Spring 2013

信號與系統
Signals and Systems

Chapter SS-10
The z-Transform

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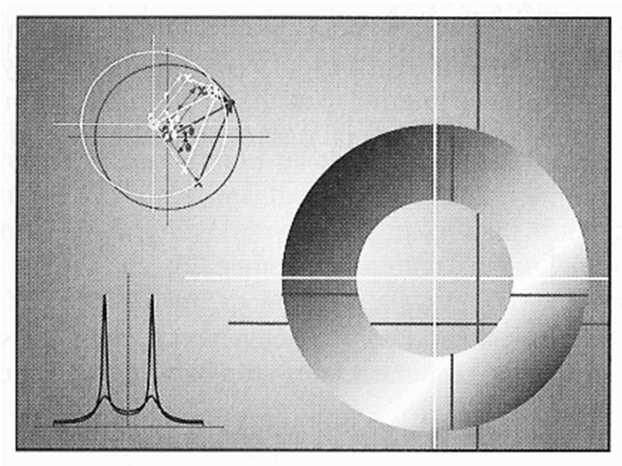
NTU-EE

Feb13 – Jun13

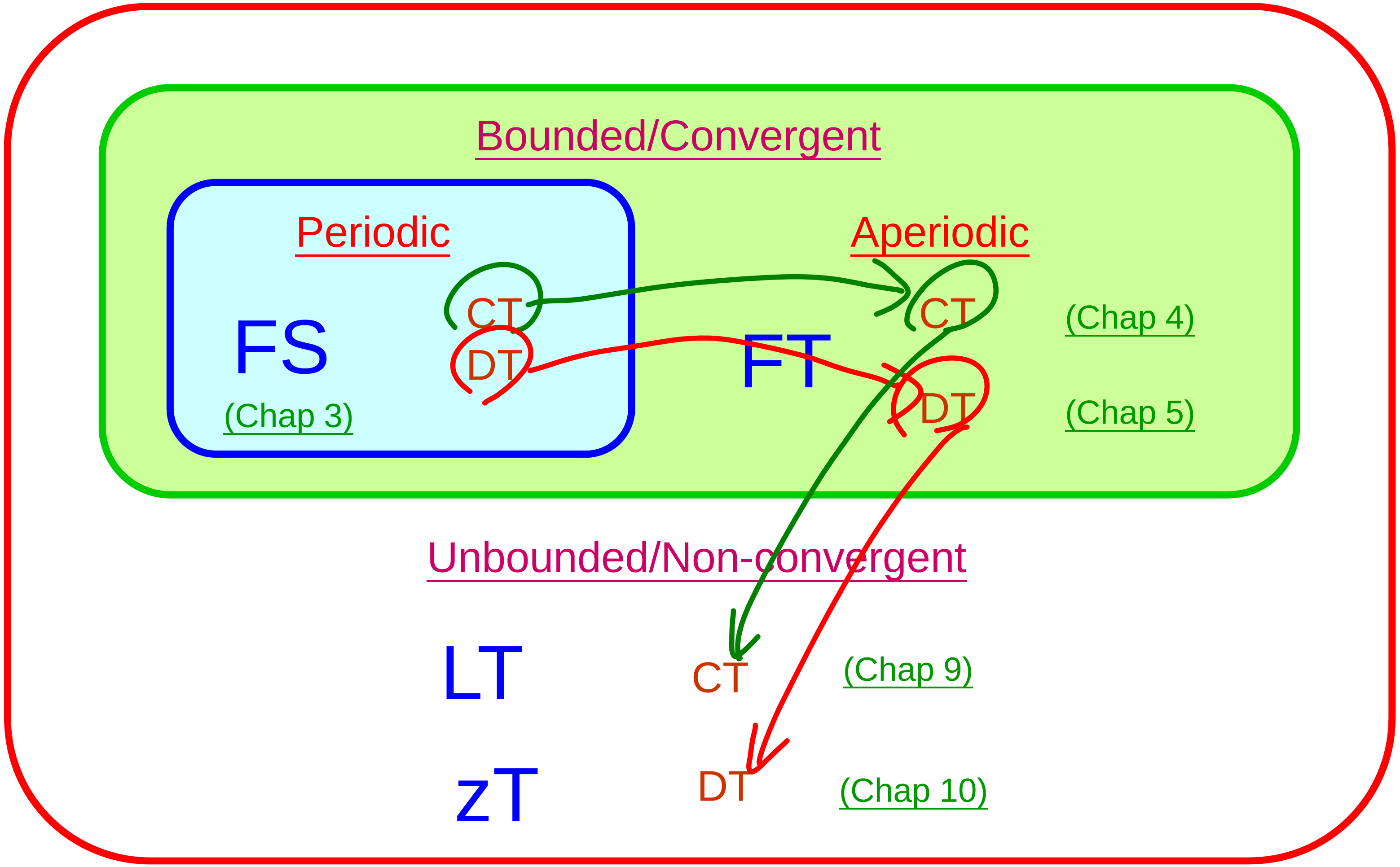


Figures and images used in these lecture notes are adopted from
"Signals & Systems" by Alan V. Oppenheim and Alan S. Willsky, 1997

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)



Time-Frequency (Chap 6)

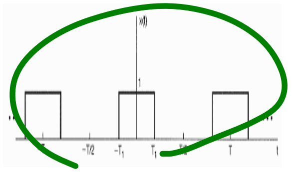
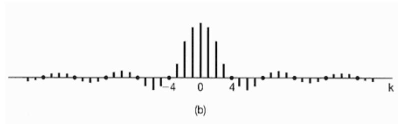
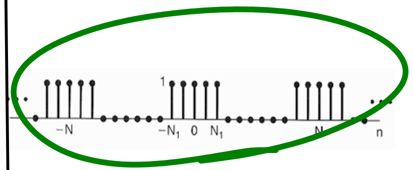
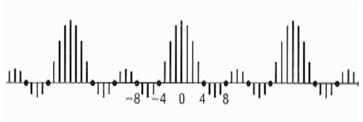
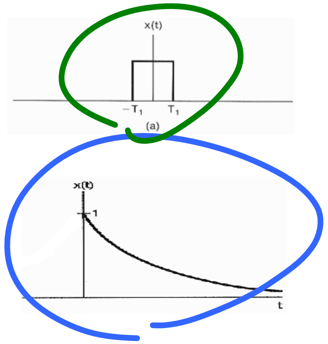
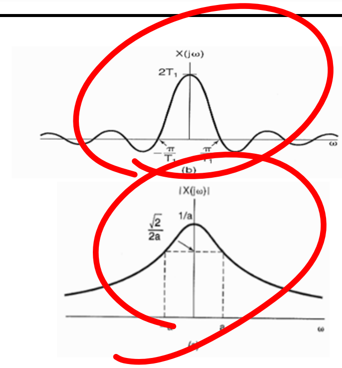
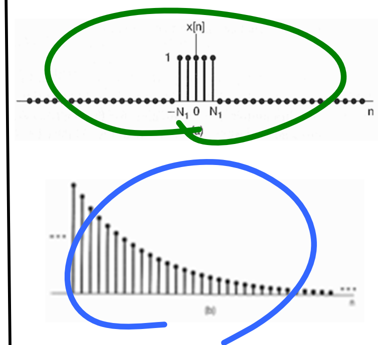
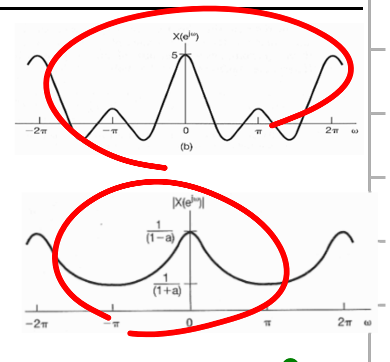
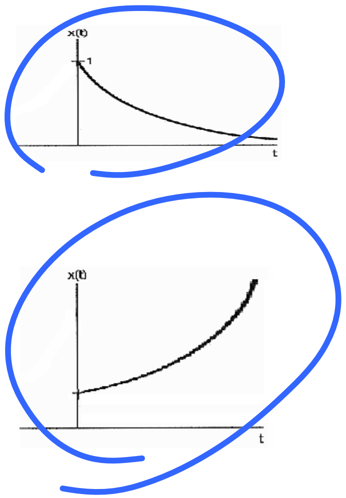
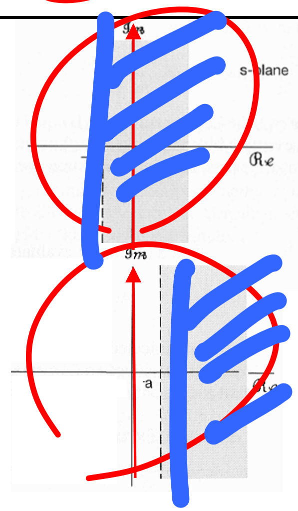
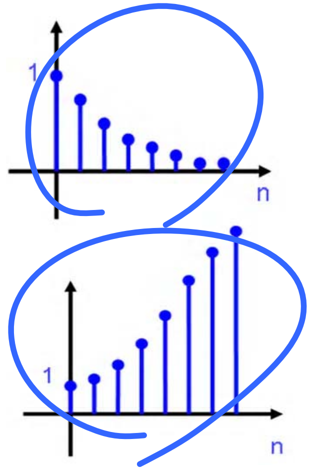
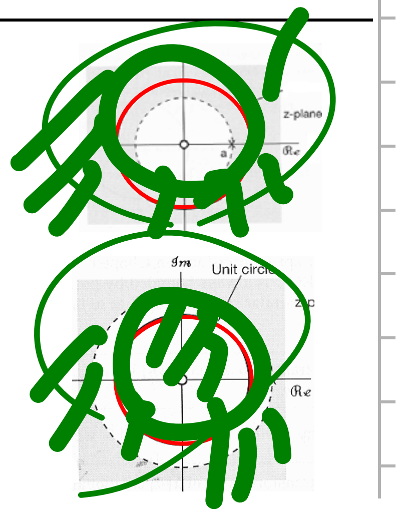
CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)

