - $\longrightarrow$ Integral curve $e^{\varepsilon v}x$ emanating from x must be contained in an integral submanifold $\mathcal{N}$
  - $\longrightarrow$ **Lemma**: $\mathcal{H}$ is rank-invariant if $\mathcal{H}$ is finitely generated.
  - $\longrightarrow$ **Generalized Frobenius Theorem**: $\mathcal{H}$ is integrable $\Leftrightarrow$ $\mathcal{H}$ is in involution & rank-invariant.

- All maximal integral submanifolds on an integrable system of vector fields is a **Foliation** on manifold $\mathcal{M}$, where each integral submanifolds are the **Leaves**
  - $\longrightarrow$ Ex 1.33: Consider $v = -y\partial_x + x\partial_y$, $w = 2xz\partial_x + 2yz\partial_y + (z^2+1-x^2-y^2)\partial_z$

$v(w) \longrightarrow v(2xz) = -y\partial_x(2xz) + x\partial_y(2xz) = -2yz \longrightarrow -2yz\partial_x + 2xz\partial_y$
$v(2yz) = 2xz$
$v(z^2+1-x^2-y^2) = 2xy - 2xy = 0$
$\longrightarrow v(w) - w(v) = [v,w] = 0$
$w(v) \longrightarrow w(-y) = -2yz \longrightarrow -2yz\partial_x + 2xz\partial_y$
$w(x) = 2xz$

$\therefore$ by the Frobenius Theorem, $\{v, w\}$ is integrable. Then as $v_{(x,y,z)}$ & $w_{(x,y,z)}$ spans a 2D space except on the z-axis $(v=0)$ & S^1 $(w=0)$, which is 1D. Rest of integral submanifolds are a 2D tori $\zeta(x,y,z) = \frac{x^2+y^2+z^2+1}{\sqrt{x^2+y^2}} = c > 2$

Then $d\zeta(v) = v(\zeta) = 0$
" $(w) = w$ " $= 0$
$\longrightarrow$ v, w are tangent to each level set of ζ where $\nabla\zeta \neq 0$

- An integrable system of vector fields $\{v_1, \dots, v_r\}$ is **Semi-Regular** if $\dim(T_x\mathcal{M})$ $\left(= \text{span}\{v_{1x}, \dots, v_{rx}\}\right)$ doesn't vary across x, & **Regular** if it's semi-regular & $\forall x \in \mathcal{M}$, $\exists$ neighborhood $x \ni$ each maximal integral submanifold intersects $\mathcal{U}$ in a pathwise-connected subset
  - $\longrightarrow$ Ex 1.34: 1.33 is semi-regular on $\mathbb{R}^3$ & regular on $\mathbb{R}^3 \setminus (\{z\text{-axis}\} \cup \{S^1\})$

Theorem: Let $\{v_1,\dots,v_r\}$ be an integrable system of vector fields $\ni \dim(\operatorname{span}\{v_{1x},\dots,v_{rx}\})$ in T_xM ($\dim M = m$) is a constant s, independent of $x\in M$. Then $\forall x_0\in M$, $\exists$ flat local coordinates y near $x_0 \ni$ integral submanifolds intersect (coordinate charts) in "slices" $\{y \mid y_1 = c_1,\dots, y_{m-s}=c_{m-s}\}$ (arbitrary constants). If the system is regular, then the (coordinate chart) can be chosen $\ni$ each integral submanifold intersects it in @ most 1 slice.

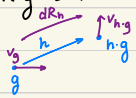
Ex1.35: From 1.33, near $(x_0,y_0,z_0) \notin$ z-axis

Flat local coordinates are given by $\tilde{x}=x$, $\tilde{y}=y$, $\tilde{z}=\zeta(x,y,z)$. Tangent space to plate $\{\tilde{z}=\text{constant}\}$ is spanned by: $\partial_{\tilde{x}} = \partial_x - \frac{x(x^2+y^2-z^2-1)}{2z(x^2+y^2)}\partial_z$, ∴ such locally "flattens out" the tori

$$\partial_{\tilde{y}} = \partial_y - \frac{y(x^2+y^2-z^2-1)}{2z(x^2+y^2)}\partial_z$$

1.4 Lie Algebras

Lemma: For Lie Group G, then $\forall g\in G$, **Right Multiplication** $R_g : G\to G$, $R_g(h) = h\cdot g$ is a diffeomorphism



→ Vector field v is **Right-Invariant** if $dR_g(v_h) = v_{R_g(h)} = v_{hg}$ a.k.a.

→ **Lemma:** If v, w are " ", so is any $av+bw$; $a,b\in\mathbb{R}$, thus forms a Vector Space

Lie Algebra of Lie Group G is $\mathfrak{g}$ (\mathfrak in LaTeX), the V.S. of all R.-invariant vector fields on G (✱)

→ All uniquely determined @ identity as $v_g = dR_g(v_e)$, $R_g(e) = g$

→ Then any tangent vector to G @ e uniquely determines a (✱) by the above

→ **Lemma:** $\mathfrak{g} \cong T_eG$, & ∴ has the same dimension as G

→ However, one can generalize Lie Algebra as a V.S. $\mathfrak{g}$ w/ bilinear operation $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, the **Lie Bracket** for $\mathfrak{g} \ni$ it satisfies: (I) Bilinearity, (II) Skew-Symmetry, & (III) Jacobi Identity

Ex1.46: If $G=\mathbb{R}$, then up to a constant, $\exists!$ (✱) being $\partial_x = \partial_x$ ∵ $R_y(x) = x+y$ w/ $dR_y(\partial_x) = \partial_x$

" " $=\mathbb{R}^+$ " " " " $\partial_x = x\partial_x$ ∵ $R_y(x) = xy$ w/ $dR_y(x\partial_x) = xy\partial_x$

" " $= SO(2)$ " " " " $\partial_\theta = \partial_\theta$ (for rot. matrix A_θ) $R_{A_\theta}(A_\phi) = A_{\theta+\phi}$, $dR_{A_\phi}(\partial_\theta) = \partial_\theta$

Proposition: $\exists!$ 1D Lie Algebra up to isomorphism.

Proof: Let $\mathfrak{g}$ be a " " ". Then for any $v,w\in\mathfrak{g}$, $\exists a,b\in\mathbb{R} \ni v = ae$, $w=be$ for basis e of $\mathfrak{g}$.

∴ $[v,w] = [ae,be] = ab[e,e] \overset{\text{use skew symmetry}}{=} 0$, ∴ $\mathfrak{g}$ must be trivial. □

Q1.36: Find the Lie Algebra of $GL(n)$.

