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CHAPTER 1: Manifold & Smooth Maps

1.1 Definitions

- $f: \underset{\text{arbitrary}}{\overset{\text{open}}{U}} \subset \mathbb{R}^n \to \mathbb{R}^m$ is **smooth** if $\exists$ partials of all orders
  " X " " " " if such can be locally extended to smooth map on open sets
  ⟶ a.k.a. $\forall x \in X$, $\exists$ open $U \subset \mathbb{R}^n$ & smooth $F: U \to \mathbb{R}^m \ni F|_{U \cap X} = f|_{U \cap X}$ (Local Property)
  ⟶ Say $X \subset \mathbb{R}^N$. X is a **k-Dimensional Manifold** if it's locally diffeomorphic to $\mathbb{R}^k$; a.k.a. $\forall x \in X$, $\exists$ neighborhood $x \in V \subset X \ni V \underset{\text{diffeomorphic}}{\cong_D} \overset{\text{open}}{U} \subset \mathbb{R}^k$

  - **Diffeomorphism** $\phi: U \to V$ is a **Parameterization** of V, & $\phi^{-1}: V \to U$ is a **Coordinate System** on V
  - For k smooth funcs. $(x_1, \dots, x_k)$ on V are **Coordinate Functions** when $\phi^{-1} = \underbrace{(x_1, \dots, x_k)}_{\text{each funcs... NOT coordinates}}$, so $v \in V \longrightarrow (x_1(v), \dots, x_k(v)) \in U$, & $\dim X = k$

- Ex: $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
  Case 1: Say (x,y) is in $\cap$. Then $\phi_1(x) = (x, \sqrt{1-x^2})$ has $W = (-1,1) \longleftrightarrow \cap$
  Case 2: " " " $\cup$. Then $\phi_2(x) = (x, -\sqrt{1-x^2})$ " " $\longleftrightarrow \cup$
  Case 3: What's left is $(1,0)$ & $(-1,0)$. So have R: $\phi_3(y) = (\sqrt{1-y^2}, y)$
  L: $\phi_4(y) = (-\sqrt{1-y^2}, y)$
  ↳ ∴ S^1 is locally diffeomorphic to $\mathbb{R}^1 \iff S^1$ is a $\boxed{\text{1D Manifold}}$

- Say $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^M$ are manifolds. If $\dim X = k$, $x \in X$, $\exists$ open $W \subset \mathbb{R}^k$ & local parameterization $\phi: W \to X$ around x. Similarly for $\dim Y = \ell$, $y \in Y$, $\exists$ open $U \subset \mathbb{R}^\ell$ & " " $\psi: U \to Y$
  ⟶ Define $\phi \times \psi: W \times U \to X \times Y$, $(\phi \times \psi)(w,u) = (\phi(w), \psi(u))$
  ⟶ $W \times U$ is open & $\phi \times \psi$ is a local parameterization of $X \times Y$ around arbitrary (x,y)
- If X & Z are manifolds in $\mathbb{R}^N$ & $Z \subset X$, then Z is a **Submanifold** of X
- **Lemma:** Any open set of X is a submanifold of X
- **Theorem:** If X, Y are manifolds, so $X \times Y$ & $\dim(X \times Y) = \dim X + \dim Y$

1.2 Derivatives & Tangents

- Say $f: \overset{\text{open}}{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ is smooth. For $x, h \in \mathbb{R}^n$, the **Derivative** of f @ x in the direction of h is:

$$Df_x(h) = \lim_{t \to 0} \frac{f(x+th)-f(x)}{t} \in \mathbb{R}^m, \text{ \& more generally: } Jf(x) = \begin{pmatrix} f^1_{x_1}(x) & \cdots & f^1_{x_n}(x) \\ \vdots & & \\ f^m_{x_1}(x) & \cdots & f^m_{x_n}(x) \end{pmatrix}$$

↳ for $f = (f_1, \dots, f_m)$

↳ **Lemma**: $Df_x : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map.

↳ **Chain Rule**: Such is commutative

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "Df_x"] \arrow[rr, bend right, "D(g\circ f)_x"'] & \mathbb{R}^m \arrow[r, "Dg_{f(x)}"] & \mathbb{R}^l \end{tikzcd}$$

↳ **Lemma**: If f is linear, then $Df_x = f$, $\forall x \in U$

- Say that $X \subset \mathbb{R}^N$ & $\phi: U \to X$ is a local parameterization around x, where $U \subset \mathbb{R}^k$ is open w/ $\phi(0) = x$

↳ Best linear approximation to ϕ @ 0 is $u \longrightarrow \phi(0) + D\phi_0(u) = x + D\phi_0(u)$

↳ **Tangent Space** of X @ x is $T_x(X) := \{D\phi_0(u) \mid u \in \mathbb{R}^k\}$, a vector subspace of $\mathbb{R}^N$ & $x + T_x(X)$ is the closest flat approximation through x on X

∘ **Tangent Vector** to $X \subset \mathbb{R}^N$ @ $x \in X$ is $v \in T_x(X)$

↳ Say $\exists$ another parameterization $\psi: V \to X$ w/ $\psi(0) = x$, $\phi(U) = \psi(V)$. Then $h = \psi^{-1} \circ \phi : U \to V$ is a diffeomorphism $\Rightarrow \phi = \psi \circ h \Rightarrow D\phi_0 = D\psi_0 \circ Dh_0$, ∴ $\mathrm{Im}(D\phi_0) \subset \mathrm{Im}(D\psi_0)$ & vice versa, ∴ $T_x(X)$ is well-defined

↳ Choose $\overset{\text{open}}{W} \subset \mathbb{R}^N$ & smooth $\Phi' : \mathbb{R}^N \to \mathbb{R}^k$ that extends ϕ^{-1}. Then $\Phi' \circ \phi$ is identity of U & ∴ $\mathbb{R}^k \xrightarrow{D\phi_0} T_x(X) \xrightarrow{D\Phi'_x} \mathbb{R}^k$ is also identity. ∴ $D\phi_0 : \mathbb{R}^k \to T_x(X)$ is an isomorphism

↳ **Proposition**: $\exists k \in \mathbb{N} \ni T_x(X) \cong \mathbb{R}^k$ & ∴ $\dim(T_x(X)) = k$

- Say $f: X \to Y$ is a smooth map between manifolds @ $x \ni f(x) = y$. Then:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r] & V \arrow[u, "\psi"'] \end{tikzcd}$$