Fourier Series, Fourier Transform, Laplace Transform, z-Transform

	CT time	CT frequency	DT time	DT frequency
FS				
FT				
LT/zT				

Signal

- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the z-Transform
- Some Common z-Transform Pairs

System

- Analysis & Characterization of LTI Systems Using the z-Transforms
- System Function Algebra and Block Diagram Representations
- The Unilateral z-Transform

Brief History of the z-Transform

- The z-transform was known to Laplace, and re-introduced in 1947 by W. Hurewicz as a tractable way to solve linear, constant-coefficient difference eqns.
- It was later dubbed "the z-transform" by Ragazzini and Zadeh in the sampled-data control group at Columbia University in 1952
- The name of "the z-transform"
  - The letter "z" being a sampled/digitized version of the letter "s" in Laplace transforms.
  - Another possible source is the presence of the letter "z" in the names of both Ragazzini and Zadeh who published the seminal paper.
- The modified or advanced z-transform was later developed and popularized by E. I. Jury in 1958, 1973.
- The idea contained within the z-transform is also known as the method of generating functions around 1730 when it was introduced by DeMoivre with probability theory.
- From a mathematical view the z-transform can also be viewed as a Laurent series where one views the sequence of numbers under consideration as the (Laurent) expansion of an analytic function (the z-transform).

s^2 s^{-1}

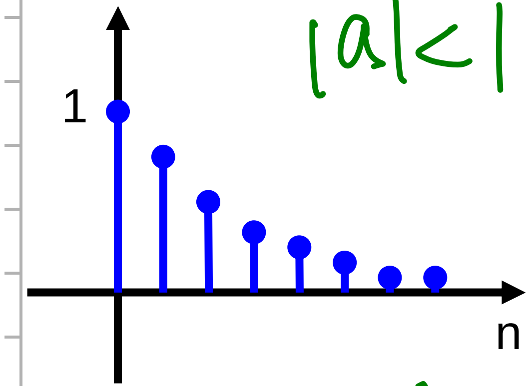
z^2 z^{-1}

- From the Fourier Transform of DT signals x[n]:

$x[n] = a^n u[n]$

FT

$|a| < 1$



$$X(e^{jw}) \triangleq \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn} = \sum_{n=-\infty}^{+\infty} x[n] \left(e^{jw}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} a^n \, u[n] \, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-jw}\right)^n = \boxed{\frac{1}{1 - a e^{-jw}}}$$

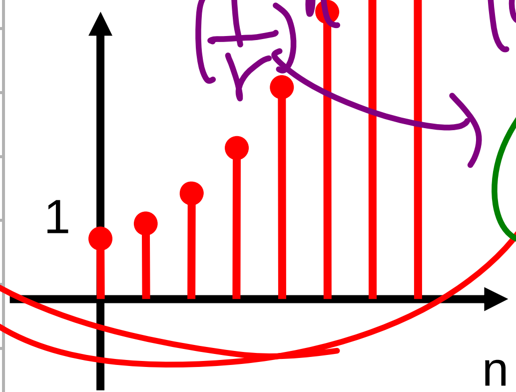
$|a| > 1$

$a^n r^{-n}$

$\left(\frac{1}{r}\right)^n$

$|ar| < 1$

r^{-n}



$$\sum_{n=-\infty}^{+\infty} \left(x[n] \, r^{-n} \right) \left(e^{jw}\right)^{-n}$$

$s = \sigma + j\omega$

$$= \sum_{n=-\infty}^{+\infty} x[n] \left(r \, e^{jw} \right)^{-n}$$

$z = r e^{j\omega}$

$$= \boxed{\sum_{n=-\infty}^{+\infty} x[n] \left(z \right)^{-n}} = X(z)$$

zT

- **The z-Transform of a General Signal x[n]:**

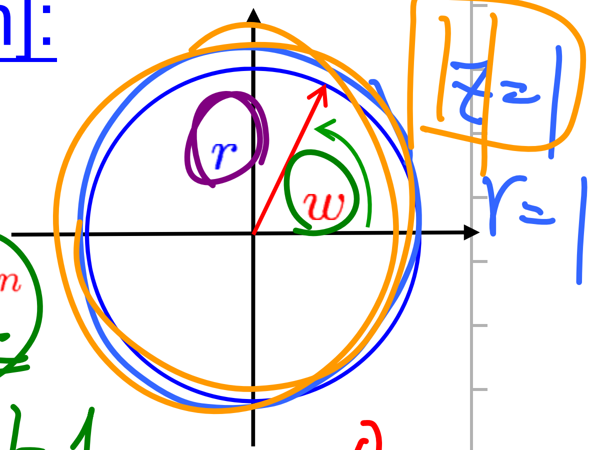
FT $\quad z = e^{jw}$ $\quad |r|=1$

$$X(e^{jw}) \triangleq \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

zT $\quad z = re^{jw}$

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$|z|=1$



$e^{jw} = \cos w + j\sin w$

$$\boxed{re^{jw} = r(\cos w + j\sin w)}$$

$s = \sigma + jw$

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jw})$$

$$X(e^{jw}) = \mathcal{F}\left\{x[n]\right\}$$

$$x[n] = \mathcal{F}^{-1}\left\{X(e^{jw})\right\}$$

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$$

$$X(z) = \mathcal{Z}\left\{x[n]\right\}$$

$$x[n] = \mathcal{Z}^{-1}\left\{X(z)\right\}$$

$$X(z)\Big|_{z=e^{jw}} = \mathcal{Z}\left\{x[n]\right\}\Big|_{z=e^{jw}} = \mathcal{F}\left\{x[n]\right\} = X(e^{jw})$$

z-Transform & Fourier Transform:

$$\mathcal{Z}\left\{x[n]\right\}\Big|_{z=re^{jw}} = X(re^{jw})$$

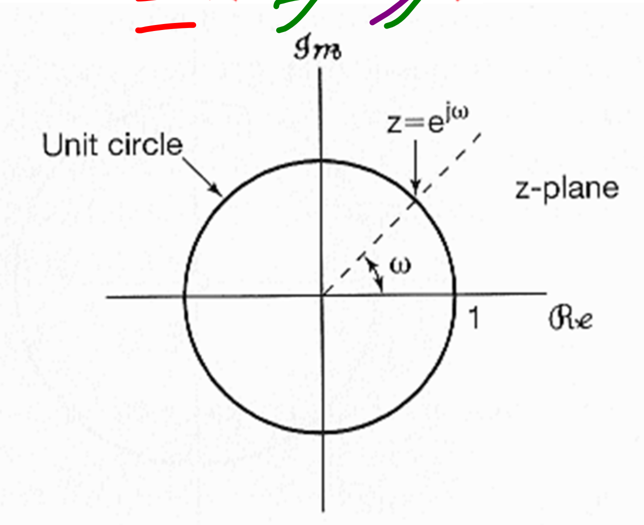
$$= \sum_{n=-\infty}^{+\infty} x[n]\left(re^{jw}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \{x[n]r^{-n}\}e^{-jwn}$$

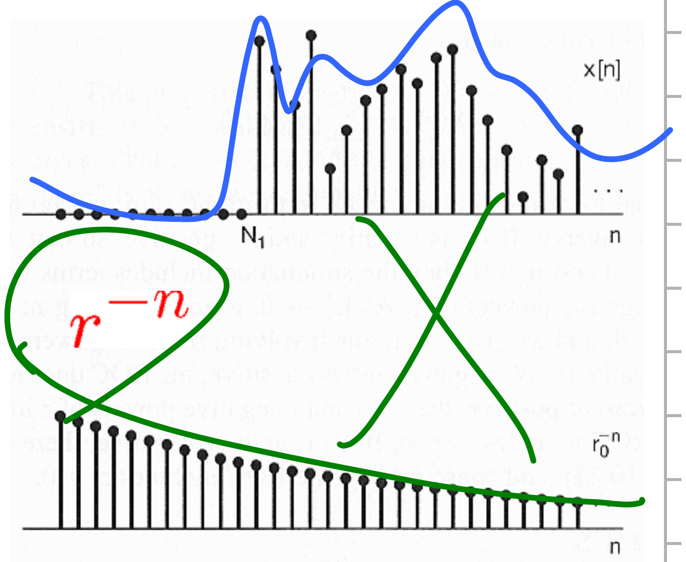
$$= \mathcal{F}\left\{x[n]r^{-n}\right\}$$