[SOLUTION] $GL(n) \simeq \mathbb{R}^{n^2} \simeq T_A(GL(n)) \simeq T_I(GL(n)) \simeq \mathfrak{g}$

$\hookrightarrow v_{AI} = \sum_{i,j} a_{ij}\, \partial_{x_{ij}}\Big|_I$

Let $Y \in GL(n)$. Then $R_Y(X) = XY$ has entries $\sum_{k=1}^{n} x_{ik} y_{kj}$. Then: $v_{AY} = dR_Y(v_{AI})$

$$= \sum_{i,m}\sum_{i,j} a_{ij}\,\partial_{x_{ij}}\Big(\sum_k x_{ik} y_{km}\Big)\partial_{x_{im}} = \sum_{i,j,m} a_{ij}\, y_{jm}\, \partial_{x_{im}} \text{ a.k.a. } v_{AX} = \sum_{i,j}\Big(\sum_k a_{ik} x_{kj}\Big)\partial_{x_{ij}}$$

$$\therefore [v_A, v_B] = \sum_{\substack{i,j,k\\ l,m,p}} \Big\{ a_{lp} x_{pm}\, \partial_{x_{lm}}(b_{ik} x_{kj}) - b_{lp} x_{pm}\, \partial_{x_{lm}}(a_{ik} x_{kj}) \Big\} \partial_{x_{ij}}$$

$$= \sum_{i,j,k}\Big[\sum_l (b_{il} a_{lk} - a_{il} b_{lk})\Big] x_{kj}\, \partial_{x_{ij}} = v_{\overset{\text{commutator}}{[A,B]}}$$

$\therefore$ Lie Algebra is $\boxed{\mathfrak{gl}(n) = M_n(\mathbb{C})}$ w/ Lie Bracket being the commutator. [ANSWER]

Proposition: Let $v \neq 0$ be a $(\star)$ on Lie Group G. Then the flow generated by v through e, $g_\varepsilon = \exp(\varepsilon v)e$, is defined $\forall \varepsilon \in \mathbb{R}$ & forms a 1-parameter subgroup of G w/ $\underbrace{g_{\varepsilon+\delta} = g_\varepsilon \cdot g_\delta}_{(\star)}$, $g_0 = e$, $g_\varepsilon^{-1} = g_{-\varepsilon}$, such being $\simeq \mathbb{R}$ or $SO(2)$. Conversely, any 1D " " " is generated by a $(\star)$ as described.

Proof: $g_\delta \cdot g_\varepsilon = R_{g_\varepsilon}(g_\delta) = R_{g_\varepsilon}[\exp(\delta v)e] = \exp[\delta \cdot dR_{g_\varepsilon}(v)]R_{g_\varepsilon}(v) = \exp(\delta v) g_\varepsilon = \exp(\delta v)\exp(\varepsilon v)e$
$= \exp[(\delta+\varepsilon)v]e = g_{\delta+\varepsilon}$. Additionally, $g_0 = e$, $g_{-\varepsilon} = g_\varepsilon^{-1}$ for small ε. W/ g_ε defined for $\varepsilon \in [-\frac{\varepsilon_0}{2}, \frac{\varepsilon_0}{2}]$, inductively define $g_{m\varepsilon_0+\varepsilon} = g_{m\varepsilon_0} \cdot g_\varepsilon$ for $m \in \mathbb{Z}$. $\therefore g_\varepsilon$ is a smooth curve in G. Thus $\forall \varepsilon, \delta$, $(\star)$ is satisfied, $\therefore$ the flow is globally defined & thus forms a subgroup.

If $\exists\, \varepsilon \neq \delta \ni g_\varepsilon = g_\delta$, then $\exists\, \varepsilon_0 > 0 \ni g_{\varepsilon_0} = g_e$ & $\therefore g_e$ is periodic w/ period ε_0. Then $\{g_e\} \simeq SO_2$; take $\theta = \frac{2\pi\varepsilon}{\varepsilon_0}$. Else $g_\varepsilon \neq g_\delta$, $\forall \varepsilon \neq \delta$, & $\therefore \{g_e\} \simeq \mathbb{R}$

Conversely, if $H \subset G$ is a 1D subgroup, let v_e be any nonzero tangent vector to H @ e. As $\mathfrak{g} \simeq T_e G$, extend v to a $(\star)$ on all of G. As $H \leq G$, then $v_h \in T_h(H)$, $\forall h \in H$, & $\therefore H$ is the integral curve of v through e. $\square$

Q1.37: What's the $(\star)$ for $A \in \mathfrak{gl}(n)$?

[SOLUTION] $\exp(\varepsilon v_A)e$ is found by integrating n^2 ODEs: $\frac{dx_{ij}}{d\varepsilon} = \sum_{k=1}^{n} a_{ik} x_{kj}$, $x_{ij}(0) = \delta_j^i$; $i,j = 1, \dots, n$

a.k.a. $\frac{dX}{d\varepsilon} = v_{AX} = dR_X(A) = AX$, $X(0) = I$

Then $\boxed{X(\varepsilon) = e^{\varepsilon A}}$ [ANSWER]

$\rightarrow$ From ex 1.17, recall in $\mathbb{T}^2$ w/ the Lie Algebra spanned by ∂_θ & ∂_ρ, let $v_\omega = \partial_\theta + \omega\partial_\rho$, $\omega \in \mathbb{R}$. Then $\exp(\varepsilon v_\omega)(0,0) = (\varepsilon, \varepsilon\omega) \bmod 2\pi$, which is H_ω. So $\omega \in \mathbb{Q} \Rightarrow H_\omega \simeq SO(2)$
$\omega \in \mathbb{I} \Rightarrow H_\omega \simeq \mathbb{R}$

- **Subalgebra** $\mathfrak{h}$ of Lie Algebra $\mathfrak{g}$ is a vector subspace that's closed under the Lie Bracket
  → ∴ any (*) v on H can be extended to G via $v_g = dR_g(v_e), g \in G$

- **Theorem**: Let G be a Lie Group w/ Lie Algebra $\mathfrak{g}$. If $H \subset G$ is a Lie Subgroup, its Lie Algebra is a subalgebra of $\mathfrak{g}$. Conversely, if $\mathfrak{h}$ is any s-dimensional subalgebra of $\mathfrak{g}$, $\exists!$ connected s-parameter Lie Subgroup H of G w/ Lie Algebra $\mathfrak{h}$. Namely: $\begin{matrix} H & \subseteq & G \\ \updownarrow & & \updownarrow \\ \mathfrak{h} & \subseteq & \mathfrak{g} \end{matrix}$