$h := \psi^{-1} \circ f \circ \phi$ by commutativity

$\Longrightarrow$

already $D\phi_0(U)$ already $D\psi_0(V)$

$$\begin{tikzcd} T_x(X) \arrow[r, "Df_x"] & T_y(Y) \\ \mathbb{R}^k \arrow[u, "D\phi_0"] \arrow[r, "Dh_0"'] & \mathbb{R}^l \arrow[u, "D\psi_0"'] \end{tikzcd}$$

say $\phi(0) = x$ say $\psi(0) = y$

best linear approx. = should linear transform & ∴ Df_x is valid map

$\Longrightarrow$ Recall $D\phi_0$, $D\psi_0$ are isomorphisms. ∴ define

$$\boxed{Df_x = D\psi_0 \circ Dh_0 \circ D\phi_0^{-1}}$$

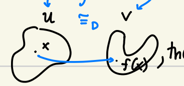
- **Chain Rule**: If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then $D(g \circ f)_x = Dg_{f(x)} \circ Df_x$

1.3 Immersions

obv. not $\forall$, just $\exists$ 2 neighborhoods

neighborhoods

$U \cong_D V$

- If X, Y are manifolds in the same dim, then if smooth $f: X \to Y$ makes  then f is a **Local Diffeomorphism** @ x

- **Inverse Function Theorem (InFT)**: Say $f: X \to Y$ is smooth w/ Df_x @ x is an isomorphism. Then f is a local diffeomorphism @ x.

just check $\det Df_x \neq 0$

a.k.a. $T_xX \cong_J T_{f(x)}Y$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "id"] & U \arrow[u, "\psi"'] \end{tikzcd}$$

parameterizations

a.k.a. if Df_x is an isomorphism, we can choose local coordinates around x & $f(x)$ $\ni$ f appears as identity @ x $\Longleftrightarrow$ diagram commutes

$\xrightarrow[\text{generally}]{\text{more}}$

$f: X \to Y$ & $f': X' \to Y'$ are **Equivalent** if $\exists$ diffeomorphisms α & $\beta \ni$ below commutes:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X' \arrow[u, "\alpha"] \arrow[r, "f'"'] & Y' \arrow[u, "\beta"'] \end{tikzcd}$$

" f & f' are same up to diffeomorphism "

a.k.a. space can twist/bend/etc. but the immersion prevents collapse, squashing, etc.

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gives criteria to when image is a manifold via embeddings

- If $Df_x: T_x(X) \to T_{f(x)}(Y)$ is injective, then f is an **Immersion** @ x

  $\longrightarrow$ **Canonical Immersion**: Inclusion map $\mathbb{R}^k \to \mathbb{R}^l$ for $k \leq l$ where $(a_1, \dots, a_k) \to (a_1, \dots, a_k, 0, \dots, 0)$

- **Local Immersion Theorem**: Say $f: X \to Y$ is an immersion @ x. Then $\exists$ local coordinates @ x & $y \ni$ $f(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$, a.k.a. f is locally equivalent to the canonical immersion near x.

Proof: Choose any local parameterizations to make such commutative:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "g"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

$\phi(0) = x$
$\psi(0) = f(x)$

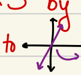
As $Dg_0: \mathbb{R}^k \to \mathbb{R}^l$ is injective, by Δ of basis in $\mathbb{R}^l$, say it has $l \times k$ $\begin{pmatrix} I_k \\ 0 \end{pmatrix}$ where I_k is $k \times k$ identity matrix.

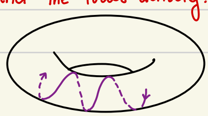
⊛

$$\begin{tikzcd} X \arrow[r] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\text{canonical}", "\text{immersion}"'] & V \arrow[u, "\psi \circ G"'] \end{tikzcd}$$

Define $G: U \times \mathbb{R}^{l-k} \to \mathbb{R}^l$, $G(x, z) = g(x) + (0, z)$. Observe G maps open set $\mathbb{R}^l \to \mathbb{R}^l$ & that $DG_0 = I_l$ $\therefore$ InFT $\Rightarrow$ G is local diffeomorphism of $\mathbb{R}^l$ @ 0. Yet $g = G \circ$ (canonical immersion). As ψ & G are local diffeomorphisms @ 0, $\therefore$ so is $\psi \circ G$, $\therefore$ such is a local parameterization of Y around y. Moreover, if U & V shrink sufficiently, ⊛ commutes. $\square$

$\longrightarrow$ **Corollary**: If f is an immersion @ x, then it's an immersion in a neighborhood of x.

- Ex: Take $g: \mathbb{R}^1 \to S^1$ by $g(t) = (\cos 2\pi t, \sin 2\pi t)$, & $G: \mathbb{R}^2 \to S^1 \times S^1$ by $G(x, y) = (g(x), g(y))$, both local diffeomorphisms. Now define $\mathbb{R}^1 \to \mathbb{T}$ by restricting G to  slope $\in \mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ as G is a local diffeomorphism, immersion wraps $\mathbb{R}^1$ around the Torus densely!



wrapping $\mathbb{R}^1$ around $\mathbb{T}$

- $f: X \to Y$ is **Proper** if $\forall$ compact $V \subset Y$, $f^{-1}(V)$ is compact in X
  - $\longmapsto$ A proper, injective immersion is an **Embedding**

- **Theorem:** An embedding $f: X \to Y$ maps X diffeomorphically onto a submanifold of Y.

  Proof: For $f(X)$ to be a manifold, WTS if $W \subset X$ is open, $f(W)$ is open in $f(X)$. BWOC, say $f(W)$ isn't open, so $\exists \{y_i\} \subset f(X) \ni \{y_i\} \not\subset f(W)$, yet $y_i \to y \in f(W)$. As $\{y, y_i\}$ is compact, (sequential compactness) (f is proper) $f^{-1}(\{y, y_i\})$ is compact. Say $y_i \xrightarrow{f^{-1}} x_i$, $y \xrightarrow{f^{-1}} x \in W$. As $\{x, x_i\}$ is compact, say $x_i \to z \in X$ (compact $\Leftrightarrow$ sequentially compact). $\therefore f(x_i) \to f(z)$ (f is immersion, $\therefore$ smooth, $\therefore$ has sequential continuity), & f is injective $\Rightarrow x = z$. Yet W is open, so for large i, $x_i \in W$. Yet $y_i \notin f(W)$, a contradiction. $\therefore f(X)$ is a manifold.