$z = e^{jw}$

$|z| = 1$



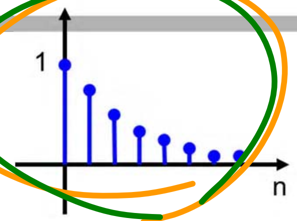
$x[n]$

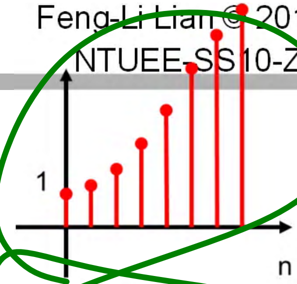


r^{-n}

Example 10.1:

$$x[n] = a^n u[n]$$





$$\Rightarrow X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^n \, u[n] \, e^{-jwn} = \sum_{n=0}^{\infty} (ae^{-jw})^n = \frac{1}{1 - ae^{-jw}}$$

|a|<1

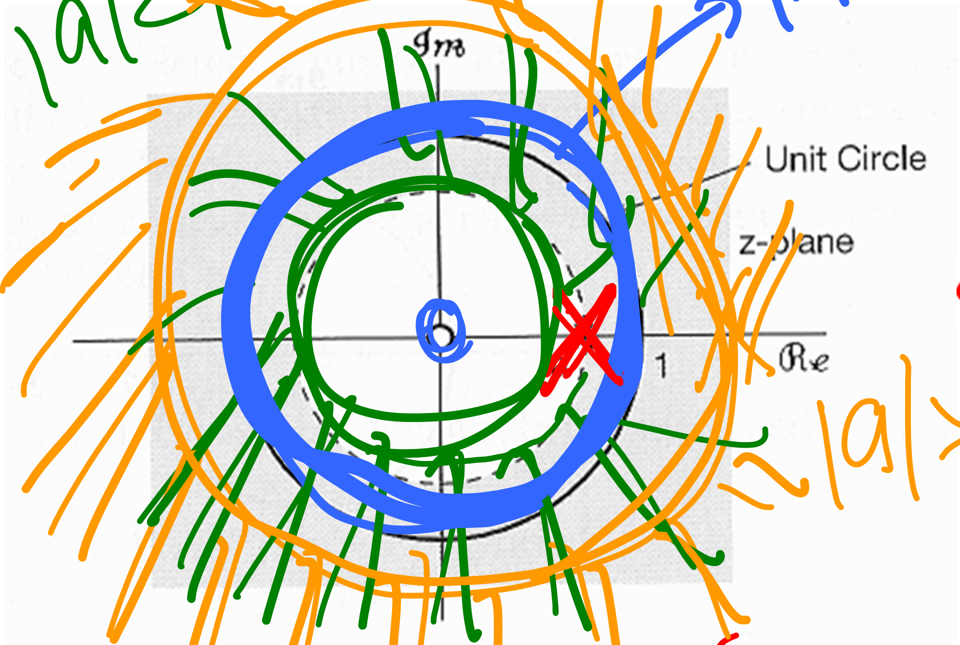
$$\Rightarrow X(z) = \sum_{n=-\infty}^{+\infty} a^n \, u[n] \, z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \qquad |az^{-1}| < 1$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|$$

ROC

|a|<1

|r|>1

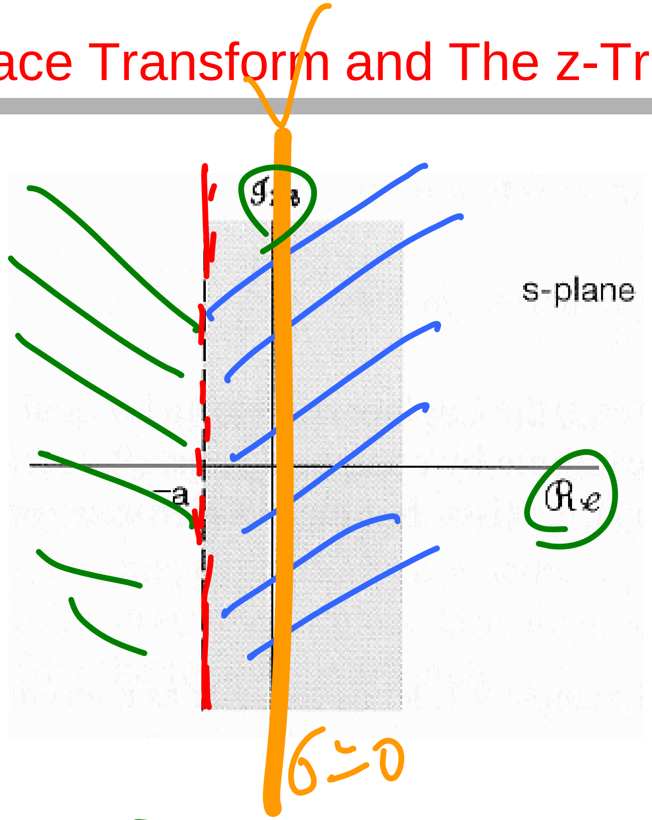


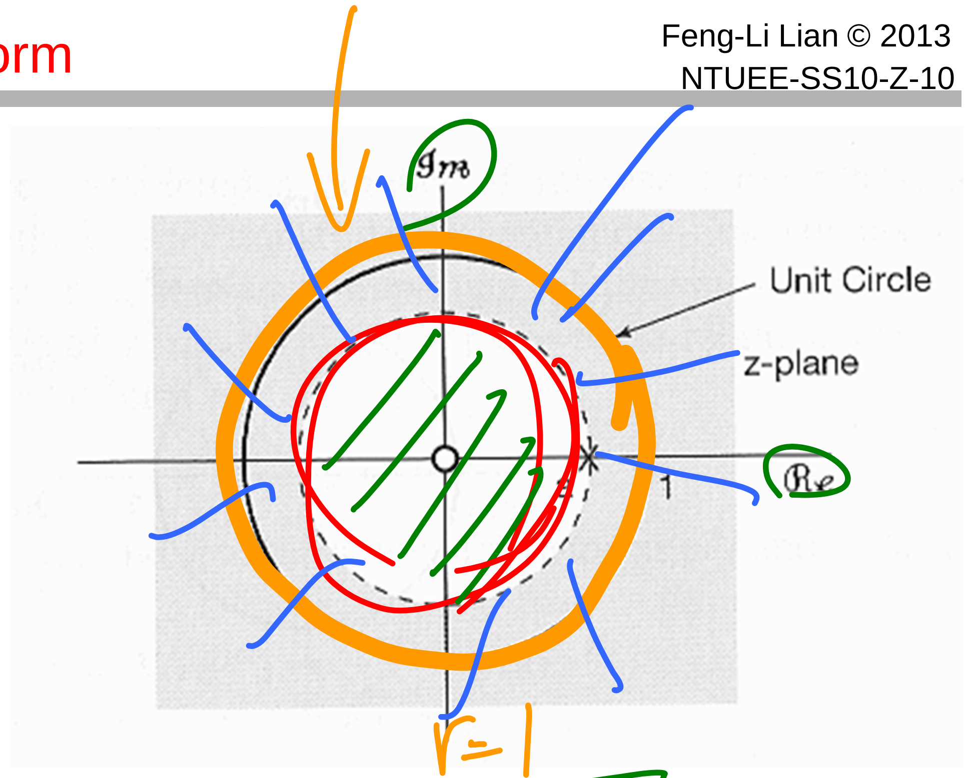
|a|>1

- For $|a| > 1$,

ROC does not include the unit circle,
$\mathcal{F}\left\{a^n u[n]\right\}$ does not converge

Laplace Transform and The z-Transform





$$\boxed{s = \sigma + jw} \in \mathbb{C} \qquad \boxed{z = r e^{jw}} \in \mathbb{C}$$

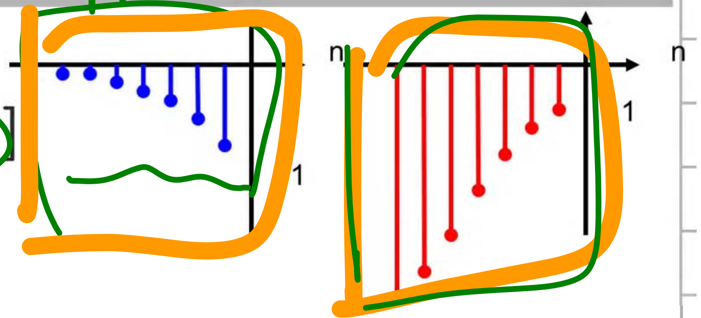
$$e^{-at}u(t) \qquad e^{-\sigma t} \boxed{e^{-jwt}} \qquad a^n u[n] \qquad r^{-n} \boxed{(e^{jw}})^{-n}$$

$$e^{-st} \qquad (z)^{-n}$$

Example 10.2:

$|a|>1$ $\quad$ $|a|<1$

$n<0$



$$x[n] = -a^n u[-n-1]$$

$$\Rightarrow X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n$$

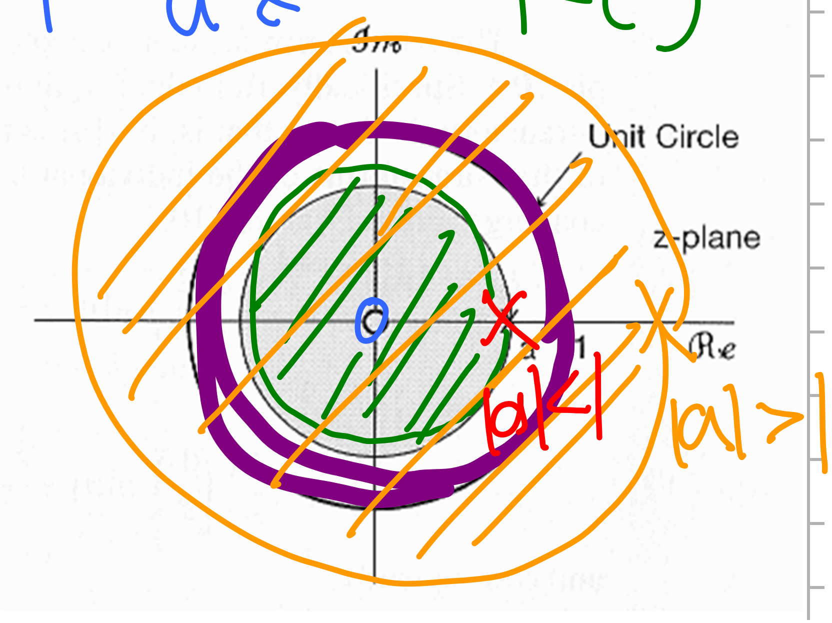
$$= - \frac{a^{-1}z^{1}}{1-a^{-1}z} = 1 - \frac{1}{1-(\ )}$$