  Pseudo-Proof: $\Rightarrow$ Let $v_1, \dots, v_s$ be a basis of $\mathfrak{h}$ as a system of vector fields on G. Then $\forall i,j$, $[h_i, h_j] \in \mathfrak{h}$, & ∴ in it's span. ∴ $\mathfrak{h}$ defines an involution system of vector fields on G. Then $\forall g \in G$, $\{v_{1g}, \dots, v_{sg}\}$ are L.I. tangent vectors, & thus is semi-regular. By Frobenius' Theorem, $\exists$ maximal s-dimensional submanifold passing through e, which this is the Lie Subgroup H corresponding to $\mathfrak{h}$. Then multiplication & inversion from G remains smooth in H. □

- Ex1.38: If $H \leq GL(n)$, then $\mathfrak{h}$ is a subalgebra of $\mathfrak{gl}(n) \cong M_n(\mathbb{C})$. W/ $\mathfrak{h} \cong T_e(H)$, then
  $\mathfrak{h} = \{A \in \mathfrak{gl}(n) \mid e^{\varepsilon A} \in H, \forall \varepsilon \in \mathbb{R}\}$
  Say $H = O(n)$. Then want $A \ni e^{\varepsilon A}(e^{\varepsilon A})^T = I \Rightarrow \frac{d}{d\varepsilon}\big|_{\varepsilon=0}$ gets $A + A^T = 0$, ∴ $\boxed{\mathfrak{so}(n) = \{A \mid A \text{ is skew symmetric}\}}$ which is the Lie Algebra of both $O(n)$ & $SO(n)$

- **Ado's Theorem**: Let $\mathfrak{g}$ be a finite-dimensional Lie Algebra. Then $\mathfrak{g} \cong$ some subalgebra of $\mathfrak{gl}(n)$, for some $n \in \mathbb{N}$

- **Theorem**: Let $\mathfrak{g}$ be a finite-dimensional Lie Algebra. Then $\exists!$ connected & simply-connected Lie Group G^* w/ Lie Algebra $\mathfrak{g}$. If G is any other connected Lie Group w/ Lie Algebra $\mathfrak{g}$, then $\pi: G^* \to G$ is the simply-connected covering group of G.

- **Exponential Map** $\exp: \mathfrak{g} \to G$, $\exp(v) = \exp(v)e$ when setting $\varepsilon = 1$
  → Then $d\exp: T_0\mathfrak{g} \cong \mathfrak{g} \to T_eG \cong \mathfrak{g}$ evaluated @ 0 is the identity map, ∴ by the InFT, $\exp$ has a local diffeomorphism from $\mathfrak{g} \to$ neighborhood of e

- **Proposition**: For g in Lie Group G, then $\exists v_1, \dots, v_k \in \mathfrak{g} \ni g = e^{v_1 + v_2 + \dots + v_k}$. More precisely, for basis $\{v_1, \dots, v_r\}$ of $\mathfrak{g}$, then $g = e^{\varepsilon_1 v_{i_1} + \dots \varepsilon_k v_{i_k}}$ for some $\varepsilon_j \in \mathbb{R}$, $1 \leq i_j \leq r$, $j = 1, \dots, k$

- Consider local Lie Group $V \subset \mathbb{R}^r$ w/ $m(x,y)$, where $R_y: V \to \mathbb{R}^r$, $R_y(x) = m(x,y)$; ∴ v on V is right-invariant $\Leftrightarrow dR_y(v_x) = v_{R_y(x)} = v_{m(x,y)}$, uniquely determined by $v_x = dR(v_0)$
  → ∴ Lie Algebra $\mathfrak{g}$ for V is an r-dimensional vector space

Proposition: Let $V \subset \mathbb{R}^r$ be a Local Lie Group w/ multiplication $m(x,y)$. Then Lie Algebra $\mathfrak{g}$ of (*) on V is spanned by vector fields $v_k = \sum_{i=1}^{r} \xi_k^i(x)\partial_{x_i}$, $k=1,\dots,r$, where $\xi_k^i(x) = \frac{\partial m_i}{\partial x_k}(0,x)$

Proof: $dR_y\left(\sum_{i=1}^{r} \xi_k^i(0)\partial_{x_i}\right) = \sum_{i,j} \xi_k^i(0)\frac{\partial m_j}{\partial x_i}(0,y)\partial_{x_j}$. Thus need to prove $\xi_k^i(0) = \delta_k^i = \frac{\partial m_i}{\partial x_k}(0,0) = \begin{cases} 1 & i=k \\ 0 & i\neq k \end{cases}$, which follows from $m(x,0)=x$ $\square$

Ex1.39: $V = \{x \mid |x|<1\} \subset \mathbb{R}$, $m(x,y) = \frac{2xy-x-y}{xy-1} \longrightarrow \left.\frac{(\cancel{xy}^{0}-1)(2y-1)-(\cancel{2xy}^{0}-\cancel{x}^{0}-y)(y)}{(\cancel{xy}^{0}-1)^2}\right|_{(0,x)}$

Observe the Lie Algebra $\mathfrak{g}$ is 1D, spanned by $\xi(x)\partial_x$ where $\xi(x) = \frac{\partial m}{\partial x}(0,x) = \frac{1-2x+x^2}{1} = (x-1)^2$

$\therefore$ $v = (x^2-1)\partial_x$ is the unique (*) on V

Say $\mathfrak{g}$ is any finite-dimensional Lie Algebra, so such is a Lie Algebra of some Lie Group G. If $\{v_1,\dots,v_r\}$ is a basis of $\mathfrak{g}$, then $[v_i,v_j] \in \mathfrak{g}$. So $\exists\, c_{ij}^k \in \mathbb{R}$; $i,j,k=1,\dots,r$, the Structure Constants of $\mathfrak{g} \ni$ $[v_i,v_j] = \sum_{k=1}^{r} c_{ij}^k v_k$; $i,j=1,\dots,r$

⟶ Then by Skew-Symmetry: $c_{ij}^k = -c_{ji}^k$ & by Jacobi Identity: $\sum_{k=1}^{r}\left(\overset{1234}{c_{ij}^k c_{kl}^m} + \overset{4132}{c_{li}^k c_{kj}^m} + \overset{2431}{c_{jl}^k c_{ki}^m}\right) = 0$

⟶ If $\mathfrak{g}$ has a new basis e.g. $\hat{v}_i = \sum_j a_{ij} v_j$, then $\hat{c}_{ij}^k = \sum_{l,m,n} a_{il}a_{jm}b_{nk}c_{lm}^n$, $B = A^{-1}$

