The Symmetry Perspective

by Ian Stewart & Martin Golubitsky

WESLEE NGUYEN

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INTRODUCTION TO SYMMETRY

- Why study symmetry? Describes patterns (passive)
  " pattern formation (active)
  - ⟶ Authors interested w/ its ties to nonlinear dynamics
  - ⟶ Focus on general, then specific cases in book

CHAPTER 1: Steady-State Bifurcation

- Q: What's the mechanism of speciation?
  - → Darwin's "survival of the fittest" failed on day-to-day level
  - → Idea 1: **Allopatric Speciation**: "Founder" pop. isolated from main pop.
    → gene flow between pops. off ⇒ now evolves
  - → Idea 2: **Sympatric Speciation**: Biological processes turn off gene flow even when new species remains mixed
  - Symmetries enter as organisms are biologically indistinguishable ∧ when broken → bifurcation = speciation event!
  - **Steady States**: $x(t) = x_0, \forall t$
    - If system has sym. group Γ, then eq. is invariant under $\Sigma \subset \Gamma$; if $\Sigma \subsetneq \Gamma$, then eq. **Breaks Symmetry**: key for pattern formation

- Q1.1: Say there's 3 clumps of identical organisms w/ phenotypes $(x,y,z) \in \mathbb{R}^3$, w/ $\dot{x} = \lambda x + x^2 - x - y$. Propose a system that models this problem.

SOLUTION w/ "internal" dynamics $\lambda x + x^2 - x$ w/ coupling y
- → Yet suggests clump x reacts to y is different from x
- → Ok, but from x's POV, y & z are identical, so instead: $-y-z$
- → "Identical" means y has same internal dynamics & similary coupling & so does z

MATH454

Note this is the normal form of a transcritical bifurcation!

$$\boxed{\begin{aligned} \dot{x} &= \lambda x - (x+y+z) + x^2 \\ \dot{y} &= \lambda y - (x+y+z) + y^2 \\ \dot{z} &= \lambda z - (x+y+z) + z^2 \end{aligned}}$$

ANSWER

Observe no matter how x,y,z are permutted, same eq. → symmetric under S_3 !

→ Steady State is **Trivial** iff ∃ full symmetry e.g. $x=y=z=\emptyset$ above

Linearize: $J(x,y,z) = \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{bmatrix}$

truncate to 1st order e.g. remove x^2

EIGENVECTORS	∧ VALUES
$v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\lambda-3$
$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$	λ
$v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$	λ

Stable when all λ's $< \emptyset$ → $\lambda < \emptyset$, else it's Unstable when $\lambda > \emptyset$

What about equilibria of $\lambda, x \to \emptyset$? $\Rightarrow \lambda x + x^2 = \lambda y + y^2 = \lambda z + z^2 = x + y + z$

By symmetry, assume $x = y \neq z$. Then: $\lambda x + x^2 = \lambda z + z^2 \longrightarrow \lambda(x-z) + \underbrace{(x+z)}(x-z) = \emptyset \Rightarrow x + z = -\lambda$

$\lambda x + x^2 = 2x + z$ $\quad \to z = -x - \lambda$

$) = x - \lambda$

$x^2 + x(\lambda - 1) - \lambda = \emptyset$

$x = y = \dfrac{-\lambda - 1 \pm \sqrt{\lambda^2 - 6\lambda + 1}}{2}$ if $\lambda \sim \emptyset$ $\hookrightarrow \approx 1 - 3\lambda - 4\lambda^2 + \mathcal{O}(\lambda^3)$

$x = \lambda + 2\lambda^2 + \mathcal{O}(\lambda^3)$

$y = \lambda + 2\lambda^2 + \mathcal{O}(\lambda^3)$

$z = -2\lambda - 2\lambda^2 + \mathcal{O}(\lambda^3)$ $\longleftarrow$ Binomial Theorem