  As f is a local diffeomorphism $X \to f(X)$ & is bijective $\Rightarrow f^{-1}: f(X) \to X$ is well-defined. Yet f^{-1} is locally smooth, so f is indeed a global diffeomorphism. □

1.4 Submersions

- Say $f: X \to Y$. If $Df_x: T_x(X) \to T_y(Y)$ is surjective, then f is a **Submersion** @ x (think of projections; useful to say when preimage @ a point is a submanifold)
  - $\longmapsto$ **Canonical Submersion:** $\mathbb{R}^k \to \mathbb{R}^l$, $k \geq l$ where $(a_1, \dots, a_k) \longrightarrow (a_1, \dots, a_l)$

- **Local Submersion Theorem:** Say $f: X \to Y$ is a submersion @ x. Then $\exists$ local coordinates around x & $f(x)$ $\ni f(x_1, \dots, x_k) = (x_1, \dots, x_l)$, $k \geq l$, a.k.a. f is locally equivalent to the canonical submersion near x.

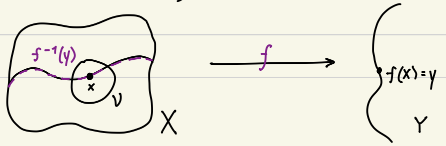
  Proof:
$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "g"] & V \arrow[u, "\psi"'] \end{tikzcd}$$
  $\phi(0) = x$
  $\psi(0) = f(x)$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U' \arrow[u, "\phi \circ G^{-1}"] \arrow[r, "\text{canonical}", "\text{submersion}"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

  As $Dg_0: \mathbb{R}^k \to \mathbb{R}^l$ is surjective, by linear Δ of coordinates in $\mathbb{R}^k$, such has $l \times k$ matrix $(\mathcal{I}_l \mid 0)$. Define $G: U \to \mathbb{R}^k$, $G(a) = (g(a), a_{l+1}, \dots, a_k)$. Thus $\det DG_0 = \mathcal{I}_k \Rightarrow G$ is a local diffeomorphism @ 0 by InFT $\therefore \exists G^{-1}$ as a diffeomorphism of some open neighborhood U' of 0 into U. By construction, $g = (\text{canonical submersion}) \circ G$, so $g \circ G^{-1}$ is a canonical submersion. $\therefore$ the L. diagram commutes. □

  - $\longmapsto$ **Corollary:** If f is a submersion @ x, f is a submersion in some neighborhood of x.

- Say $f: X \to Y$ is a submersion of $x \in f^{-1}(y)$. Select local coordinates around x & $y \ni f(x_1, \dots, x_k) = (x_1, \dots, x_l)$ w/ $y \to (0, \dots, 0)$; a.k.a. near x, $f^{-1}(y)$ is set of pts. $(0, \dots, 0, x_{l+1}, \dots, x_k)$
  - $\longmapsto$ a.k.a. let V denote neighborhood of x where $(x_1, \dots, x_k)$ is defined. Then $f^{-1}(y) \cap V$ is set of pts. where $x_1 = \dots = x_k = 0$, $\therefore x_{k+1}, \dots, x_l$ form a coordinate system on $f^{-1}(y) \cap V$



- For smooth $f: X \to Y$ between manifolds, $y \in Y$ is a **Regular Value** for f if $Df_x: T_x(X) \to T_y(Y)$ is surjective $\forall x \in X \ni f(x) = y$. Else y is a **Critical Value**

- **Preimage Theorem:** If y is a regular value of $f: X \to Y$, then $f^{-1}(y)$ is a submanifold of X with $\dim f^{-1}(y) = \dim X - \dim Y$ (a.k.a. if every pt. on level set is non-degen, it's a submanifold)

- Say y is regular w/ $f: X \to Y$ smooth:
  - $\longrightarrow$ Case 1: $\dim X > \dim Y$; then f is a submersion $\forall x \in f^{-1}(y)$
  - $\longrightarrow$ Case 2: $\dim X = \dim Y$; then f is a local diffeomorphism $\forall x \in f^{-1}(y)$
  - $\longrightarrow$ Case 3: $\dim X < \dim Y$; then x is a critical value $\forall x \in f(X)$, & x' is regular $\forall x' \in X / f(X)$

- By the Preimage Theorem, constructing submanifolds is significantly easier
  - $\longrightarrow$ Ex: Consider $f: \mathbb{R}^k \to \mathbb{R}$, $f(x) = \|x\|^2$

Observe $Df_a = (2a_1, \dots, 2a_k)$ for $a = (a_1, \dots, a_k)$. $\therefore Df_a$ is surjective unless $f(a) = 0$, so $x, \forall x \in \mathbb{R} \setminus \{0\}$, is a regular value of f. $\therefore S^{k-1} = f^{-1}(1)$ is a $k-1$ dimensional manifold.

  - $\longrightarrow$ Ex: Consider $O(n) = \{A \mid AA^T = I\} \subset M_{n \times n} \cong \mathbb{R}^{n^2}$. Show $O(n)$ is a manifold.

Proof: AA^T is symmetric & V.S. $S(n)$ of all symmetric $n \times n$ matrices is a submanifold of $M(n) \ni S(n) \cong_D \mathbb{R}^k$ for $k = \frac{n(n+1)}{2}$ w/ $f: M(n) \to S(n)$, $f(A) = AA^T$ smooth. Observe $O(n) = f^{-1}(I)$, so WTS I is regular.

$$\begin{aligned} Df_A(B) &= \lim_{s \to 0} \frac{f(A+sB) - f(A)}{s} \\ &= \lim_{s \to 0} \frac{(A+sB)(A+sB)^T - AA^T}{s} \\ &= \lim_{s \to 0} \frac{AA^T + sBA^T + sAB^T + s^2 BB^T - AA^T}{s} \\ &= \lim_{s \to 0} BA^T + AB^T + sBB^T \\ &= BA^T + AB^T \end{aligned}$$

Note $T_A M(n) = M(n)$ & $T_{f(A)} S(n) = S(n)$ as subspaces of Euclidean Spaces. I is regular $\iff Df_A: M(n) \to S(n)$ is surjective $\forall A \in O(n) \iff \forall C \in S(n), \exists B \in M(n)$ that solves $Df_A(B) = C$ or $BA^T + AB^T = C$. W/ C symmetric $\Rightarrow C = \frac{C}{2} + \frac{C^T}{2}$. $\therefore BA^T = \frac{C}{2}$. Yet as $A^T A = I$ $\Rightarrow B = \frac{CA}{2}$. Then:

$$Df_A(B) = \frac{(CA)}{2} A^T + A\left(\frac{CA}{2}\right)^T = \frac{C}{2}(AA^T) + \frac{C^T}{2}(AA^T) = \frac{C}{2} + \frac{C^T}{2} = C$$

$\therefore f$ is a submersion $\forall A \in f^{-1}(I) \Rightarrow O(n)$ is a submanifold of $M(n)$. Moreover:
$\dim O(n) = \dim M(n) - \dim S(n) = n^2 - \frac{n(n+1)}{2} = \boxed{\frac{n(n-1)}{2}}$ $\square$

- **Lie Group:** A manifold & group w/ a smooth b.o.