If $\mathfrak{g}$ is an r-dimensional Lie Algebra w/ basis $v_1,\dots,v_r$, the Commutator Table is an $r\times r$ table whose ij^{th} entry is $[v_i,v_j]$, which must be Skew-Symmetric as $[v_i,v_j] = -[v_j,v_i]$ w/ all diagonals 0

⟶ Ex1.40: Let $\mathfrak{g} = \mathfrak{sl}(2)$ be the Lie Algebra of $SL(2)$. For basis $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

	A_1	A_2	A_3
A_1	0	A_1	$-2A_2$
A_2	$-A_1$	0	A_3
A_3	$2A_2$	$-A_3$	0

Commutator Table

e.g.
$[A_1,A_3] = A_3A_1 - A_1A_3 = -2A_2$

$c_{12}^1 = c_{23}^1 = 1 = -c_{21}^1 = -c_{32}^3$; $c_{13}^2 = -2 = c_{31}^2$

Say G is a local group of transformations acting on manifold M via $g\cdot x = \Psi(g,x)$ for $(g,x) \in \mathcal{U} \subset G\times M$.

Then there's an Infinitesimal Action of $\mathfrak{g}$ on M, where if $v \in \mathfrak{g}$, then $\psi(v)$ is the vector field on M whose flow coincides w/ the action of the 1-parameter subgroup $e^{\varepsilon v}$ of G on M

⟶ a.k.a. for $x \in M$, $\psi(v)_x = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Psi(e^{\varepsilon v},x) = d\Psi_x(v_e)$ for $\Psi_x(g) := \Psi(g,x)$,

w/ $\Psi_x \circ R_g(h) = \Psi(h\cdot g,x) = \Psi(h,g\cdot x) = \Psi_{g\cdot x}(h) \Rightarrow d\Psi_x(v_g) = d\Psi_{g\cdot x}(v_e) = \psi(v)_{g\cdot x}$

⟶ Then $[\psi(v),\psi(w)] = \psi([v,w])$, $\therefore$ $\{\psi(v) \mid v\in\mathfrak{g}\}$ forms a Lie Algebra of vector fields on M

Lie Bracket is a Lie Algebra Homomorphism — which " $\simeq \mathfrak{g}$ & has the same structural constants

∘ Similarly, given a finite-dimensional Lie Algebra of vector fields on M, $\exists$ local group of transformations whose infinitesimal action is generated by the given Lie Alg.

Theorem: Let $w_1,\dots,w_r$ be vector fields on manifold $M \ni [w_i,w_j] = \sum_{k=1}^{r} \overset{\text{constants}}{c_{ij}^k} w_k$; $i,j=1,\dots,r$. Then $\exists$ Lie Group G whose Lie Algebra has structure constants c_{ij}^k relative to some basis $v_1,\dots,v_r$ & a local group action G on $M \ni \psi(v_i) = w_i$ for $i=1,\dots,r$ where $\psi(v)_x := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \underbrace{\Psi(e^{\varepsilon v},x)}_{\text{multiplication map}}$

→ $v \in \mathfrak{g}$ is an Infinitesimal Generator of group action G

→ a.k.a. if $w_1,\dots,w_r$ are " " that forms a basis for Lie Algebra $\mathfrak{g}$, then exponentiate to get a local group of transformations where $v_x = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{\varepsilon v}x$, $v \in \mathfrak{g}$

Proposition: There's 3 finite-dimensional Lie Algebras of vector fields on $\mathbb{R}$ up to diffeomorphism. They're:

→ 1D Lie Algebra $v = \partial_x$ $\quad v_x = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{\varepsilon\partial_x}x = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sum_{i=0}^{\infty} \frac{\varepsilon^i \partial_x^i}{i!}x = \underbrace{\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \underbrace{(x+\varepsilon)}_{\text{flow}}}_{\text{vector field}} = 1 \to \partial_x$, ∴ $\mathbb{R}$ acts on M via $x \to x+\varepsilon$

→ 2D Lie Algebra $v_1 = \partial_x$, $v_2 = x\partial_x$ $\quad v_1$ has $x \to x+\varepsilon$ & v_2 has $e^{\varepsilon v_2}x = \sum_{i=0}^{\infty} \frac{\varepsilon^i (x\partial_x)^i x}{i!} = x\sum_{i=1}^{\infty}\frac{\varepsilon^i}{i!} = e^{\varepsilon}x$
∴ $x \to \lambda x$ for $\lambda = e^{\varepsilon}$ ∴ Lie Algebra $\cong$ Lie Algebra span$\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}\right\} = \left\{A = \begin{pmatrix}\alpha&\beta\\0&1\end{pmatrix} \,\middle|\, \alpha>0\right\}$,
∴ $\mathbb{R}$ acts on M via affine $x \to \alpha x+\beta$

→ 3D Lie Algebra $v_1 = \partial_x$, $v_2 = x\partial_x$, $v_3 = x^2\partial_x$ $\quad x \to \frac{x}{1-\varepsilon x}$, $|\varepsilon| < \frac{1}{x}$

→ Commutator Table:

	v_1	v_2	v_3
v_1	0	v_1	$2v_2$
v_2	$-v_1$	0	v_3
v_3	$-2v_2$	$-v_3$	0

If $v_3 \to -v_3 = -x^2\partial_x$, then get same table as $\mathfrak{sl}(2)$ from ex 1.40 ∴ $\exists$ local action of $SL(2)$ on $\mathbb{R}$ w/ ∂_x, $x\partial_x$, $-x^2\partial_x$ as infinitesimal generators

- Group action is the Projective Group $x \to \frac{\alpha x+\beta}{\gamma x+\delta}$, $\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix} \in SL(2)$

1.5 Differential Forms

100% AWFUL Introduction, but great for review! Refer to Alg II presentation notes for a refresher.