$\to x + y + z = 2\lambda^2 + \mathcal{O}(\lambda^3)$

$\hookrightarrow$ Observe that when $\lambda = \emptyset$, branches are tangent to $x + y + z = \emptyset$ @ origin, so is kernel of linearization @ $\lambda = x = y = z = \emptyset$, which is S_3-invariant

- **Theorem**: Solutions of symmetric ODEs may not be symmetric, but the set of solutions must be, where sym.-breaking soln can bifurcate w/ Δ of param

- Q 1.2: Let $\dot{x}_i = \lambda x_i - \sum_{j=1}^{N} x_j + x_i^2 - x_i^3$, $1 \leq i \leq N$

Observe ODE is S_N-invariant. One way to study is by plotting trials w/ λ amped a tiny amount per run, termed Ramping the Bifurcation Parameter

- Finite Groups in Book: D_m, $\mathbb{Z}_m$, S_m
- Compact Continuous Groups in Book:
  - $\hookrightarrow$ Circle Group: $S^1 = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$
  - $\hookrightarrow$ Special Orthogonal Group: $SO(2)$, w/ all rotations $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad \det R_\theta = 1$
  - $\hookrightarrow$ Orthogonal Group: $\mathcal{O}(n)$ is all rotations in $\mathbb{R}^n$, $n \times n$ $A \ni AA^T = I$ (invertible)
    - ■ More generally, $SO(n)$ is $O(n)$ w/ $\det A = 1$
  - $\hookrightarrow$ m-Torus: $\mathbb{T}^m = S^1 \times \dots \times S^1$

- Non-Compact Continuous Lie Groups:
  - $\hookrightarrow$ Euclidean Group in Plane: $E(2)$; all rotations, reflections, translations (all isometries)
  - $\hookrightarrow$ Special Euclidean Group: $SE(2) \leq E(2)$, no reflections (preserves chirality)

- Lie Group: Smooth manifold w/ multiplication $m(g,h) = gh$ are smooth (a.k.a. C^∞) inversion $i(g) = g^{-1}$

Hausdorff & 2nd Countable + $\forall x \in M$, $\exists$ neighborhood $N \ni x$ s.t. $N \cong$ homeomorphic $\mathbb{R}^n$

- **Group Action** of Group G on set S is a group homomorphism $S \to S$, $S \circlearrowleft^{G}$
  - $\longrightarrow$ w/ lie group Γ & V.S. V, we say Γ acts on $V \longleftrightarrow \exists$ smooth homomorphism $\rho: \Gamma \to GL(V)$; denote $\underbrace{\gamma v}_{\text{simplified}} = \underbrace{\rho(\gamma)(v)}_{\rho \text{ turns } \gamma \text{ into a homomorphism } V \to V}$, where
    (1) $(\gamma\delta)v = \gamma(\delta v)$ (associativity)
    (2) $1v = v$ (where $1 \in \Gamma$)

- **Propositions**: $\rho(\Gamma) = \hat{\Gamma} \subset GL(V)$ & if ρ is injective, then $\hat{\Gamma} \cong \Gamma$
  - $\longrightarrow$ **Corollary**: If Γ is compact, so is $\hat{\Gamma}$

- Ex. 1.3: Let $x = (x_1, \dots, x_N)$ w/ $\sigma x = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(N)})$

Proof of Associativity: Let $(y_1, \dots, y_N) = \sigma x$; $y_j = x_{\sigma^{-1}(j)}$. Then:

$$\gamma(\sigma x) = \gamma y = (y_{\gamma^{-1}(1)}, \dots, y_{\gamma^{-1}(N)}) = (x_{\sigma^{-1}\gamma^{-1}(1)}, \dots, x_{\sigma^{-1}\gamma^{-1}(N)})$$
$$= (x_{(\gamma\sigma)^{-1}(1)}, \dots, x_{(\gamma\sigma)^{-1}(N)}) = (\gamma\sigma)x \quad \square$$

- **Theorem**: Every compact Lie group Γ acting on $\mathbb{R}^n$ has $\Gamma \cong \underbrace{M \leq O(n)}_{\text{some subgroup}}$