Q: Say $g_1, \dots, g_l$ are smooth, real-valued funcs. on manifold X w/ dimension $k \geq l$. When is $Z := \{\text{common zeros}\}$ a manifold?

SOLUTION: Let $g := (g_1, \dots, g_l): X \to \mathbb{R}^l$. As $Z = g^{-1}(0)$, Z is a submanifold if 0 is a regular value of g.
More precisely: each g_i is smooth $X \to \mathbb{R}$ w/ $D(g_i)_x : T_x(X) \to \mathbb{R}$, a linear functional on a V.S.
Then $D(g_i)_x$ is surjective $\iff$ l functionals $D(g_1)_x, \dots, D(g_l)_x$ are L.I. on $T_x(X)$, a.k.a. the l functions $g_1, \dots, g_l$ are **Independent** @ x. $\therefore$

Proposition: If smooth, real-valued functions $g_1, \dots, g_l$ on X are independent $\forall z \in Z := Ker(g_1, \dots, g_n)$, then Z is a submanifold of X w/ $\dim Z = \dim X - l$

$\longrightarrow$ **Codimension** of arbitrary submanifold Z of X is $\operatorname{codim} Z := \dim X - \dim Z$

partial converses of the last prop.

Proposition: If y is a regular value of smooth $f: X \to Y$, then $f^{-1}(y)$ (submanifold by Preimage Thm) can be cut out by independent funcs.
Proof: Choose diffeomorphism h of neighborhood W of y w/ a neighborhood of origin in $\mathbb{R}^l$ where $h(y) = 0$. Let $g = h \circ f$ & check that 0 is a regular value of g. $\therefore$ coordinate funcs. $g_1, \dots, g_l$ works! $\square$

Proposition: Every submanifold of X is locally cut out by independent functions.
Proof: Let submanifold Z have codimension l w/ $z \in Z$. Then $\exists\, l$ ind. funcs. $g_1, \dots, g_l$ defined on open neighborhood W of z in $X \ni Z \cap W$ is the common vanishing set of g_i by the converse of the Local Immersion Theorem. $\square$

Proposition: Let $f: X \to Y$ be smooth w/ regular $y \in Y$, $Z := f^{-1}(y)$. Then $Ker(Df_x) = T_x(Z)$, $\forall x \in Z$
Proof: As f is constant on $Z \implies Df_x = 0$ on $T_x(Z)$. Yet $Df_x : T_x(X) \to T_y(Y)$ is surjective, $\therefore$ $\dim(Ker(Df_x)) = \dim(T_x(X)) - \dim(T_y(Y)) = \dim X - \dim Y = \dim Z$, $\therefore$ $T_x(Z)$ is a subspace of the kernel w/ the same dimension as the complete kernel $\implies T_x(Z)$ is the kernel! $\square$

1.5 Transversality

Q: If $f: X \to Y$ is smooth w/ $Z \subset Y$ a submanifold, what are properties of $f^{-1}(Z)$?

by definition; locally diffeomorphic to open in $\mathbb{R}^n \leftrightarrow \forall x, \exists$ neighborhood diffeomorphic

Recall $f^{-1}(Z)$ is a manifold $\iff \forall x \in f^{-1}(Z)$ has a neighborhood U in $X \ni f^{-1}(Z) \cap U$ is a manifold

$\longrightarrow$ If $|Z| = 1$, then a neighborhood of Z is the zero set of independent funcs. $g_1, \dots, g_l$ w/ $l = \operatorname{codim} Z$

- Near x, then $f^{-1}(Z)$ is zero set of $g_1 \circ f, \dots, g_l \circ f$ $\quad \mathbb{R}^l \to W$
- Let $g = (g_1, \dots, g_l)$ be a submersion of y. Then $(g \circ f)^{-1}(0)$ is a submanifold if 0 is regular
- As $D(g \circ f)_x = Dg_{f(x)} \circ Df_x$, then $D(g \circ f)_x : T_x(X) \to \mathbb{R}^l$ is surjective $\iff Dg_{f(x)}$ carries $Im(Df_x) \to \mathbb{R}^l$. Yet $Dg_{f(x)} : T_{f(x)}(Y) \to \mathbb{R}^l$ is surjective w/ $Ker = T_y(Z)$
- $\therefore$ $Dg_{f(x)}$ carries a subspace of $T_{f(x)}(Y) \to \mathbb{R}^l$ if that subspace & $T_{f(x)}(Z)$ span all of $T_{f(x)}(Y)$

$\longrightarrow$ $g \circ f$ is a submersion @ $x \in f^{-1}(Z) \iff Im(Df_x) + T_y(Z) = T_y(Y)$

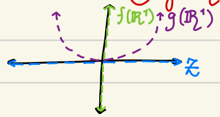
→ Say f is transversal to submanifold Z w/ $f \pitchfork Z$ ("clean" intersections enables to say that preimage of submanifold is indeed a submanifold)

Theorem: If $f: X \to Y$ is smooth w/ $f \pitchfork Z \subset Y$ submanifold, then $f^{-1}(Z) \subset X$ is a submanifold. Moreover, $\text{codim}\, f^{-1}(Z) = \text{codim}\, Z$

→ Recall $f^{-1}(Z)$ is zero set of l ind. funcs. $g_1 \circ f, \dots, g_l \circ f$, then $\text{codim}\, f^{-1}(Z) = l = \text{codim}\, Z$