Let M be a smooth manifold. Then $\Lambda^k T_x^* M = \{$all k-linear, alternating funcs. $\omega: \times_{i=1}^{k} T_xM \to \mathbb{R}\}$ is a Differential k-Form; denote $\omega(v_1,\dots,v_k) := \langle \omega; v_1,\dots,v_k\rangle$. Then:

Alternating: $\langle\omega;\dots,cv_i + c'v_i',\dots\rangle = c\langle\omega;\dots,v_i,\dots\rangle + c'\langle\omega;\dots,v_i',\dots\rangle$ for $c,c' \in \mathbb{R}$

k-Linear: $\langle\omega; v_{\pi 1},\dots,v_{\pi k}\rangle = (-1)^{\pi}\langle\omega; v_1,\dots,v_k\rangle$ for some $\pi \in S^k$

→ $\Lambda^k T_x^* M$ is a V.S.; by convention a 0-form is a smooth, real-valued function $f: M \to \mathbb{R}$

→ 1-Form, a.k.a. Cotangent Space to M @ x: $\Lambda T_x^* M =$ Dual V.S. to T_xM $= \mathrm{Hom}(T_xM, \mathbb{R})$

→ Say $\langle\omega; v_1,\dots,v_k\rangle(x) := \langle\omega_x; v_{1x},\dots,v_{kx}\rangle$

→ For local coordinates $(x_1,\dots,x_m)$, then T_xM has Tangent Basis $\{\partial x_1,\dots,\partial x_m\}$ & the Cotangent Space has a Cotangent Basis $\{dx_1,\dots,dx_m\}$ where $\langle dx_i; \partial x_j\rangle = \delta_j^i = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

→ ∴ differential 1-form $\omega = h_1(x)dx_1 + \dots + h_m(x)dx_m$ locally w/ each $h_j(x)$ smooth

- So for any vector field $v = \sum_{i=1}^{m}\xi_i(x)\partial_{x_i}$, $\langle\omega; v\rangle = \sum_{i=1}^{m} h_i(x)\xi_i(x)$ is smooth
- If $f: M \to \mathbb{R}$ is a 0-form, then $df = \sum_{i=1}^{m} f_{x_i}dx_i$, $\langle df; v\rangle = v(f)$

- For k differential 1-forms $\omega_1, \dots, \omega_k$, can form a k-form $\omega_1 \wedge \dots \wedge \omega_k$ via $\langle \omega_1 \wedge \dots \wedge \omega_k ; v_1, \dots, v_k \rangle = \det(\underbrace{\langle \omega_i ; v_j \rangle}_{k\times k \text{ matrix @ } ij^{\text{th}} \text{ entry}})$, noting that $\omega_{\pi 1} \wedge \dots \wedge \omega_{\pi k} = (-1)^{\pi} \omega_1 \wedge \dots \wedge \omega_k$

- **Proposition:** In local coordinates, $\Lambda^k T^*_x \mathcal{M} = \operatorname{span}\{dx_I := dx_{i_1} \wedge \dots \wedge dx_{i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq m\}$ (all possible "volumes of dim k" that can be made in dim m), $\therefore$ $\dim(\Lambda^k T^*_x \mathcal{M}) = \binom{m}{k}$ & $\therefore$ $\Lambda^k T^*_x \mathcal{M} \cong \{0\}$ if $k > m$
  - $\longrightarrow$ Any smooth differential k-form on $\mathcal{M}$ has local coordinate expression $\omega = \sum_I \alpha_I(x) dx_I$ e.g. 2-form in $\mathbb{R}^3$: $\omega = \alpha\, dy \wedge dz + \beta\, dx \wedge dy + \gamma\, dx \wedge dz$

- Let $F: \mathcal{M} \to \mathcal{N}$ be a smooth map between manifolds. Then $dF: T_x\mathcal{M} \to T_x\mathcal{N}$, & $\therefore$ $\exists$ induced linear $F^*: \Lambda^k T^*_{F(x)} \mathcal{N} \to \Lambda^k T^*_x \mathcal{M}$ as the **Pullback**, a.k.a. **Codifferential** of F, where $\langle F^*(\omega); v_1, \dots, v_k \rangle = \langle \omega; dF(v_1), \dots, dF(v_k) \rangle$ for $v_1, \dots, v_k \in T_x\mathcal{M}$, $\omega \in \Lambda^k T^*_{F(x)} \mathcal{N}$
  - $\longrightarrow$ See that if $x = (x_1, \dots, x_m)$ are local coords. on $\mathcal{M}$, $y = (y_1, \dots, y_n)$ are local coords. on $\mathcal{N}$, then $F^*(dy_i) = \sum_{j=1}^{m} \frac{\partial y_i}{\partial x_j} dx_j$, $y = F(x)$, $\therefore$ $F^*(\omega) = F^*\left(\sum_I \alpha_I(y) dy_I\right) = \sum_{I,J} \alpha_I(F(x)) \frac{\partial y_I}{\partial x_J} dx_J \in \Lambda^k T^*_x \mathcal{M}$ & $\therefore$ F^* is well-defined
  - $\longrightarrow$ **Lemma:** F^* is linear w.r.t the wedge product, namely $F^*(\omega \wedge \theta) = F^*(\omega) \wedge F^*(\theta)$

- If ω is a differential k-form & v is a smooth vector field, then the **Interior Product** $v \lrcorner \omega$ is a $k-1$ form $\ni \langle v \lrcorner \omega; v_1, \dots, v_{k-1} \rangle = \langle \omega; v, v_1, \dots, v_{k-1} \rangle$
  - $\longrightarrow$ The basis is $\partial_{x_i} \lrcorner (dx_{j_1} \wedge \dots \wedge dx_{j_k}) = \begin{cases} (-1)^{k-1} dx_{j_1} \wedge \dots \wedge dx_{j_{k-1}} \wedge dx_{j_{k+1}} \wedge \dots \wedge dx_{j_k} & i = j_k \\ 0 & i \neq j_k, \forall k \end{cases}$ (\lrcorner) (a.k.a. get rid of dx_k if there's ∂x_k & alternate)
    - Ex1.41: $\partial_x \lrcorner dx \wedge dz = dx$, $\partial_x \lrcorner dz \wedge dx = -dz$ (swap $-dx \wedge dz$), $\partial_x \lrcorner dy \wedge dz = 0$
      For 2-form $\omega = \alpha\, dy \wedge dz + \beta\, dz \wedge dx + \gamma\, dx \wedge dy$ & $v = \xi \partial_x + \eta \partial_y + \zeta \partial_z \dots$
      $v \lrcorner \omega = \eta\alpha\, dz - \beta\xi\, dz + \gamma\xi\, dy - \zeta\alpha\, dy + \beta\zeta\, dx - \eta\gamma\, dx = \boxed{(\zeta\beta - \eta\gamma)dx + (\gamma\xi - \zeta\alpha)dy + (\eta\alpha - \beta\xi)dz}$
      $(\zeta \partial_z)(\alpha\, dy \wedge dz) = (\zeta \partial_z)(-\alpha\, dz \wedge dy) = -\zeta\alpha\, dy$
  - $\longrightarrow$ For algebra A over field $\mathbb{K}$, a **$\mathbb{K}$-Derivation** is a $\mathbb{K}$-linear map $D: A \to A$ that satisfies **Leibniz's Law:** $D(ab) = aD(b) + D(a)b$ (product rule)
    - If $D(ab) = D(a)b + (-1)^{|a||D|} aD(b)$, then D is an **Anti-Derivation**
  - $\longrightarrow$ **Lemma:** The interior product acts as an anti-derivation on forms, where $v \lrcorner (\omega \wedge \theta) = (v \lrcorner \omega) \wedge \theta + (-1)^k \omega \wedge (v \lrcorner \theta)$ for k-form ω, l-form θ