- Say we have parameterized ODEs $\dot{x} = f(x, \lambda)$, $f: \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$, $f \in C^\infty$, $\lambda \in \mathbb{R}^r$. Then:
  - $\longrightarrow$ $\gamma \in O(n)$ is a **Symmetry** if $\forall$ solns $x(t)$, $\gamma x(t)$ is also a soln

- **Theorem**: γ is a symmetry iff $f: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is **γ-Equivariant**:
  $f(\gamma x, \alpha) = \gamma f(x, \alpha)$, $\forall \gamma \in \Gamma$, $x \in \mathbb{R}^n$

Proof: Let $y(t) = \gamma x(t)$. Then: $\dot{y}(t) = f(y(t)) = f(\gamma x(t))$
$$= \gamma \dot{x}(t) = \gamma f(x(t)) \quad \square$$

- For $v \in \mathbb{R}^n$, the $\underbrace{\textbf{Isotropy}}_{\text{same in all orientations}}$ **Subgroup** of v is $\Sigma_v = \underbrace{\{\gamma \in \Gamma \mid \gamma v = v\}}_{\text{all symmetries leaving } v \text{ invariant, essentially stabilizer, } Stab(v)} \leq \Gamma$

· **Ex. 1.4:** In Q1.1, recall $\exists\, 4$ eq. solns: $(0,0,0)$, $\begin{matrix}(u,u,v)\\(u,v,u)\\(v,u,u)\end{matrix}$: $\quad u = \dfrac{-\lambda+1-\sqrt{\lambda^2-6\lambda+1}}{2}$, $\quad v = \dfrac{-\lambda+1+\sqrt{\lambda^2-6\lambda+1}}{2}$

What are each of its Isotropy Subgroups?

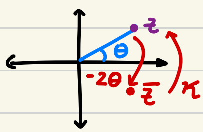
$(0,0,0)$: clearly all of $\boxed{S_3}$

(u,u,v): clearly (12) leaves it invariant, but that's it, so $\boxed{\{e,(12)\}}$

(u,v,u): $\boxed{\{e,(13)\}}$; (v,u,u): $\boxed{\{e,(23)\}}$; observe $\left.\begin{matrix}(12)\\(23)\\(13)\end{matrix}\right)$ conjugate in S_3!

· **Ex. 1.5:** Let $\Gamma = O(2)$ act on $\mathbb{C}$ w/ actions $\theta z = e^{i\theta}z$ (rotate)
$\kappa z = \bar{z}$ (reflect)

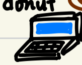
- Say $z=\emptyset$. Then $\Sigma_0 = O(2)$ (TRIVIAL)
- $z \in \mathbb{R}^\times$. Then $\Sigma_z = \{\emptyset, \kappa\}$ $(\cong \mathbb{Z}_2)$
- Say $z \neq \emptyset$, $z = re^{i\theta}$. Then $\Sigma_z = \{\emptyset, \underbrace{\kappa(-2\theta)}_{\text{conjugate to } \kappa}\}$ $(\cong \mathbb{Z}_2)$



· **Ex. 1.6:** Recall Q1.2; that has numerical soln x w/ values $\begin{matrix}a>0\\b<0\end{matrix}$. If $A = \{j \mid x_j = a\}$, $B = \{j \mid x_j = b\}$, w/ $N = 100 \longrightarrow |A| = 16$, $|B| = 84$. What's Σ_x?