→ Also recall that if $|Z| = 1$, then $T_y(Z)$ is zero subspace of $T_y(Y)$, ∴ $f \pitchfork y$ if $Df_x(T_x(X)) = T_y(Y), \forall x \in f^{-1}(y)$, ∴ transversality generalizes regularity

Ex: Consider $f: \mathbb{R} \to \mathbb{R}^2, f(t) = (0, t)$, w/ $Z \subset \mathbb{R}^2$ the x-axis. Let $g: \mathbb{R}^1 \to \mathbb{R}^2, g(t) = (t, t^2)$. Then $f \pitchfork Z$, but g isn't transversal to Z.



submanifolds $\left\{ X \subset Y,\ Z \subset \right.$ (i: inclusion)

$X \subset Y$ means $x \in i^{-1}(Z) \Rightarrow x \in X \cap Z$ & that $Di_x: T_x(X) \to T_x(Y)$ is also an inclusion map,

∴ $i \pitchfork Z \Leftrightarrow \forall x \in X \cap Z, T_x(X) + T_x(Z) = T_x(Y)$

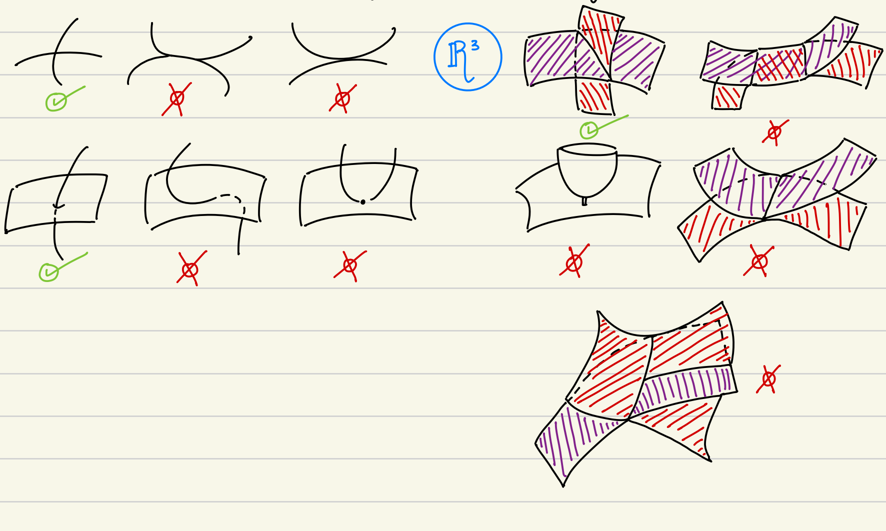
→ Then say submanifolds X & Z are transversal, denoted $X \pitchfork Z$

Theorem: For submanifolds $X, Z \subset Y$, then $X \pitchfork Z \Rightarrow X \cap Z$ is a submanifold of Y. Moreover: $\text{codim}(X \cap Z) = \text{codim}\, X + \text{codim}\, Z$

→ Say X is cut out of Y by $k = \text{codim}\, X$. Then these $k + l$ funcs. are the independent funcs. for $X \cap Z$
" Z " " " $l = \text{codim}\, Z$

→ Recall transversality also depends on space Y e.g. 2 intersecting curves is a submanifold ✓ $\mathbb{R}^2$ ✗ $\mathbb{R}^3$

Ex: $\mathbb{R}^2$



✓ Transversal
✗ Transversal

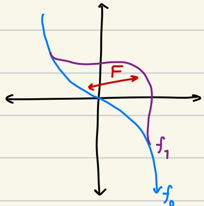
Curves & Surfaces

$\boxed{1.6}$ Homotopy & Stability

Let $\mathcal{I}=[0,1]$ & $f_0, f_1 : X \to Y$ be smooth. They're **Homotopic** w/ $f_0 \sim f_1$ if $\exists$ smooth $F: X\times \mathcal{I} \to Y \ni$ $F(x,0)=f_0(x)$, $F(x,1)=f_1(x)$, where F is the **Homotopy** between f_0 & f_1. If $\exists F$, then say they're equivalent as part of the same **Homotopy Class**

$\longrightarrow$ A property is **Stable** if whenever $f_0 \sim f_{t,\, t\in I}$ both possesses the property, then $\exists \epsilon>0 \ni f_t$, $\forall t\in[0,\epsilon)$, also has the property

$\longrightarrow$ Ex:



Is property that ______ is stable?

... such passes through the origin ✗

... the intersection w/ the x-axis being transversal ✓

Stability Theorem: The following classes of smooth maps of compact manifold $X \to$ manifold Y are stable:
(I) local diffeomorphisms, (II) diffeomorphisms, (III) immersions, (IV) submersions,
(V) embeddings, (VI) maps transversal to some submanifold $Z \subset Y$

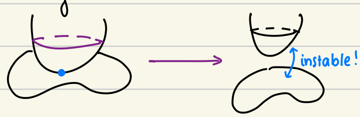
⊛ vacuously true if $=\emptyset$

$\longrightarrow$ So recall $X \pitchfork Z$ in Y if $\forall x\in X\cap Z$, $T_x(X)+T_x(Z)=T_x(Y)$. Then for 2 curves in $\mathbb{R}^3$, they can't intersect transversally unless they never intersect for

algebraically: $1+1<3$

geometrically: any homotopy $\longrightarrow$ prevents intersection stability

- Explains why other intersections are nontransversal even when "arithmetically-ok": instability
- Ex:



Proof: I, III, IV, VI have same proof as local diffeomorphisms are just immersions when $\dim X=\dim Y$. If $f_0\sim f_t$, X is compact $\Rightarrow$ any open neighborhood $X\times\{0\}\subset X\times \mathcal{I}$ has $X\times[0,\epsilon]$ for small ϵ. $\therefore$ WTS each $(x_0,0)$ has neighborhood $\mathcal{U}\subset X\times\mathcal{I} \ni Df_x$ is injective for $(x,t)\in\mathcal{U}$ for III. As claim is local, only need to prove for open pieces $X\subset\mathbb{R}^k$, $Y\subset\mathbb{R}^l$. Injectivity of $D(f_0)_{x_0} \Rightarrow$ its $l\times k$ Jacobian $\left(\frac{\partial(f_0)_i}{\partial x_j}(x_0)\right)$ has a $k\times k$ submatrix w/ nonzero determinant. Yet each $\frac{\partial (f_t)_i}{\partial x_j}(x)$ as a func. on $X\times \mathcal{I}$ is cont. As the determinant is also continuous, $k\times k$ submatrix must be nonsingular $\forall (x,t)$ in a neighborhood $(x_0,0)$.