- If $\omega = \sum_I \alpha_I(x) dx_I$ is a k-form on manifold $\mathcal{M}$, its **Differential**, a.k.a. **Exterior Derivative** is the $k+1$ form $d\omega = \sum_I d\alpha_I \wedge dx_I = \sum_{I,j} \frac{\partial \alpha_I}{\partial x_j} dx_j \wedge dx_I$

Properties of the exterior derivative, (I) **Linearity**: $d(c\omega + c'\omega') = c\,d\omega + c'\,d\omega'$

(II) **Anti-Derivation**: $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k \omega \wedge d\theta$

(III) **Closure**: $dd\omega = 0$

(IV) **Commutation of Pullback**: $F^*(d\omega) = d(F^*\omega)$

Ex1.42: Calculate $d(\lambda dx + \mu dy + \nu dz) = (\mu_x - \lambda_y)dx\wedge dy + (\nu_x - \lambda_z)dx \wedge dz + (\nu_y - \mu_z) dy \wedge dz$

$= \nabla \times (\lambda, \mu, \nu)$ curl

$\lambda_x dx^2 = 0 \quad \mu_x dx\wedge dy \quad \nu_x dx \wedge dz$

$\lambda_y dy \wedge dx \quad \mu_y dy^2 = 0 \quad \nu_y dy \wedge dz$

$\lambda_z dz \wedge dx \quad \mu_z dz \wedge dy \quad \nu_z dz^2 = 0$

Calculate $d(\alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy) = (\alpha_x + \beta_y + \gamma_z) dx \wedge dy \wedge dz = \nabla \cdot (\alpha, \beta, \gamma)$ **Divergence**

Denote $\Lambda^k := \Lambda^k \mathcal{M} = \{\text{all } k\text{-forms on manifold } \mathcal{M}\}$. Then the **de Rham Complex** of $\mathcal{M}$ is:

$$0 \longrightarrow \mathbb{R} \xrightarrow{\text{constant map}} \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{m-1} \xrightarrow{d} \Lambda^m \longrightarrow 0$$

↳ **Complex**: Sequence of V.S. + linear maps between them $\ni$ composition pair is 0 (closure: $dd\omega = 0$)

↳ Recall $A \xrightarrow{f} B \xrightarrow{g} C$ is **Exact** if $\operatorname{Ker} g = \operatorname{Im} f$, which in this case means a **Closed** differential k-form ω, a.k.a. $d\omega = 0 \Leftrightarrow$ exact, e.g. $\exists\, k-1$ form $\theta \ni \omega = d\theta$

- $\Leftarrow$ is clear as $d\omega = dd\theta = 0$, but not $\Rightarrow$!
- Ex1.43: Let $\mathcal{M} = \mathbb{R}^2 \setminus \{0\}$, $\omega = \left(\frac{1}{x^2+y^2}\right)(y\,dx - x\,dy)$

$dy \wedge dx = -dx \wedge dy$

$$d\omega = \left(\frac{(x^2+y^2)(-1) - (-x)(2x)}{(x^2+y^2)^2} \ominus \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}\right)(dx \wedge dy) = 0,\ \therefore\ \omega \text{ is closed. Yet } \nexists\, \theta \text{ on } \mathcal{M} \text{ } k-1 \text{ form} \ni d\theta = \omega,\ \therefore\ \cancel{\text{exact}}$$

$x^2 - y^2 \quad - \quad (x^2 - y^2) = 0$

Star-Shaped Domain: If $x \in \mathcal{M} \subset \mathbb{R}^m$, then $\{\lambda x \mid \lambda \in [0,1]\} \subset \mathcal{M}$

Poincaré Lemma: Let $\mathcal{M} \subset \mathbb{R}^m$ be star-shaped. Then the de Rham complex over $\mathcal{M}$ is exact.

Ex1.43: For $\mathcal{M} \subset \mathbb{R}^m$, any m-form ω is uniquely determined by 1 smooth $f \ni \omega = f\,dx_1 \wedge \ldots \wedge dx_m$

Any $m-1$ form ζ depends on m-tuple of smooth $p \ni \zeta = \sum_{j=1}^{m} (-1)^{j-1} p_j(x)\, dx_{\hat{j}}$ where $dx_{\hat{j}} = dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_m$. As $d\omega = 0$ as $\nexists\, m+1$ form, then $\mathcal{M}$ star-shaped $\Rightarrow \exists p \ni f = \nabla \cdot p$ & an $m-1$ form η is determined by $\frac{m(m-1)}{2}$ funcs. $q_{jk}(x)$; $j,k = 1, \ldots, m$, $q_{jk} = -q_{kj} \ni \eta = \sum_{\substack{j,k=1 \\ j<k}}^{m} (-1)^{j+k-1} q_{jk}(x)\, dx_{\widehat{jk}}$ (omits dx_j & dx_k in $dx_1 \wedge \ldots \wedge dx_m$), where $p_j(x) = \sum_{k=1}^{m} \frac{\partial q_{jk}}{\partial x_k}$. For ☆ $\mathcal{M}$, then not only is $\nabla \cdot p = 0$, but $\exists q \ni \underbrace{\nabla \times q}_{\text{generalized curl}} = p$

- Let σ be a differential form or vector field over $\mathcal{M}$. Then @ $x\in\mathcal{M}$ after small time ε, it moved to $e^{\varepsilon v}x$
  - $\longrightarrow$ Want to compare, but PROBLEM: $\sigma_{e^{\varepsilon v}x}$ & σ_x are different V.S. (for forms) ; $T_{e^{\varepsilon v}x}\mathcal{M}\neq T_x\mathcal{M}$ (if σ is a vector field)
  - $\longrightarrow$ Case 1: If vector field, use Inverse Differential: $\phi_\varepsilon^* := d\exp(-\varepsilon v): T_{e^{\varepsilon v}x}\mathcal{M}\to T_x\mathcal{M}$