SOLUTION

$$\Sigma_x = \boxed{S_{16} \times S_{84}} \leq S_{100} = \Gamma$$

· Say $x = (x_1, x_2, x_3, x_4, x_5)$ w/ $\begin{matrix}x_1, x_2, x_3\\ \& \ x_4, x_5\end{matrix}$ "equal" e.g. $\begin{matrix}\text{chocy donut}\\ \text{laptop}\end{matrix}$  . Then

$\Sigma_x = S_3 \times S_2$. More generally w/ $x \in \mathbb{R}^N$, partition $\{1,\dots,N\}$ into disjoint blocks $B_1, \dots, B_\kappa$ where $x_i = x_j \Leftrightarrow \exists\, \alpha \ni i,j \in B_\alpha$. Let $b_i = |B_i|$. Then

$$\boxed{\Sigma_x = \prod_{n=1}^{\kappa} S_{b_n}}$$

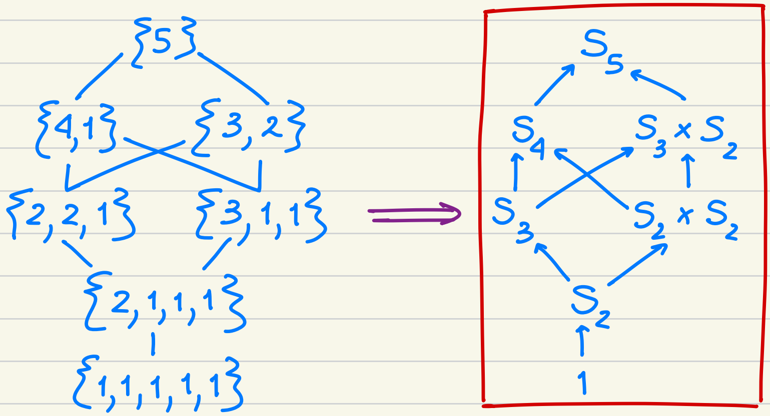
· If $x \in \mathbb{R}^n$ is an eq. of $\dot{x} = f(x,\lambda)$ & $\gamma \in \Gamma$, then so is γx as $f(\gamma x) = \gamma f(x) = \gamma\emptyset = \emptyset$, so the Group Orbit of x is $\Gamma x = \{\gamma x \mid \gamma \in \Gamma\}$

$\longrightarrow$ **Proposition:** x & γx has conj. isotropy subgroups w/ $\Sigma_{\gamma x} = \gamma \Sigma_x \gamma^{-1}$

- Let $H=\{H_i\}$, $K=\{K_j\}$ be 2 conj. classes of isotropy subgroups of Γ. Define **Partial Ordering**, transitive antisymmetric reflexive relation on *some* (not ALL!), $H \precsim K \Leftrightarrow H_i \subset K_j$. The **Isotropy Lattice** of Γ in its action on $\mathbb{R}^n$ is the set of all conj. classes of isotropy subgroups partially ordered by $\precsim$ (not strictly a "lattice")

- **Q1.7. What's the isotropy lattice of S_5?**

$\boxed{\text{SOLUTION}}$ Say $P_1=\{5,3\}$ & $P_2=\{4,2,1,1\}$. Then P_2 is a refinement of P_1, so $P_2 \precsim P_1$. Then we extend the logic to $\{5\}$:



- W/ $\Sigma \le \Gamma$, the **Fixed-Point Subspace** of Σ is $\mathrm{Fix}\,\Sigma = \{v \in \mathbb{R}^n \mid \sigma v = v, \forall \sigma \in \Sigma\}$

- **Q1.8: What's $\underbrace{\mathrm{Fix}\,S_3}_{\{(0,0,0)\}}$ & $\underbrace{\mathrm{Fix}\,\langle(12)\rangle}_{\{(u,u,v)\}}$ from Q1.1?**

- **Theorem**: Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be Γ-equivariant w/ $\Sigma \le \Gamma$. Then $f(\mathrm{Fix}\,\Sigma) \subseteq \mathrm{Fix}\,\Sigma$

Proof: Let $v \in \mathbb{R}^n$ & $\sigma \in \Sigma$. Then $\sigma v = v \Rightarrow \sigma f(v) = f(\sigma v) = f(v) \in \mathrm{Fix}\,\Sigma$ ∎

$\longrightarrow$ **Corollary**: Let $x(t)$ be a soln trajectory of an equivariant ODE. Then $\Sigma_{x(t)} = \Sigma_{x(0)}$, $\forall t \in \mathbb{R}$. Namely, isotropy subgroups remain constant along trajectories.