$|a^{-1}z| < 1$

$$= 1 - \frac{1}{1-a^{-1}z}$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$|z| < |a|$



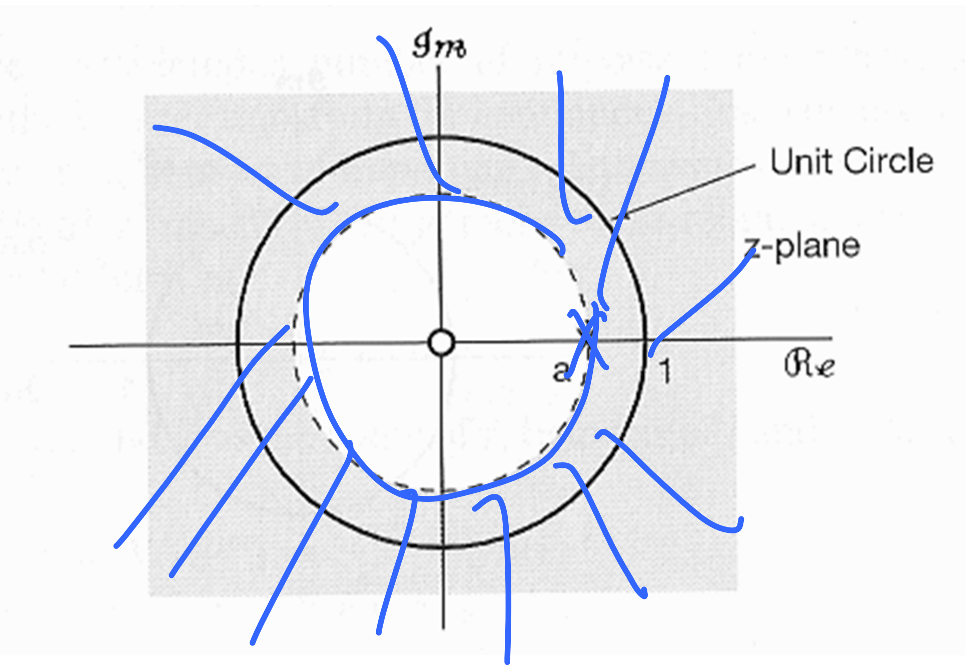
- **Region of Convergence (ROC):**

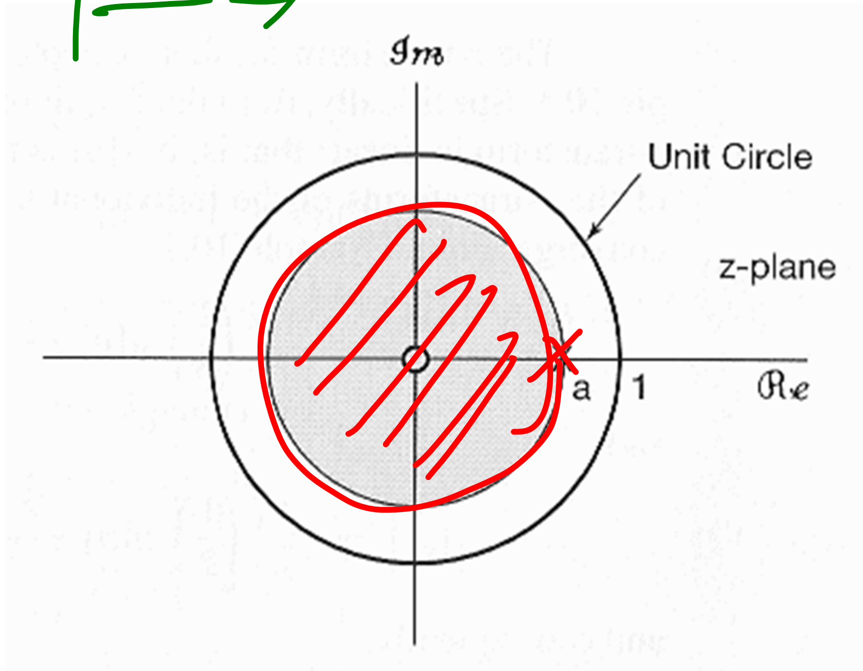
$z = r e^{jw}$

R $\quad a^n u[n] \quad \overset{z}{\longleftrightarrow} \quad \boxed{\dfrac{z}{z-a}}, \quad |z| > |a|$ outer

L $\quad -a^n u[-n-1] \quad \overset{z}{\longleftrightarrow} \quad \boxed{\dfrac{z}{z-a}}, \quad |z| < |a|$ inner

where Fourier transform of $x[n] r^{-n}$ converges





- Example 10.3:

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \left\{7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]\right\} z^{-n}$$

$$= 7\sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n u[n] z^{-n} - 6\sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n}$$

$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}$$

ROC

$$7 \cdot \boxed{\left(\frac{1}{3}\right)^n u[n]} \overset{z}{\longleftrightarrow} 7 \cdot \boxed{\frac{1}{1-\frac{1}{3}z^{-1}}} = 7 \cdot \boxed{\frac{z}{z-\frac{1}{3}}}, \quad |z| > \frac{1}{3}$$

$$6 \cdot \boxed{\left(\frac{1}{2}\right)^n u[n]} \overset{z}{\longleftrightarrow} 6 \cdot \boxed{\frac{1}{1-\frac{1}{2}z^{-1}}} = 6 \cdot \boxed{\frac{z}{z-\frac{1}{2}}}, \quad |z| > \frac{1}{2}$$

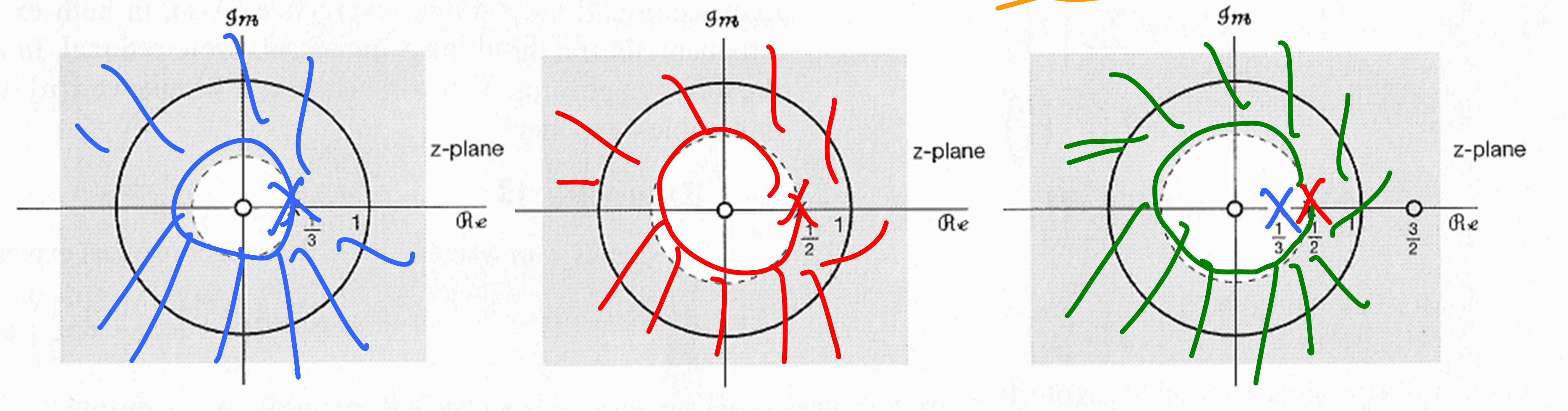
$\cap$

- Example 10.3:

$$7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})} \quad |z| > \frac{1}{2}$$

$$\overset{\mathcal{Z}}{\longleftrightarrow} \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| > \frac{1}{2}$$



(a) (b) (c)

- Example 10.4:

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n]$$

$$\sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j}\left(e^{j\pi/4n} - e^{-j\pi/4n}\right)$$

$$= \frac{1}{2j}\left(\left(e^{j\pi/4}\right)^n - \left(e^{-j\pi/4}\right)^n\right)$$

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j}\left(\left(\frac{1}{3}e^{j\pi/4}\right)^n - \left(\frac{1}{3}e^{-j\pi/4}\right)^n\right)$$

$$\frac{z}{z-A} \leftarrow A^n \qquad B^n \rightarrow \frac{z}{z-B}$$

Example 10.4:

$$a^n u[n] \stackrel{z}{\longleftrightarrow} \frac{z}{z-a}, \quad |z| > |a|$$

$$\left(\frac{1}{3} e^{j\pi/4}\right)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{z}{z - \frac{1}{3}e^{j\pi/4}}, \quad |z| > \frac{1}{3}$$