    Case 2: If diff. form, " Pull-Back Map: $\phi_\varepsilon^* := \exp(\varepsilon v)^* : \Lambda^k T^*_{e^{\varepsilon v}x}\mathcal{M}\to\Lambda^k T^*_x\mathcal{M}$ (a.k.a. bring σ back to original position & compare Δ)

    $\phi_\varepsilon^*(\sigma_x) = \sigma_{e^{\varepsilon v}x}\circ d\exp(\varepsilon v)x$

- Let v be a vector field on $\mathcal{M}$ & σ a vector field or differential form on $\mathcal{M}$. The Lie Derivative of σ w.r.t v @ $x\in\mathcal{M}$ is $(\mathcal{L}_v\sigma)(x) = \lim_{\varepsilon\to 0}\frac{\Phi_\varepsilon^*(\sigma_{e^{\varepsilon v}x})-\sigma_x}{\varepsilon} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Phi_\varepsilon^*(\sigma_{e^{\varepsilon v}x})$
  - $\longrightarrow$ Helps answer how forms/vector fields infinitesimally Δ under flow $e^{\varepsilon v}x$ induced by v
  - $\longrightarrow$ Proposition: Let v,w be vector fields on $\mathcal{M}$. Then $\mathcal{L}_v = [v,w]$

Proof: Let $x=(x_1,\dots,x_m)$ be local coordinates w/ $v=\xi(x)\cdot\partial_x$, $w=\eta\cdot\partial_x$. Then $w_{e^{\varepsilon v}x} = \sum_{i=1}^m(\eta_i(x)+\varepsilon v(\eta_i)+\mathcal{O}(\varepsilon^2))\partial_{x_i}$ $\therefore$ $d\exp(-\varepsilon v)\,w_{e^{\varepsilon v}x} = \sum_{i=1}^m(\eta_i(x)+\varepsilon[v(\eta_i)-w(\xi_i)]+\mathcal{O}(\varepsilon^2))\partial_{x_i}$ $\square$

  - $\longrightarrow$ Properties: (A) Linearity: $v(c\omega+c'\omega') = cv(\omega)+c'v(\omega')$

    (B) Derivation: $v(\omega\wedge\theta) = v(\omega)\wedge\theta+\omega\wedge v(\theta)$

    (C) Commutation w/ Differential: $v(d\omega) = dv(\omega)$
  - $\longrightarrow$ Proposition: $v(w\lrcorner\omega) = [v,w]\lrcorner\omega + w\lrcorner v(\omega)$

- Proposition: A differential k-form on $\mathcal{M}$ is invariant under the flow of vector field v, a.k.a. $\omega_{e^{\varepsilon v}x} = (e^{-\varepsilon v})^*\omega_x \Longleftrightarrow (\mathcal{L}_v w)(x)=0, \forall x\in\mathcal{M}$

Proof: $(e^{\varepsilon v})^*(\mathcal{L}_v\omega)(e^{\varepsilon v}x) = \frac{d}{d\varepsilon}(e^{\varepsilon v})^*(\omega_{e^{\varepsilon v}x})$ $\square$

- Proposition: Let ω be k-form & v a vector field on $\mathcal{M}$. Then Cartan's Formula $\mathcal{L}_v(\omega) = d(v\lrcorner\omega)+v\lrcorner d\omega$ holds.
  - $\longrightarrow$ Corollary #1: Let f be a 0-form. Then $\mathcal{L}_v(f) = v\lrcorner df = \langle df; v\rangle = v(f)$

    $v\lrcorner f$ is a $k-1$ form but $\nexists$ -1 form, so must be $0\Rightarrow d(v\lrcorner f)=0$
  - $\longrightarrow$ Corollary #2: $\mathcal{L}_v d\omega = d(v\lrcorner d\omega) = d\mathcal{L}_v(\omega)$
  - $\longrightarrow$ Ex1.44: Let $\omega = dx\wedge dy$, $v = -y\underbrace{\partial_x}_{(\circledast)}+x\underbrace{\partial_y}_{(*)}$, $\mathcal{M}=\mathbb{R}^2$

    Observe that $\mathcal{L}_v\omega = d(v\lrcorner\omega)+v\lrcorner d(\omega) = \boxed{0}$, which makes sense as rotation preserves area

    $(\circledast) = d(-y\partial_x(dx\wedge dy)+x\partial_y(-dy\wedge dx))$

    $= d(-xdx-ydy) = -1dx^2+0dx\wedge dy+0dx\wedge dy-1dy^2 = 0$

    $(*) = v\lrcorner 0$ as $\nexists$ 3-form $=0$

    Try $f(x,y)=x^2+y^2$: $\mathcal{L}_v f = v(f) = -y(2x)+x(2y) = \boxed{0}$

- Observe that $(e^{\varepsilon v})^*(\omega_{e^{\varepsilon v}x}) - \omega_x = \int_0^\varepsilon (e^{\tilde\varepsilon v})^*(\mathcal{L}_v(\omega)(e^{\tilde\varepsilon v}x))d\tilde\varepsilon = \int_0^\varepsilon d[(e^{\tilde\varepsilon v})^*(v\lrcorner\omega_{e^{\tilde\varepsilon v}x})]+(e^{\tilde\varepsilon v})^*[v\lrcorner d\omega_{e^{\tilde\varepsilon v}x}]d\tilde\varepsilon$

  Define $h_v^\varepsilon(\omega)_x := \int_0^\varepsilon (e^{\tilde\varepsilon v})^*[v\lrcorner\omega_{e^{\tilde\varepsilon v}x}]d\tilde\varepsilon$, then $(e^{\varepsilon v})^*\omega_{e^{\varepsilon v}x}-\omega_x = dh_v^\varepsilon(\omega)_x + h_v^\varepsilon(d\omega)_x$

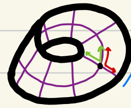
- Then $h: \Lambda^k \mathcal{M} \longrightarrow \Lambda^{k-1}\mathcal{M}$ has $\omega = dh(\omega) + hd(\omega)$, & is a Homotopy Operator

⋮

math words — very poorly written & too dense to understand

⋮

- Intuitively want consistent coordinate systems that varies "nicely" across $x \in \mathcal{M}$



as ↳ moves across the Möbius strip, it "flops" to ↰ ∴ is non-orientable

- For differentiable manifold $\mathcal{M}$ of dimension n, such is Orientable if $\exists$ n-form $\omega \ni \omega_x \neq 0, \forall x \in \mathcal{M}$ OR $\exists$ atlas $\{(\mathcal{U}_\alpha, \varphi_\alpha)\} \ni \forall$ transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ have $\oplus$ Jacobian determinants
  - ⟶ Ex1.45: Construct the Möbius Strip $[0,1]\times[-1,1]$ via $(0,y)\sim(1,-y)$. Such is non-orientable.