Proof: As $\mathrm{Fix}\,\Sigma_{x(0)}$ is flow-invariant, then $x(t) \in \mathrm{Fix}\,\Sigma_{x(0)}$, $\forall t \in \mathbb{R}$. Thus $\Sigma_{x(0)} \le \Sigma_{x(t)}$. As $x(0)$ lies on the trajectory through $x(t)$, then $\Sigma_{x(t)} \le \Sigma_{x(0)} \Rightarrow \Sigma_{x(0)} = \Sigma_{x(t)}$ ∎

- Thus, we find an eq. soln w/ isotopy subgroup Σ by restricting vector field to subspace $\mathrm{Fix}\,\Sigma$. The bigger Σ is $\longrightarrow$ smaller dim $\mathrm{Fix}\,\Sigma$ is. So find solns starting w/ largest isotopy subgroups, then go down the lattice

- $V \subset \mathbb{R}^n$ is a **Γ-Invariant Subspace** if $\gamma V = V, \forall \gamma \in \Gamma$

  $\hookrightarrow$ Ex: $V = \{0\}$, $\mathbb{R}^n$, $\mathrm{Fix}\,\Gamma$

  $\hookrightarrow$ Ex: Let $SO(2)$ act on $\underbrace{\mathbb{R}^4}_{\mathbb{R}^2 \times \mathbb{R}^2}$ by $\theta(v,w) = (R_\theta(v), R_\theta(w))$, $\theta \in [0, 2\pi)$. Fix $\psi \in SO(2)$ & let $V_\psi = \{(v, R_\psi(v)) \mid v \in \mathbb{R}^2\}$. Then V_ψ is $SO(2)$-Invariant

  Pf: $\theta(v, R_\psi(v)) = (R_\theta(v), R_\theta R_\psi(v)) = (R_\theta(v), R_\psi R_\theta(v)) \in V_\psi$ $\square$

  ⊛ $\hookrightarrow$ Ex: Let S_N act on $\mathbb{R}^n$ by permuting coordinates. Then:
  $$\left.\begin{aligned} V_0 &= \mathbb{R}(1, \dots, 1) \\ V_1 &= \{(x_1, \dots, x_N) \mid \textstyle\sum x_i = 0\} \end{aligned}\right) \text{ are } S_N\text{-Invariant}$$

- $W \subset \mathbb{R}^n$ is **Irreducible** if the only Γ-invariant subspaces of W are W & $\{0\}$

  $\hookrightarrow$ So if Γ acts irreducibly on $\mathbb{R}^n$, then $\mathrm{Fix}\,\Gamma = \{0\}$ (f(0)=0 by last theorem) or $\mathbb{R}^n$ (if acts trivially).

- **Theorem:** Let $\Gamma \subset O(n)$ be a compact Lie group. Then $\exists\, \Gamma$-irreducible subspaces $V_1, \dots, V_s \ni \mathbb{R}^n = \bigoplus_{i=1}^{s} V_i$

  Proof: $\exists\, \Gamma$-invariant inner product $(\cdot,\cdot)$. If Γ acts irreducibly on $\mathbb{R}^n$, done! Else, $\exists\, \Gamma$-invariant subspace $W \neq \{\emptyset\}$ or $\mathbb{R}^n$. Then $\exists\, W^\perp$ is Γ-invariant as $v \in W^\perp$, $w \in W$, & $\gamma \in \Gamma$ has $(\gamma v, w) = (v, \gamma^{-1} w) = (v, w') = 0$ for $w' \in W$. So $\gamma v \in W^\perp \Longrightarrow \mathbb{R}^n = W \oplus W^\perp$. As $n \in \mathbb{N}$, the decomposition will terminate. $\square$