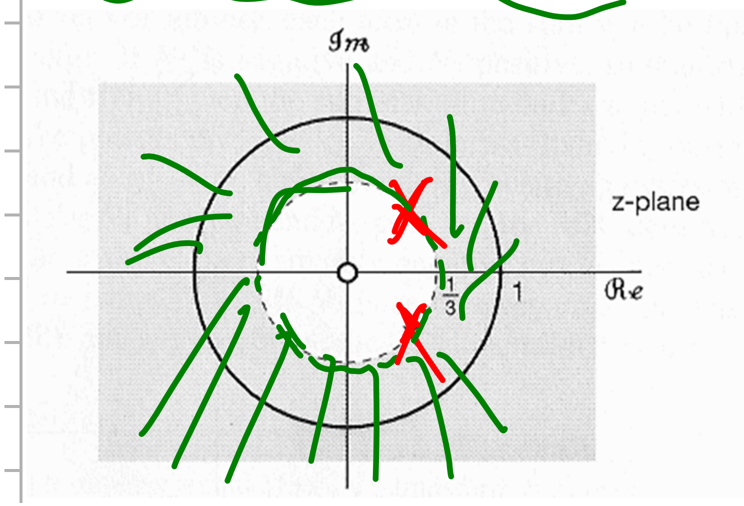
$$|Z| > |A| = \frac{1}{3}$$

$$\left(\frac{1}{3} e^{-j\pi/4}\right)^n u[n] \stackrel{z}{\longleftrightarrow} \frac{z}{z - \frac{1}{3}e^{-j\pi/4}}, \quad |z| > \frac{1}{3}$$

$$|Z| > |B| = \frac{1}{3}$$

$$\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right) u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{2j}\left(\frac{z}{z - \frac{1}{3}e^{j\pi/4}} - \frac{z}{z - \frac{1}{3}e^{-j\pi/4}}\right), \quad \boxed{|z| > \frac{1}{3}}$$

$re^{j\theta}$ $\quad re^{-j\theta}$



$$\frac{\frac{1}{3\sqrt{2}}z}{\left(z - \frac{1}{3}e^{j\pi/4}\right)\left(z - \frac{1}{3}e^{-j\pi/4}\right)}$$

$re^{j\theta}$ $\quad re^{-j\theta}$

$e^{j\theta}$ $\quad e^{-j\theta}$

$$z^2 - 2r\cos\theta z + (r\cancel{e^{j\theta}})^2$$

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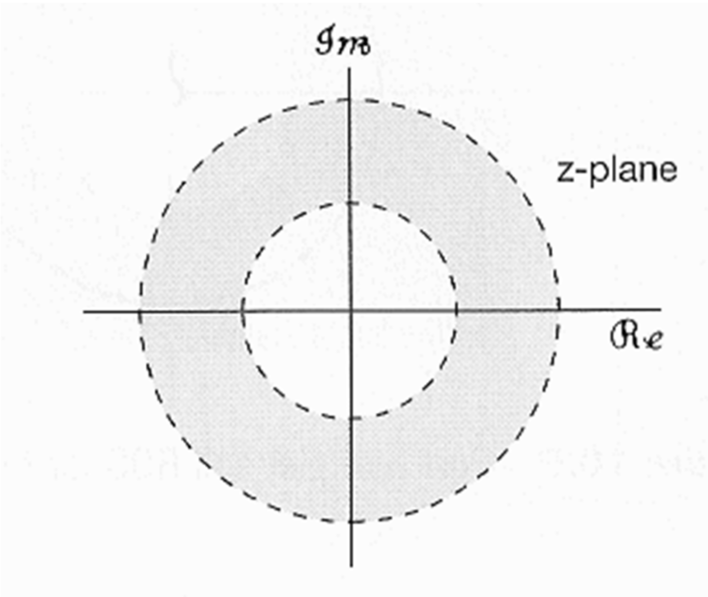
Properties of ROC:

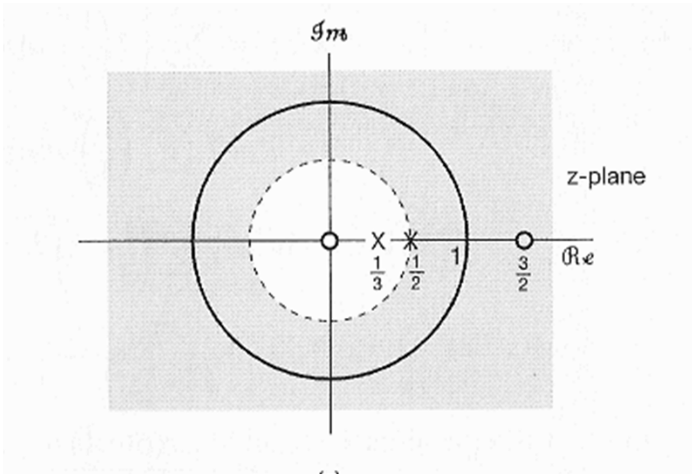
1. The ROC of X(z) consists of a ring in the z-plane centered about the origin

$z = re^{j\omega}$

2. The ROC does not contain any poles

$$\frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \qquad |z| > \frac{1}{2}$$





(c)

Properties of ROC:

3. If x[n] is of finite duration,
then the ROC is the entire z-plane,
except possibly z = 0 and/or z = ∞

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$= \sum_{n=N_1}^{N_2} x[n] z^{-n} \quad \text{is bounded}$$

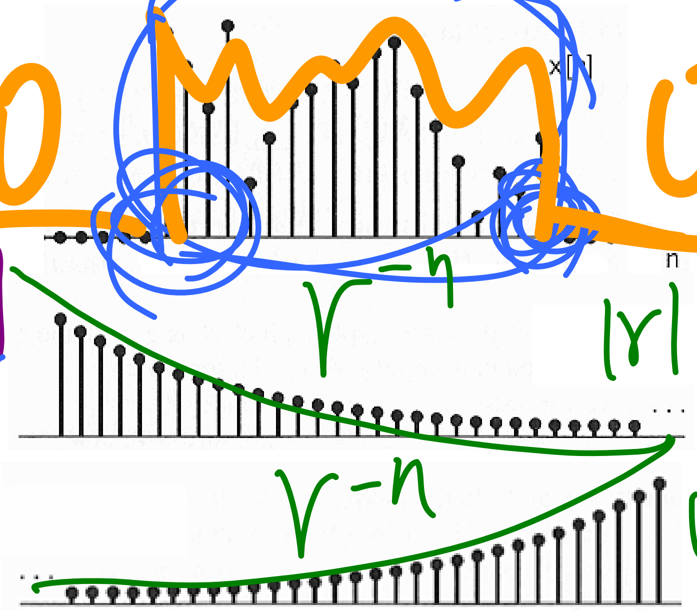
$$= \cdots + x[-3] z^{+3} + \cdots + x[5] z^{-5}$$

$$= \cdots + x[-3] z^{+3} + \cdots + x[5] \left(\frac{1}{z}\right)^5$$

$|z| \to 0$

- However, $|z| \to 0 \Rightarrow |z|^N \to \infty$ if N is negative

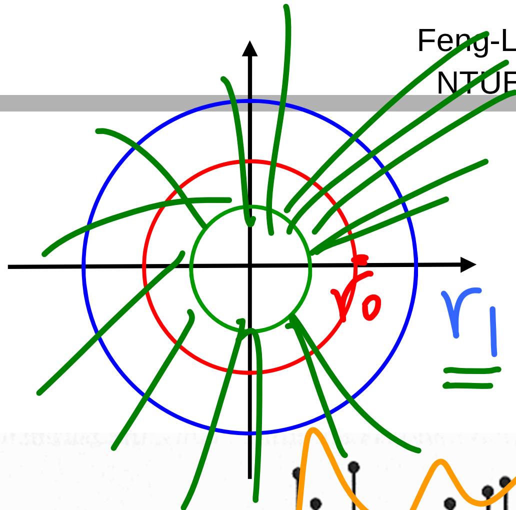
$|z| \to \infty \Rightarrow |z|^N \to \infty$ if N is positive



$x[n]\, r^{-n}$

- **Properties of ROC:**

4. If x[n] is right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC



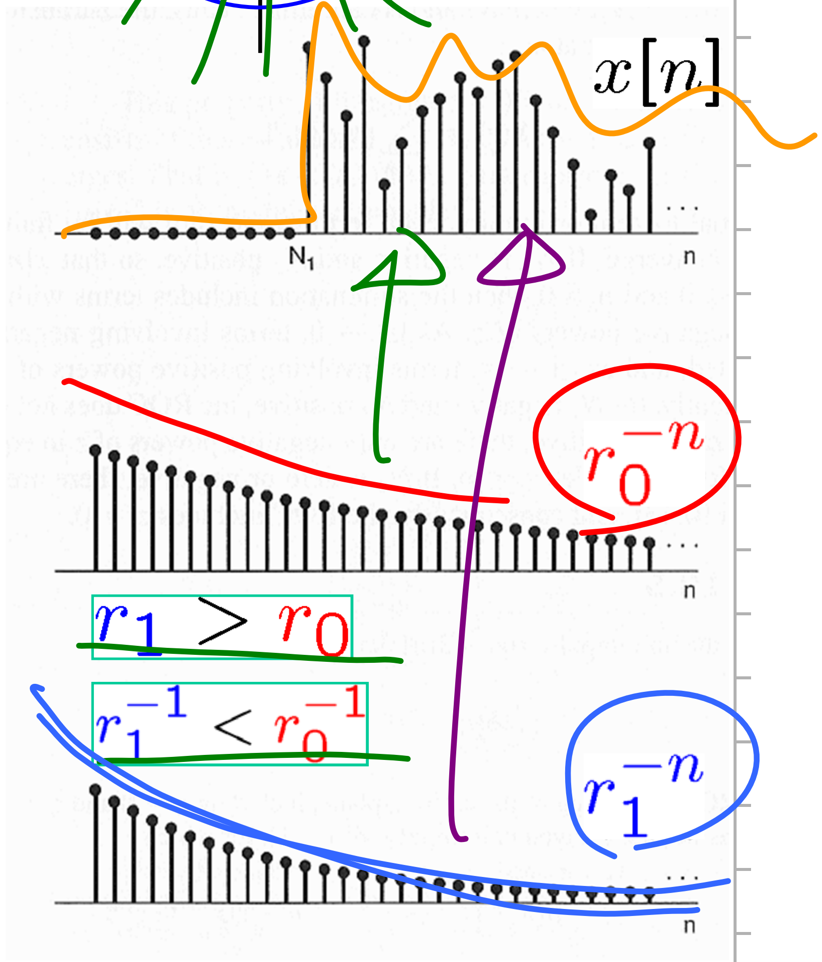
$$X(r_0 e^{jw}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$

$$X(r_1 e^{jw}) = \sum_{n=N_1}^{\infty} \left\{ x[n] r_1^{-n} \right\} e^{-jwn}$$

?