Pf for IV & VI are nearly identical. For V, define $G: X\times\mathcal{I}\to Y\times\mathcal{I}$, $G(x,t)=(f_t(x),t)$. BWOC, $\exists$ $t_i\to 0$ & $x_i\neq y_i \in X \ni G(x_i,t_i)=G(y_i,t_i)$. W/ X compact $\Rightarrow$ sequentially compact $\Rightarrow$ rewrite subsequence s.t. $x_0\to x_i$, $y_0\to y_i$. Thus $G(x_0,0)=\lim G(x_i,t_i)=\lim G(y_i,t_i)=G(y_0,0) \Rightarrow f(x_0)=f_0(y_0)$, so if f is injective, $x_0=y_0$.

As $DG_{(x_0,0)} = \begin{pmatrix} D(f_0)_{x_0} & \begin{matrix}a_1\\ \vdots \\ a_l\end{matrix} \\ 0\cdots 0 & 1\end{pmatrix}$ w/ $D(f_0)_{x_0}$ is injective, $\exists k$ ind. rows. $\therefore DG_{(x_0,0)}$ has $k+1$ ind. rows, making such injective. $\therefore G$ is an immersion around $(x_0,0)$ & $\therefore$ injective on a neighborhood. Yet for large i, (x_i,t_i) & (y_i,t_i) lie inside, a contradiction. Finally, II is left as an exercise. □

1.7
Sard's & Morse

- $A \subset \mathbb{R}^l$ has **Measure Zero** if $\forall \varepsilon > 0$, $\exists$ countably many rectangles $\{S_i\} \ni A \subset \bigcup_{i\in I} S_i$, $\sum_{i \in I} Vol(S_i) < \varepsilon$
  - $\longrightarrow$ For manifold Y, $C \subset Y$ also has **Measure Zero** if $\forall$ local parameterization ψ of Y, then $\psi^{-1}(C)$ has measure zero
  - $\longrightarrow$ **Lemma**: If $A \subset \mathbb{R}^l$ has measure zero & $g: \mathbb{R}^l \to \mathbb{R}^l$ is smooth, then $g(A)$ has measure zero

- **Sard's Theorem**: The set of critical values of smooth map between manifolds $X \to Y$ has measure zero.
  - $\longrightarrow$ **Corollary**: The regular values of any " " " " " " are dense in Y.

countable union of measure zero still has measure zero, so take $C_i = \{\text{critical for } C\}$ & have $C = \bigcup C_i$

$\left\{ \right.$ $\therefore$ if $f_i : X_i \to Y$ w/ $i \in I$ countable are all smooth maps between manifolds, then all $y \in Y$ that's regular to all f_i are dense in Y.

- Say $f: X \to Y$ is smooth. Then $x \in X$ is a **Regular Point** of f if $Df_x : T_x(X) \to T_y(Y)$ is surjective, else x is a **Critical Point** of f
  - $\longrightarrow$ $\therefore$ y is a regular value$^{\in Y}$ $\Longleftrightarrow$ $\forall x \in f^{-1}(y)$ is a regular point$^{\in X}$

    " critical value $\Longleftrightarrow$ " " " critical point
  - $\longrightarrow$ Ex: Let $f: X \to Y$ be the constant map between manifolds. Say $\dim X$ & $Y > 0$, $f(x) = y_0$

Observe Df_x is zero map, $\therefore$ critical values (in Y) has measure zero, as that's y_0

critical points (in X) do $\boxed{\text{not}}$!!! have measure zero, as that's X

- Let $Hf_{ij} = (f_{x_i x_j})$ be the **Hessian Matrix**. If Hf_x is nonsingular ($\det H(x) \neq 0 \Leftrightarrow$ invertible) @ critical pt. x, then x is a **Non-Degenerate Critical Point** of f
  - $\longrightarrow$ Let $g: \mathbb{R}^k \to \mathbb{R}^k$, $g = (f_{x_1}, \dots, f_{x_k})$. Then $Df_x = 0 \Longleftrightarrow g(x) = 0$ & $Dg_x = Hf_x$
  - $\longrightarrow$ $\therefore$ if x is nondegenerate, then $g(x) = 0$ & g maps a neighborhood of x diffeomorphically onto a neighborhood of x, so g can't be 0 anywhere else in the " "

- **Morse Lemma**: Say $a \in \mathbb{R}^k$ is a nondegenerate critical point of f. Then $\exists$ local coordinate system $(x_1, \dots, x_k)$ around $a \ni f = f(a) + \sum (Hf_a)_{ij} x_i x_j$ near a (quadratic!)
  - $\longrightarrow$ **Corollary**: All nondegenerate critical points of f are isolated.

- Say $f: X \to \mathbb{R}$ has a critical point @ x & that ϕ is a local parameterization of the origin $\to x$. $\therefore$ 0 is a critical point for $f \circ \phi$ as $D(f \circ \phi)_0 = Df_x \circ D\phi_0 = 0$. Say x is **Nondegenerate** for f if 0 is too for $f \circ \phi$
  - $\longrightarrow$ **Lemma**: Say f on $\mathbb{R}^k$ has a nondegenerate critical point @ 0 & say ψ is a diffeomorphism w/ $\phi(0) = 0$. Then $f \circ \psi$ also has a " " " ".

Proof: Let $f' = f \circ \psi$. Then $f'_{x_i}(x) = \sum_\alpha f_{x_\alpha}[\psi(x)] (\psi_\alpha)_{x_i}(x)$. $\therefore$ $f'_{x_i x_j}(0) = \sum_\alpha \sum_\beta f_{x_\alpha x_\beta}(0)(\psi_\alpha)_{x_i}(0)(\psi_\beta)_{x_j}(0) + \sum_\alpha f_{x_\alpha}(0)(\psi_\alpha)_{x_i x_j}(0)$. 0 is critical pt. of f $\Rightarrow$ 2^{nd} sum is 0. a.k.a. $Hf' = (D\psi_0)^T \cdot H(D\psi_0)$. $\therefore$ w/ $\det(D\psi_0) \neq 0$ & $\det(D\psi_0)^T = \det(D\psi_0)$. $\therefore$ $\det(Hf) \neq 0 \Rightarrow \det(Hf') \neq 0$ ☐

Morse Functions: Funcs. whose critical points are all nondegenerate

Say manifold $X \subset \mathbb{R}^N$ w/ usual coordinate funcs. $x_1, \dots, x_N$. If f is a func. on X & $a = (a_1, \dots, a_N)$, define $f_a := f + a_1x_1 + \dots + a_Nx_N$

Theorem: No matter $f: X \to \mathbb{R}$, for almost every $a \in \mathbb{R}^n$, f_a is a Morse function on X.

a.k.a. $\{a \mid \text{Thm false}\}$ has measure zero

Lemma: Let f be smooth on open $U \subset \mathbb{R}^k$. Then for almost all $a \in \mathbb{R}^k$, f_a is Morse on U.