Proof: For 2-form, observe that $dx\wedge dy = dx \wedge(-dy) = -dx\wedge dy$, so the only valid 2-form is $0 dx\wedge dy$ □

- $f: \tilde{\mathcal{M}} \longrightarrow \mathcal{M}$ between 2 oriented manifolds is Orientation-Preserving if pullback of orientation form on $\mathcal{M}$ determines the same orientation on $\tilde{\mathcal{M}}$ as the given one

- Let $\mathcal{M}$ be an orientable m-dimensional manifold w/ atlas $\{(\mathcal{U}_i, \phi_i)\}$ that admits a partition of unity $\{\theta_i\}$. Then the Integral of ω over $\mathcal{M}$ is $\int_{\mathcal{M}} \omega := \sum_i \int_{\mathcal{U}_i} \theta_i \cdot \omega = \sum_i \int_{\phi_i(\mathcal{U}_i)} \underbrace{(\phi_i^{-1})^*}_{\text{parameterization of } \phi_i(\mathcal{U}_i)} (\theta_i \cdot \omega)$ } regular Riemann Integral
  - ⟶ Ex1.46: Let $\mathcal{M} = \mathbb{S}^1 \subset \mathbb{R}^2$, covered by $\Phi: \mathcal{U} \subset \mathbb{S}^1 \longrightarrow \mathbb{R}$, $\Phi(\cos\theta, \sin\theta) = \theta \in (0, 2\pi)$ & 1-form $\omega = x dy - y dx$. Observe $\begin{matrix} x = \cos\theta \\ y = \sin\theta \end{matrix} \Rightarrow \begin{matrix} dx = -\sin\theta d\theta \\ dy = \cos\theta d\theta \end{matrix}$, $\therefore (\phi^{-1})^* \omega = \cos^2 d\theta - (-\sin^2\theta) d\theta = d\theta$

    $\therefore \int_0^{2\pi} d\theta = \boxed{2\pi}$

- Let $\mathbb{H}^m := \{(x_1, \dots, x_m) \mid x_m \geq 0\} \subset \mathbb{R}^m$ be the Upper Half-Space w/ $\partial\mathbb{H}^m = \{(x_1, \dots, x_{m-1}, 0)\} \cong \mathbb{R}^{m-1}$
  - ⟶ Then for atlas $\{(\mathcal{U}_\alpha, \chi_\alpha)\}$, $\chi_\alpha: \mathcal{U}_\alpha \longrightarrow \mathcal{V}_\alpha \subset \mathbb{H}^m$ open, where $\mathcal{V}_\alpha = \mathbb{H}^m \cap \underbrace{\tilde{\mathcal{V}}_\alpha}_{\text{open}}$, then $\partial\mathcal{U}_\alpha = \chi_\alpha^{-1}[\partial\mathcal{V}_\alpha]$ where $\partial\mathcal{V}_\alpha = \mathcal{V}_\alpha \cap \partial\mathbb{H}^m$. ∴ the Boundary of $\mathcal{M}$ is $\partial\mathcal{M} = \bigcup_\alpha \partial\mathcal{U}_\alpha$, which is a smooth m-1 dimension w/o boundary (a.k.a. $x \in \partial\mathcal{M}$ if $\exists$ chart $\chi' \ni \chi'(x) \in \partial\mathbb{H}^m$)

- Generalized Stokes' Theorem: Let $\mathcal{M}$ be a compact, oriented, m-dimensional manifold. Let ω be a smooth m-1 form defined on $\mathcal{M}$. Then $\int_{\partial\mathcal{M}} \omega = \int_{\mathcal{M}} d\omega$

Modeling Notes

music notation

a	b	c	α	β	γ	δ	Gait
0.02	0.2	0.44	-0.02	-0.002	-0.025	0.015	1→2→3→4, no valid gait
"	"	"	"	"	0.025	"	4→3→2→1→ —, no valid gait ⊛
"	"	"	"	0.002	"	"	same as above
"	"	"	0.02	"	"	"	Standing
"	"	"	"	"	"	0.02	"
"	"	"	"	0.02	"	"	"
"	"	"	"	"	0.02	"	"
"	"	"	0.03	"	"	"	"
"	"	"	"	0.2	"	"	"
"	"	"	0.2	"	"	"	"
"	"	"	-0.2	"	"	"	⊛
"	"	"	0.025	0.02	-0.01	-0.01	1,3→2,4, pace!
"	"	"	-0.01	0.0102	-0.025	0.02	1→4→2→3, not valid
"	"	"	"	"	"	-0.02	1,3→2,4, pace!
"	"	"	0.025	-0.02	-0.01	-0.01	1→3→2→4 ∿, walk! (not perfect duty cycle)
"	"	"	-0.025	"	"	"	1→4→3→2, not valid
"	"	"	"	"	0.01	0.01	1→2→3→4, not valid
"	"	"	0.01	0.01	-0.01	-0.01	1,3→2,4, pace!
"	"	"	-0.01	-0.01	0.01	0.01	1,4→2,3, trot? doesn't converge...
"	0.002	"	0.01	0.01	-0.01	-0.01	Still pace
0	0	"	"	"	"	"	" "
"	"	1	"	"	"	"	" " but now like ∧ instead of ∩ vs.
"	"	2	"	"	"	"	" " w/ same weird shape
0.02	0.2	0.44	0.01	-0.01	-0.01 (or 0.01)	-0.01	2→4→1→3→ ⋛, pseudo-canter
"	"	"	"	0.01	0.01	"	Standing
"	"	"	"	"	-0.01	0.01	"
"	"	"	"	-0.01	0.01	"	"
"	"	"	-0.01	0.01	"	"	"
"	"	"	0.01	"	"	"	"
"	"	"	"	-0.01	-0.01	"	1→2→3→4→ —, Transverse Gallop
"	"	"	0.02	-0.02	-0.01	0.01	" " " " " "

Toyed a lot more but nothing interesting arose

Tested all $(\alpha,\beta,\delta,\gamma)=\frac{1}{100}(1,1,1,1)$ where such was $(-1^a,-1^b,-1^c,-1^d)$, $a,b,c,d\in\mathbb{Z}_2$

Discoveries:

#1) Scaling parameters uniformly ✗ Δ gait
✓ ↑ time of convergence

#2) Strength of parameters ∝ gait
e.g. ↑ ipsi + ↓ contra = pace!