- Q1.8: Let S_N act on $\mathbb{R}^N$ by permutting coordinates. Define V_0, V_1 as ⊛.
  Let S_N act on V_1. Then the isotropy subgroups of S_N are the same, yet $\dim \mathrm{Fix}\,\Sigma = k-1$ when Σ corresponds to a partition into k blocks (df of $x_1 + \dots + x_n = 0$ is -1)

- Let Γ act on V w/ linear map $A: V \to V$. A **Commutes** w/ Γ if $A\gamma = \gamma A, \forall \gamma \in \Gamma$, namely A is **Γ-Equivariant**

- **Proposition**: Let A be Γ-Equivariant. Then:

  (1) $\ker A$ is a Γ-Invariant Subspace

  Pf: $Av = 0 \Rightarrow A(\gamma v) = \gamma(Av) = 0$ $\square$

  $\longrightarrow$ If Γ acts Irreducibly, then $\ker A = \{0\}$ or $\mathbb{R}^n$

  (2) If A commutes w/ Γ & $\exists A^{-1}$, then A^{-1} also commutes w/ Γ

  Pf: $A^{-1}\gamma = (\gamma^{-1}A)^{-1} = (A\gamma^{-1})^{-1} = \gamma A^{-1}$ $\square$

- **Proposition**: Let D be the set of commuting linear maps when the group acts on an irreducible subspace. Then D is a linear subspace & is a skew field. So $D = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

  $\hookrightarrow$ Γ acts Absolutely Irreducibly on $\mathbb{R}^n$ if $D = \mathbb{R}$

- A Steady-State Bifurcation occurs @ an eq. where the linearized eq. has a zero λ & no λ lies on the imaginary axis

- **Theorem**: Say there's a branch of group-invariant equilibria $y(\lambda)$ to ODE $\dot{y} = F(y, \lambda)$ & $\exists$ steady-state bifurcation @ $\lambda = \lambda_0$, a.k.a. $A_0 = (dF)_{y(\lambda_0), \lambda_0}$

  Then: (1) $\emptyset$ is the only eigenvalue of A_0 on the imaginary axis

  (2) The generalized eigenspace corresponding to $\emptyset$ is $\ker A_0$

  (3) Γ acts absolutely irreducibly on $\ker A_0$

- We want to find bifurcations

  $\longrightarrow$ One method, Center Manifold Reduction, reduces eq. to $\dot{x} = g(x, \lambda)$ where $g: \ker A_0 \times \mathbb{R} \to \ker A_0$

  $\longrightarrow$ Another is this: consider ODE system $\dot{y} = F(y, \alpha)$, $y \in \mathbb{R}^N$, $\alpha \in \mathbb{R}^k$ w/ $F(0,0) = 0$ & $(dF)_{0,0}$ singular. Let $K = \ker(dF)_{0,0} \neq \{0\}$ & $R = \text{range}(dF)_{0,0}$. Consider $\hat{K} = \text{coker}(dF)_{0,0}$ w/ $\hat{R} = \text{corange}(dF)_{0,0}$

  linearization @ the origin

  det = 0

  $\mathbb{R}^N = K \oplus \hat{K}$
  $= R \oplus \hat{R}$. Then $F(x, w, \alpha) = \emptyset$ $\begin{pmatrix} x \in K \\ w \in \hat{R} \end{pmatrix}$ ...

$\Leftrightarrow$ (1) $EF(x,w,\alpha)=\emptyset$ w/ $E:\mathbb{R}^N\to\mathcal{R}$ is the projection w/ kernel
(2) $(I-E)F(x,w,\alpha)=\emptyset$ $\text{Ker}E=\hat{\mathcal{R}}$. Then:

identity map; lies in $\mathcal{R}$; lies in $\hat{\mathcal{R}}$

$$\frac{d}{dw}EF(x,w,\alpha)\Big|_0 = E(dF)_{0,0}\,\Big|\hat{\mathcal{K}} = (dF)_{0,0}\,\Big|\hat{\mathcal{K}}$$

nonsingular $\longrightarrow$ Implicit Function Theorem

$\longrightarrow \exists!\, w:\mathcal{K}\times\mathbb{R}^k\to\hat{\mathcal{K}}$ w/ $w(0,0)\ni EF(x,w(x,\alpha),\alpha)\equiv\emptyset$.