Proof: Let $g: U \to \mathbb{R}^k$, $g = (f_{x_1}, \dots, f_{x_k})$. Then $D(f_a)_p = ((f_a)_{x_1}(p), \dots, (f_a)_{x_k}(p)) = g(p) + a$. ∴ p is a critical pt. of $f_a \iff g(p) = -a$. As f_a & f have the same second partials, then $Hf_p = Dg_p$. Say a is a regular value of g. Then $g(p) = -a$ has Dg_p nonsingular. Then every critical point of f_a is nondegenerate. By Sard's Theorem, $-a$ is a regular value of g for almost all $a \in \mathbb{R}^k$. □

Proof: Say $x \in X$ w/ standard coordinate funcs $x_1, \dots, x_N$ on $\mathbb{R}^N$. Restricting to some k of $x_{i_1}, \dots, x_{i_k}$ to X is a coordinate system in a neighborhood of x. ∴ we can cover X w/ open U_α ∋ for each α, some k of $x_1, \dots, x_N$ form a coordinate system. By the Second Axiom of Countability, ∃ countably many U_α. Choose U_α w/ coordinate system $(x_1, \dots, x_k)$. For each $c = (c_{k+1}, \dots, c_N)$, consider $f_{(0,c)} + c_{k+1}x_{k+1} + c_Nx_N$. The lemma means for almost all $b \in \mathbb{R}^k$, $f_{(b,c)} = f_{(0,c)} + b_1x_1 + \dots + b_kx_k$ is a Morse function on U_α. Let $S_\alpha = \{a \in \mathbb{R}^N \mid f_a \text{ isn't a Morse func. on } U_\alpha\}$. As each horizontal slice $S_\alpha \cap \mathbb{R}^k \times \{c\}$ has measure zero in $\mathbb{R}^k$, S_α has measure zero in $\mathbb{R}^N$, $\forall \alpha$ by Fubini's Theorem. A func. has a degenerate critical point on $X \iff$ it has one in U_α. ∴ $\{a \in \mathbb{R}^N \mid f_a \text{ isn't a Morse func. on } X\} = \bigcup S_\alpha$. As union of measure zero has measure zero, follows. □

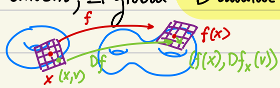
1.8 Embedding Manifolds

Q: k-dim manifold X is locally diffeomorphic to subsets in $\mathbb{R}^k$, which can be embedded in $\mathbb{R}^N$, $N \geq k$. So does there exist an N ∋ all k-dim manifolds lie in? Ex: S^1 is 1D but embeds in $\mathbb{R}^2$

⟶ Yes! Suggests ∃ "limit" to how complex manifolds can be

Tangent Bundle of manifold X in $\mathbb{R}^N$ is $T(X) = \{(x,v) \in X \times \mathbb{R}^N \mid v \in T_x(X)\}$, a.k.a. all tangent spaces which includes a copy of $X = \{(x,0)\}$

⟶ If $f: X \to Y$ is smooth, ∃ global **Derivative Map** $Df: T(X) \to T(Y)$, $Df(x,v) = (f(x), Df_x(v))$



; if $X \subset \mathbb{R}^N \to T(X) \subset \mathbb{R}^{2N}$ & $Y \subset \mathbb{R}^M \to T(Y) \subset \mathbb{R}^{2M}$

⟶ **Lemma:** Df is smooth. Proof: f smooth extends to $F: \underset{\text{open}}{U} \subset \mathbb{R}^N \to \mathbb{R}^M \Rightarrow DF: T(U) \to \mathbb{R}^{2M}$ extends Df. Let $T(U) = \underset{\text{open}}{U} \times \mathbb{R}^N \subset \mathbb{R}^{2N}$, ∴ DF is smooth. ∴ Df can be locally extended to a smooth map. □

⟶ Chain rule says smooth $X \xrightarrow{f} Y \xrightarrow{g} Z \Rightarrow Dg \circ Df : T(X) \to T(Z)$ equals $D(g \circ f)$

- $\therefore$ f diffeomorphism $\Rightarrow Df$ is too as $Df \circ Df^{-1} = D(id|_Y) = id_{T(Y)}$ & $Df^{-1} \circ Df = D(id|_X) = id_{T(X)}$ + Df is smooth, $\therefore$
- a.k.a. diffeomorphic manifolds has diffeomorphic tangent bundles
- **Lemma:** $X \cong_D Y \Rightarrow T(X) \cong_D T(Y)$
- **Lemma:** If U is open in manifold X, then U is a submanifold.

Proof: Let $x \in U$ w/ neighborhood $U_x \cong_D V_x \subset \mathbb{R}^k$ via diffeomorphism φ_x. Restrict $\varphi_x|_{A \cap U}$, which is still a diffeomorphism $A \cap U \cong_D \varphi_x(A \cap U)$, $\varphi_x|_{A \cap U}(x) = 0$, & $x \in A \cap U$. Et voilà! $\square$

Proposition: If X is a manifold w/ $f : X \to Y$ smooth, then $T(X)$ is a manifold w/ $\dim T(X) = 2 \dim X$

Proof: $W \underset{open}{\subset} X \Rightarrow W$ is a ". Then $T(W) \subset T(X) \cap (W \times \mathbb{R}^N) \subset T(X)$. As $W \times \mathbb{R}^N \underset{open}{\subset} X \times \mathbb{R}^N$, $T(W) \underset{open}{\subset} T(X)$. Say W is image of local parameterization $\Phi : U \underset{open}{\subset} \mathbb{R}^k \to W$. Then $D\Phi : T(U) \to T(W)$ is a diffeomorphism. Yet $T(U) = U \times \mathbb{R}^k \underset{open}{\subset} \mathbb{R}^{2k}$, $\therefore$ $D\Phi$ parameterizes $T(W)$ in $T(X)$. $\square$

Theorem: Every k-dimensional manifold admits an injective immersion in $\mathbb{R}^{2k+1}$

Proof: Let $X \subset \mathbb{R}^N$ be k-dimensional & $N > 2k+1$. Claim if $f : X \to \mathbb{R}^N$ is an injective immersion, $\exists$ unit $a \in \mathbb{R}^M \ni a \to a^{\perp}$ is an injective immersion too. PROOF: Define $h : X \times X \times \mathbb{R} \to \mathbb{R}^N$, $h(x,y,t) = t[f(x) - f(y)]$ & $g : T(X) \to \mathbb{R}^N$, $g(x,v) = Df_x(v)$. As $N > 2k+1$, Sard's Theorem means $\exists a \in \mathbb{R}^N$ has $0 \neq a \notin \operatorname{Im} f \cup \operatorname{Im} g$. Let $H = \{ b \in \mathbb{R}^N \mid b \perp a \}$ & π project $\mathbb{R}^N \to H$. Then $\pi \circ f : X \to H$ is injective (say $\pi \circ f(x) = \pi \circ f(y) \Rightarrow f(x) - f(y) = ta$ for some scalar t. $x \neq y \Rightarrow t \neq 0$ as f is injective, yet $h(x, y, \frac{1}{t}) = a \to \leftarrow$) & an immersion (say $0 \neq v \in T_x(X)$ s.t. $D(\pi \circ f)_x(v) = 0$. π is linear $\Rightarrow D(\pi \circ f)_x = \pi \circ Df_x(v) = 0 \Rightarrow Df_x(v) = ta$ for some scalar t. f is an immersion $\Rightarrow t \neq 0$. Yet $g(x, \frac{1}{t}) = a \to \leftarrow$) ■

As H is a $N-1$ dimensional vector subspace, follows $H \cong_T \mathbb{R}^{N-1}$, $\therefore$ $a \to a^{\perp}$ is an injective immersion. $\square$

⟶ **Corollary:** Compact manifolds has injective immersions $\Leftrightarrow$ embeddings, $\therefore$ all embed to $\mathbb{R}^{2k+1}$

Theorem: Let $X \subset \mathbb{R}^N$ be arbitrary. For open covering $\{U_\alpha\}$, $\exists$ sequence of smooth funcs $\{\theta_i\}$ on X, a Partition of Unity subordinate to $\{U_\alpha\}$, s.t. (I) $0 \leq \theta_i(x) \leq 1$, $\forall x \in X$, $i \in \mathbb{N}$, (II) each $x \in X$ has a neighborhood where all but finitely many funcs. are 0, (III) each θ_i is 0 except on some closed set contained in one U_α, & (IV) $\forall x \in X$, $\sum_i \theta_i(x) = 1$

→ **Corollary**: On any manifold X, $\exists$ proper $\rho : X \to \mathbb{R}$

Proof: Let $\{U_\alpha\}$ open cover X w/ compact closure & have $\{\theta_i\}$ be a subordinate partition of unity. Then $\rho = \sum_{i=1}^{\infty} i\theta_i$ is a well-defined smooth func. So if $\rho(x) \le j$, then @ least 1 of first j funcs. $\theta_1, \dots, \theta_j$ is nonzero @ x. $\therefore \rho^{-1}[-j,j] \subset \bigcup_{i=1}^{j} \{x \mid \theta_i(x) \neq 0\}$, a set w/ compact closure. Yet every compact set in $\mathbb{R}$ lies in some interval $[-j,j]$ $\square$

Whitney's Embedding Theorem: Every k-dimensional manifold embeds in $\mathbb{R}^{2k+1}$.

Proof: Say $X \to \mathbb{R}^{2k+1}$ is an injective immersion. Compose w/ any diffeomorphism of $\mathbb{R}^{2k+1}$ into unit ball, say $z \longrightarrow \frac{z}{1+|z|^2}$, we get " " " $f : X \to \mathbb{R}^{2k+1} \ni |f(x)| < 1, \forall x \in X$. Let $\rho : X \to \mathbb{R}$ be proper. Define injective immersion $F : X \to \mathbb{R}^{2k+2}$, $F(x) = (f(x), \rho(x))$. Drop to $\mathbb{R}^{2k+1}$ via $\pi : \mathbb{R}^{2k+2} \xrightarrow{\text{orthogonal projection}} H$ (linear space $\perp$ to $a \in \mathbb{R}^{2k+2}$)

Recall $\pi \circ F : X \to H$ is an injective immersion for almost every $a \in S^{2k+1}$, so pick a that's neither poles. Then $\pi \circ F$ is proper. Claim given bound c, $\exists$ number $d \ni \{x \in X \mid |\pi \circ F(x)| \le c\} \subset \{x \in X \mid |\rho(x)| \le d\}$. PROOF: As ρ is proper, latter is compact subset of X. $\therefore$ claim suggests preimage under $\pi \circ F$ of every closed ball in H is a compact subset of X. So BWOC, $\exists \{x_i\} \subset X \ni |\pi \circ F(x_i)| < c$ but $\rho(x_i) \to \infty$. Then by definition, $\forall z \in \mathbb{R}^{2k+2}$, $\pi(z) \in H$ where $z - \pi(z)$ is a multiple of a. $\therefore F(x_i) - \pi \circ F(x_i)$ is a multiple of a for each i, $\therefore w_i = \frac{1}{\rho(x_i)}[F(x_i) - \pi \circ F(x_i)]$. Then $i \to \infty$ has $\frac{F(x_i)}{\rho(x_i)} = \left(\frac{f(x_i)}{\rho(x_i)}, 1\right) \longrightarrow (0, \dots, 0, 1)$ as $|f(x_i)| < 1$, $\forall i$. Then $\frac{\pi \circ F(x_i)}{\rho(x_i)}$ has norm $\le \frac{c}{\rho(x_i)}$ that converges 0. $\therefore w_i \to (0, \dots, 0, 1)$. Yet each w_i is a multiple of a, $\therefore$ so is the limit. $\therefore a$ is N or S pole of $S^{2k+1} \longrightarrow \longleftarrow$ ■ Thus $\pi \circ F$ embeds X into $\mathbb{R}^{2k+1}$. $\square$