$\therefore$ solving $F=\emptyset \Leftrightarrow$ solving $f(x,\alpha)=(I-E)F(x,w(x,\alpha),\alpha)=\emptyset$ ⊛
where $f:\mathcal{K}\times\mathbb{R}^k\to\hat{\mathcal{R}}$, $f(0,0)=0$, $(df)_{0,0}=0$

$=\mathbb{R}^n$ dim of original space $= \dim\mathcal{K}=\dim\text{Ker}(dF)_{0,0}$

· **Theorem:** Solns to $F(y,\alpha)=0$, $y\in\mathbb{R}^N$ are in 1-to-1 correspondence w/ solns to $F(x,\alpha)=0$, $x\in\mathbb{R}^n$ near the origin, termed the **Lyapunov-Schmidt Reduction**

· **Theorem:** Say $\Gamma\subset O(n)$ is a compact Lie group w/ $F:\mathbb{R}^N\times\mathbb{R}^k\to\mathbb{R}^N$ Γ-Equivariant. Then $f:\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$ from ⊛ where $\mathbb{R}^n=\text{Ker}(dF)_{0,0}$ can be chosen to be Γ-Equivariant

Proof. $(dF)_{0,0}$ commutes w/ Γ & so $\mathcal{K}$ & $\mathcal{R}$ are both Γ-invariant subspaces. We can choose $\hat{\mathcal{K}}$ & $\hat{\mathcal{R}}$ to be Γ-invariant e.g. $\hat{\mathcal{K}}=\mathcal{K}^\perp$. Compute:

$$\begin{aligned} EF(x,w_\gamma(x,\alpha),\alpha) &= EF(x,\gamma^{-1}w(\gamma x,\alpha),\alpha)) \\ &= E\gamma^{-1}F(\gamma x, w(\gamma x,\alpha),\alpha) \\ &= \gamma^{-1}EF(\gamma x, w(\gamma x,\alpha),\alpha)=0 \end{aligned}$$

By uniqueness of IFT, $w_\gamma=w$ or $\gamma w(x,\alpha)=w(\gamma x,\alpha)$. Then by ⊛, follows that f is Γ-Equivariant. $\square$

· An isotopy subgroup Σ is Axial if $\dim\text{Fix}\,\Sigma=1$

· Ex. 1.9: What are the axial subgroups of S_N acting on $V=\{x\in\mathbb{R}^n \mid \sum_{i=1}^N x_i=0\}$

SOLUTION Up to conjugacy are the form $\boxed{S_p\times S_q}$ w/ $p+q=N$, $1\le p\le N/2$ as N is partitioned into 2 blocks $\longrightarrow$ fixed-point space has dim 1

Equivariant Branching Lemma: Let $\Gamma \subset O(n)$ be a compact Lie group where Γ acts absolutely irreducibly on $\mathbb{R}^n$ & $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be Γ-equivariant. Then $f(0,\lambda) \equiv 0$ & $(df)_{0,\lambda} = c(\lambda) I$

Additionally, say $c(0) = 0$ (bifurcation condition), $c'(0) \neq 0$ (λ crossing condition), & $\Sigma \leq \Gamma$ is axial. Then $\exists!$ branch of solns to $f(x,\lambda) = 0$ from $(0,0)$ where the symmetry of the solns is Σ

Proof: As $c'(0) \neq 0 \Rightarrow \dim \operatorname{Fix} \Sigma = 1$. Let $0 \neq v \in \operatorname{Fix}$. Then $\operatorname{Fix} \Sigma = \mathbb{R}\{v\}$.
So $f(tv,\lambda) = h(t,\lambda) v$ w/ $h: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ & $f(0,\lambda) = h(0,\lambda) = 0$.
By Taylor's Thm: $h(t,\lambda) = t k(t,\lambda)$ where $k(0,0) = c(0) = 0$ & $k_\lambda(0,0) = c' \neq 0$. By the IFT, $\exists!\ \lambda(t) \ni \lambda(0) = 0$ & $k(t,\lambda(t)) \equiv 0$.
Thus $f(tv,\lambda(t)) = 0$. Finally, the solns lie in $\operatorname{Fix}\Sigma$, which have symmetries including Σ, so $\Sigma \leq \Sigma_v$. As Σ is an isotropy subgroup: $\Sigma_v = \Sigma$ $\square$

Ex. 1.10: Let $\Gamma = \mathbb{Z}_2$ w/ $x \to -x$

For $\Sigma = 1$, we have $\operatorname{Fix}\Sigma = \mathbb{R}$, $\dim \operatorname{Fix} \Sigma = 1$.
By the Equivariant Branching Lemma, $\exists$ branch of solns w/ trivial symmetry, w/ $f(-x,\lambda) = -f(x,\lambda) \Rightarrow f(0,\lambda) = -f(0,\lambda)$
$\Rightarrow f(0,\lambda) = 0$. By Taylor's Thm: $f(x,\lambda) = a(x,\lambda)x$ &
$a(-x,\lambda) = a(x,\lambda) = b(x^2,\lambda)$

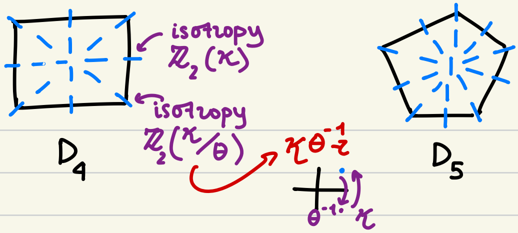
W/ $b_{x^2} \neq 0 \neq b_\lambda$, $f(x,\lambda) \sim (x^2 \pm \lambda)x + \dots$, the normal form of a Pitchfork Bifurcation

Ex. 1.11: Let $\Gamma = O(2)$ act on $\mathbb{R}^2 \cong \mathbb{C}$ by $\theta z = e^{i\theta} z$, $\theta \in [0, 2\pi)$ & $\kappa z = \bar{z}$

For $\Sigma = \mathbb{Z}_2(\kappa) = \{1, \kappa\}$, we have $\operatorname{Fix}\Sigma = \mathbb{R}$. Then at an $O(2)$ steady-state bifurcation, $\exists$ equilibria w/ reflectional symmetry

Ex. 1.12: Let $\Gamma = D_m$ act on $\mathbb{C}$ by $\theta z = e^{i\theta} z$, $\theta = \frac{2\pi}{m}$, & $\kappa z = \bar{z}$

For $\Sigma = \mathbb{Z}_2(\kappa) = \{1, \kappa\}$, we still have $\operatorname{Fix}\Sigma = \mathbb{R}$ & $\exists$ solns w/ reflectional symmetry



- Speciation was traditionally thought to be allopatric as discont. effects stem from discont. causes, but by end of 1990s, considered cont. causes: sympatric
  - ⟶ Could have N animals but rapidly Δ, so instead interpreted as **Placeholders for Organism Dynamics (PODs)**: Coarse-grained phenotypically related clusters
  - ⟶ Assumes sympatric speciation occurs in a **Panmictic Population**: All organisms can interbreed, & that environment is uniform, a.k.a. $\exists$ diversity but each organism experiences the same diversity #
  - ⟶ Consider N PODs. State of POD j is described by x_j belongs to a **Phenotypic Space** $\mathbb{R}^r$, where x_j^i is a phenotypic character & x_j is the entire phenotype
    - Ex: $N = 100$ PODs w/ $x_j \in \mathbb{R}^5$ e.g. (weight, ear angle, head angle, wing span, call rate)