$\boxed{r_1 > r_0}$ $\boxed{r_1^{-1} < r_0^{-1}}$

$$< \sum_{n=N_1}^{\infty} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$

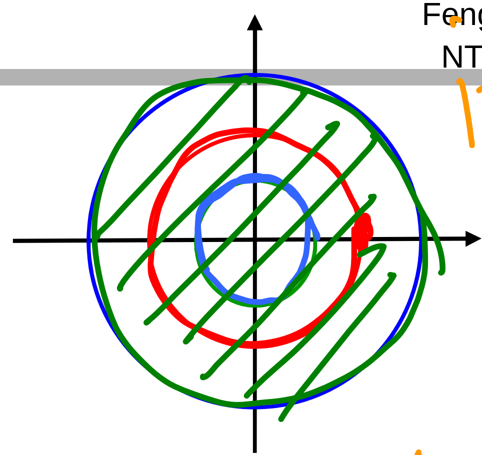


- Properties of ROC:

5. If x[n] is left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which

$0 < |z| < r_0$

will also be in the ROC

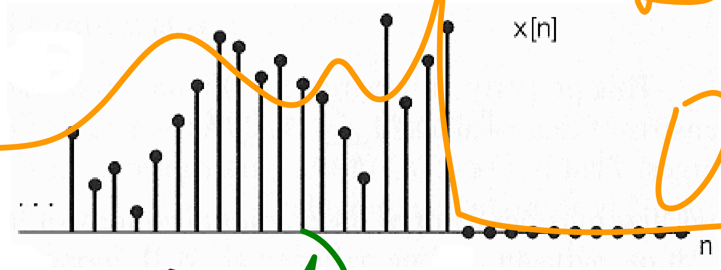


inner r_0

r_1

L

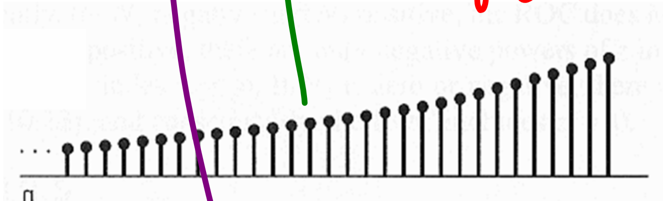
0



$$X(r_0 e^{jw}) = \sum_{n=-\infty}^{N_2} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$

r_0^{-n}

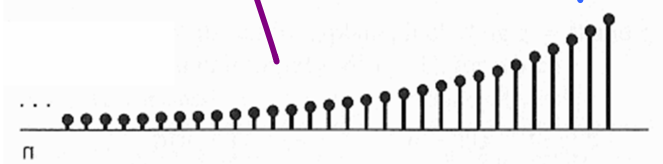
$$X(r_1 e^{jw}) = \sum_{n=-\infty}^{N_2} \left\{ x[n] r_1^{-n} \right\} e^{-jwn}$$



$$0 < r_1 < r_0 \qquad r_1^m < r_0^m, \quad m > 0$$

r_1^{-n}

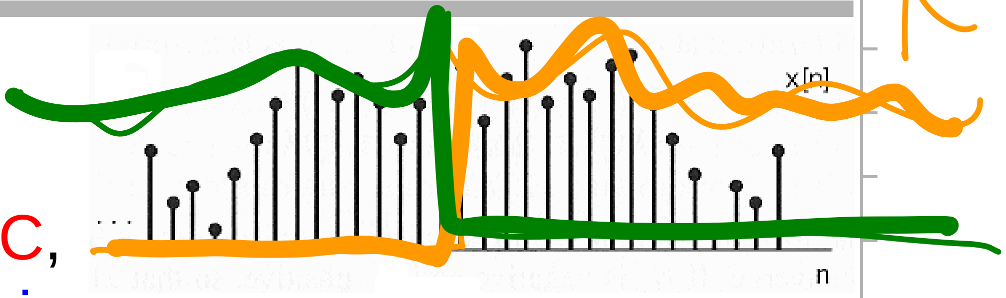
$$< \sum_{n=-\infty}^{N_2} \left\{ x[n] r_0^{-n} \right\} e^{-jwn} < \infty$$



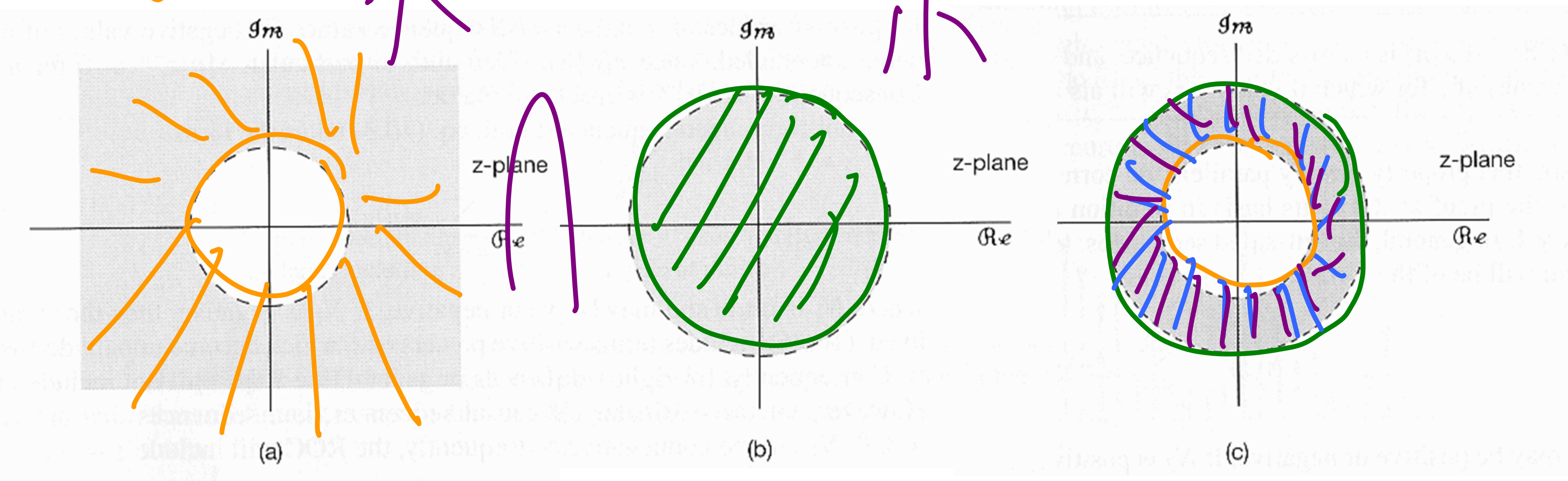
Properties of ROC:

6. If x[n] is two-sided, and
   if the circle $|z| = r_0$ is in the ROC,
   then the ROC will consist of a ring
   in the z-plane that includes the circle $|z| = r_0$

L R

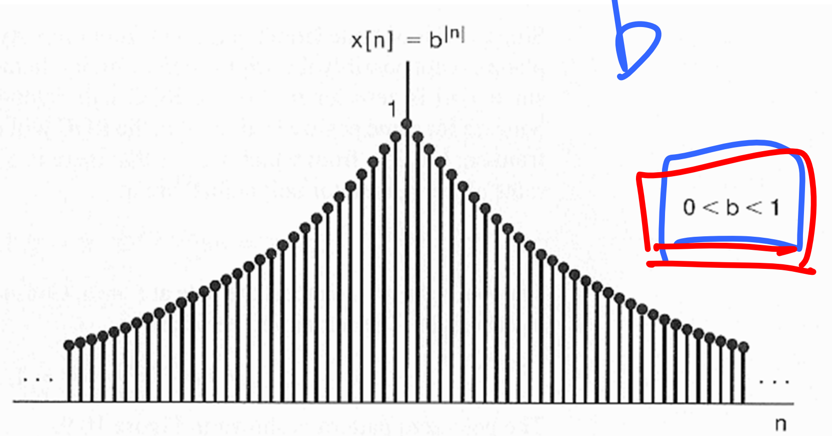


outer 大 inner 小

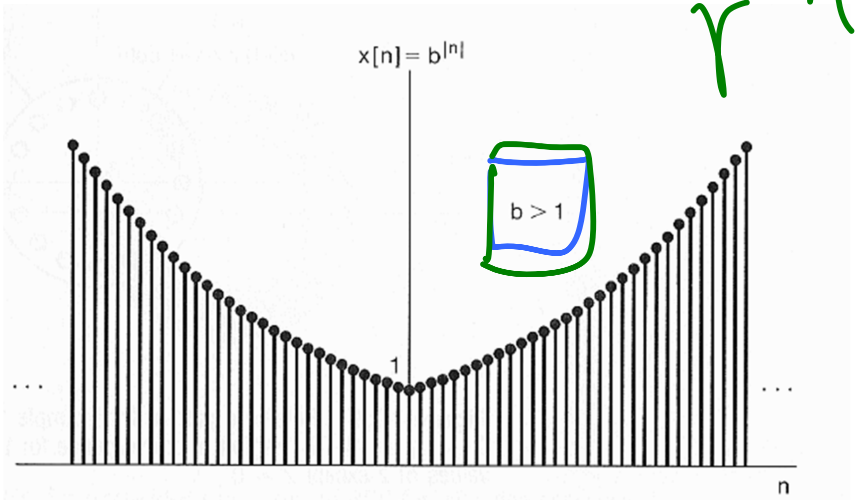


- Example 10.7:

$b^{|n|}$ $\qquad\qquad$ r^{-n}



(a)



(b)

$$x[n] = b^{|n|}, \quad b > 0$$

$$= b^n u[n] + b^{-n} u[-n-1]$$

R $\qquad$ L

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(z) = \frac{1}{1 - bz^{-1}} - \frac{1}{1 - b^{-1}z^{-1}}, \quad \boxed{b < |z| < \frac{1}{b}}$$

$$= \boxed{\left(\frac{b^2 - 1}{b}\right)} \boxed{\frac{z}{(z-b)(z-b^{-1})}}, \quad \boxed{b < |z| < \frac{1}{b}}$$

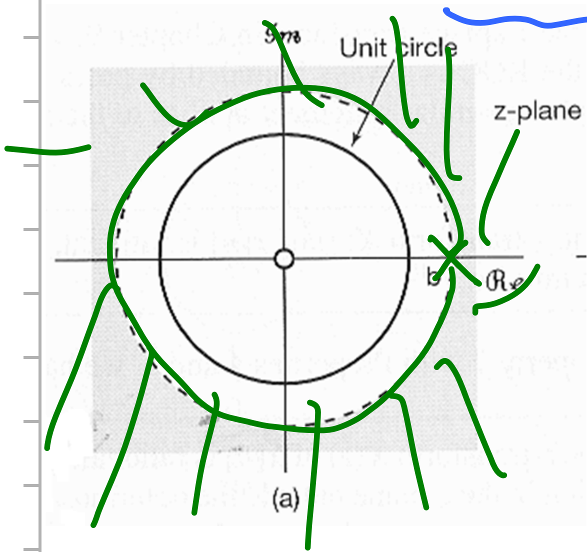
- **Example 10.7:**

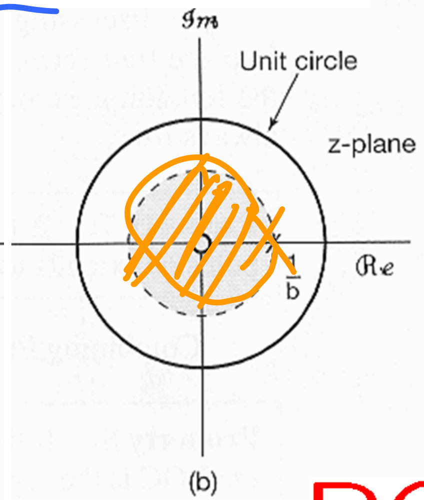
$$X(z) = \left(\frac{b^2-1}{b}\right)\frac{z}{(z-b)(z-b^{-1})}, \quad b < |z| < \frac{1}{b}$$

$z=b \quad z=b^{-1}$

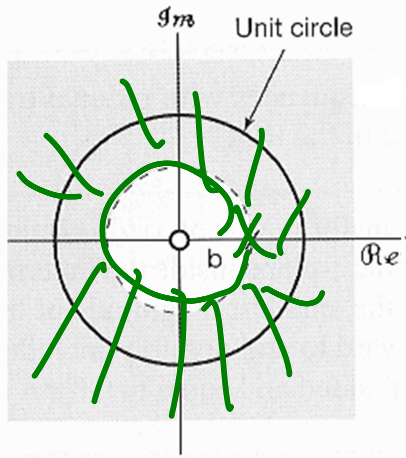
$b > 1$

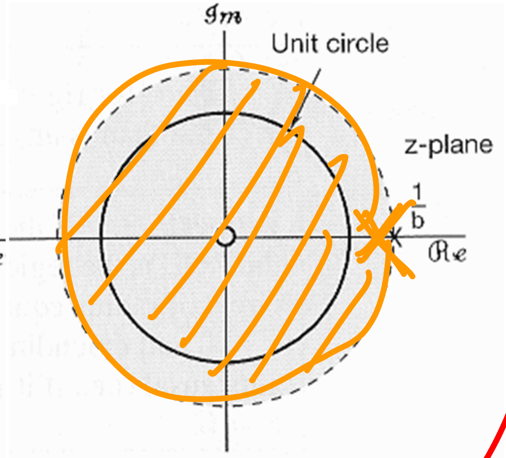
$0 < b < 1$

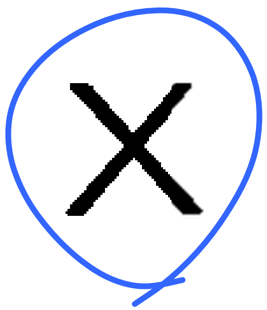


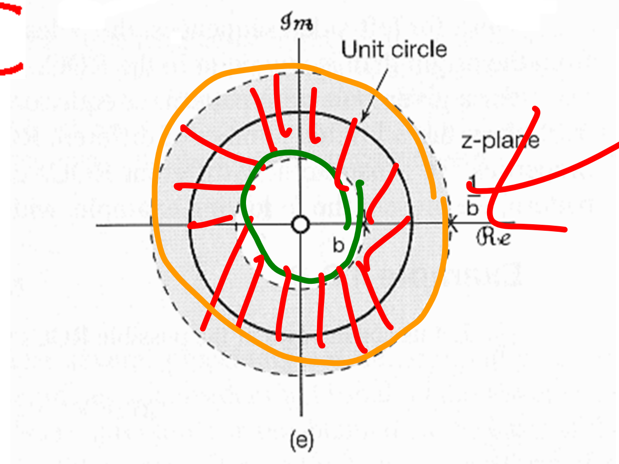


ROC



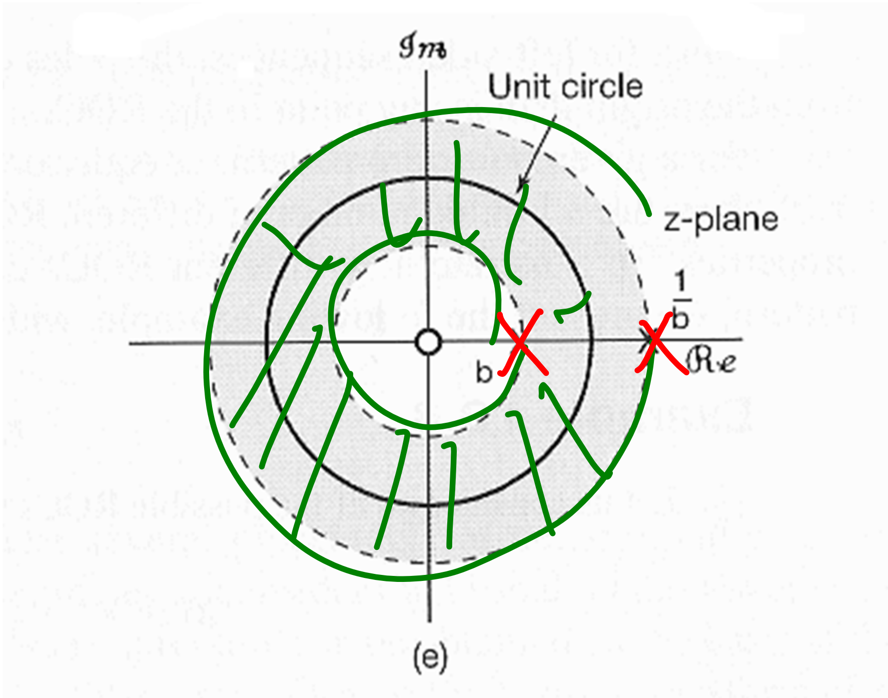




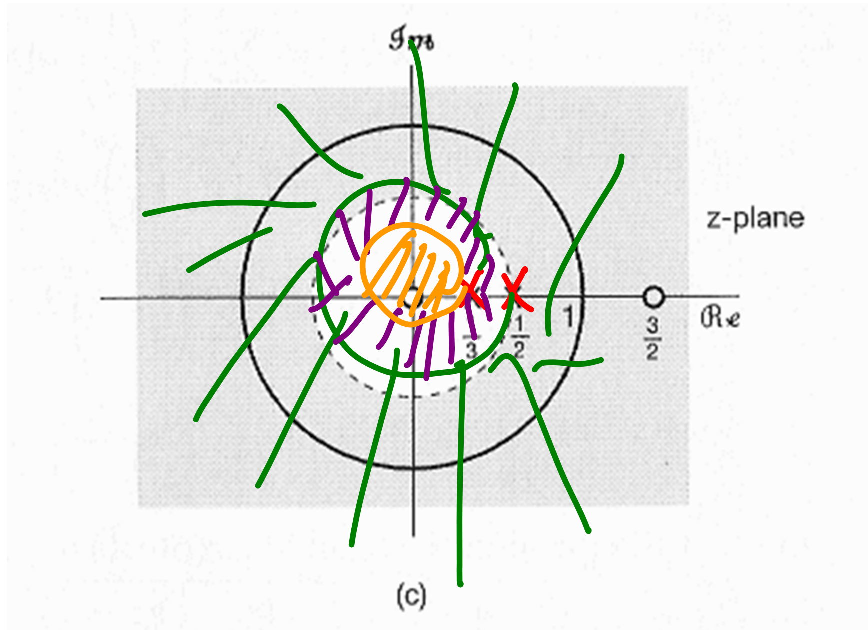


Properties of ROC:

7. If the z-transform X(z) of x[n] is rational, then its ROC is bounded by poles or extends to ∞



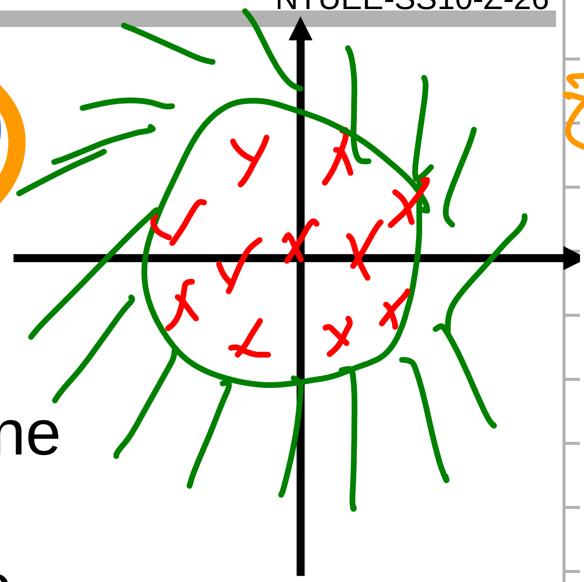
(e)



(c)

$$X(z) = \left(\frac{b^2-1}{b}\right)\frac{z}{(z-b)(z-b^{-1})}, \quad b < |z| < \frac{1}{b}$$

$$\frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| > \frac{1}{2}$$



$z = \infty$

▪ Properties of ROC:

8. If the z-transform X(z) of x[n] is rational

– If x[n] is right sided,
then the ROC is the region in the z-plane
outside the outermost pole ---
i.e., outside the circle of radius equal to
the largest magnitude of the poles of X(z)

– Furthermore, if x[n] is causal,
then the ROC also includes z = ∞

$z^m \quad \left(\frac{1}{z}\right)^m$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = \sum_{n=0}^{+\infty} x[n] \left(\frac{1}{z}\right)^n$$

$z = \infty$

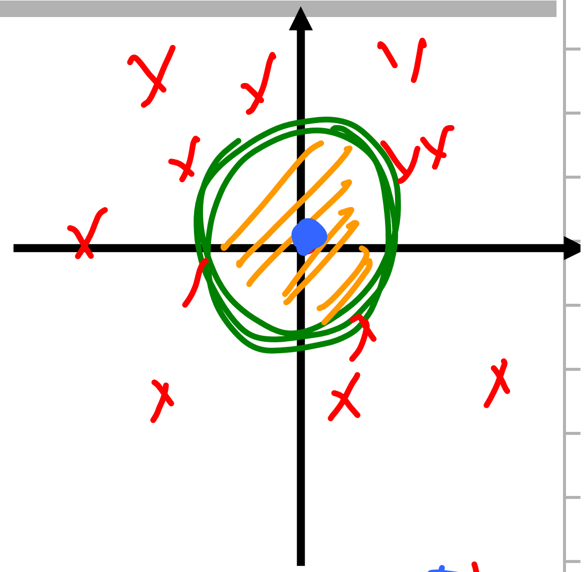
- Properties of ROC:

  9. If the z-transform X(z) of x[n] is rational and if x[n] is left sided,

     then the ROC is the region in the z-plane inside the innermost pole ---

     i.e., inside the circle of radius equal to the smallest magnitude of the poles of X(z) other than any at z = 0 and extending inward and possibly including z = 0

     - In particular, if x[n] is anti-causal, (i.e., if it is left sided and = 0 for n > 0), then the ROC also includes z = 0



$z^{-(n=5)} = z^{-5} = \left(\frac{1}{z}\right)^5$

$z=0$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{0} x[n] z^{-n}$$

$$= \sum_{m=0}^{\infty} x[-m] z^{m}$$

$z=0$

- Example 10.8:

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

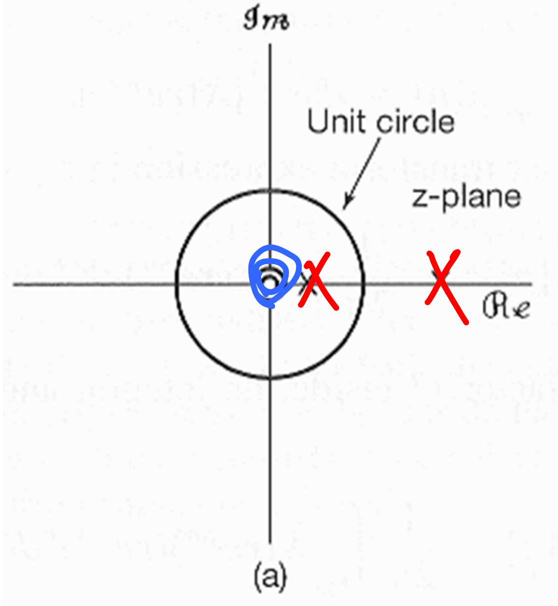
$$= \frac{z^2}{(z-\frac{1}{3})(z-2)}$$

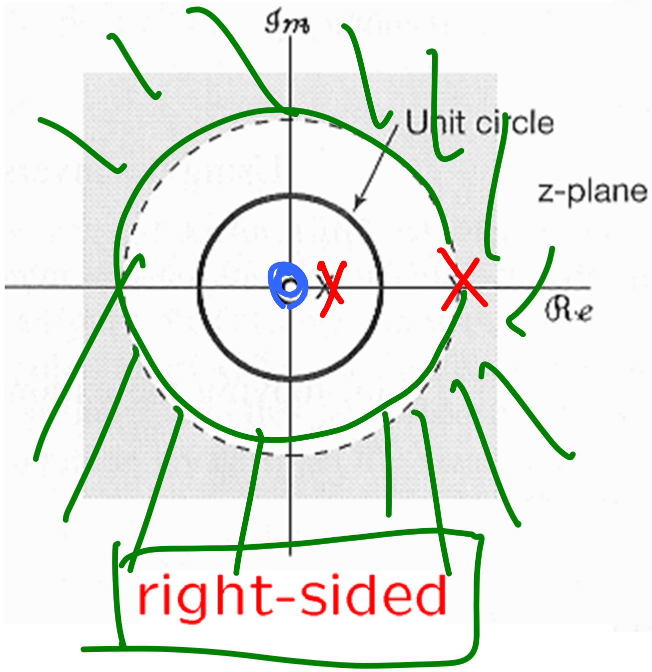
$= 0$

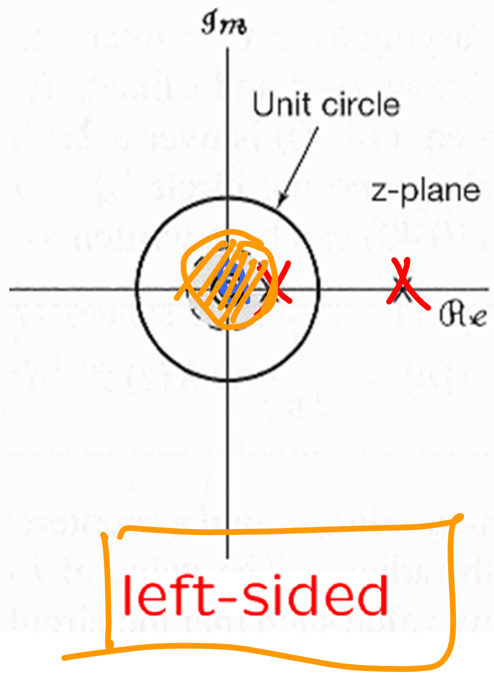
$= 0$

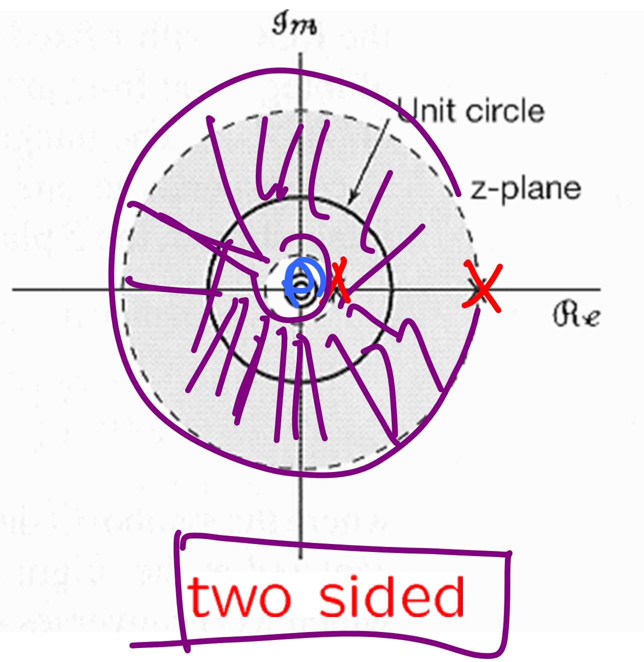
$= 0$

$= 0$









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The Inverse z-Transform:

- By the use of contour integration

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(re^{jw}) = \mathcal{F}\left\{ x[n] r^{-n} \right\}$$

$$\forall z = re^{jw} \text{ in the ROC}$$

$$x[n] r^{-n} = \mathcal{F}^{-1}\left\{ X(re^{jw}) \right\}$$

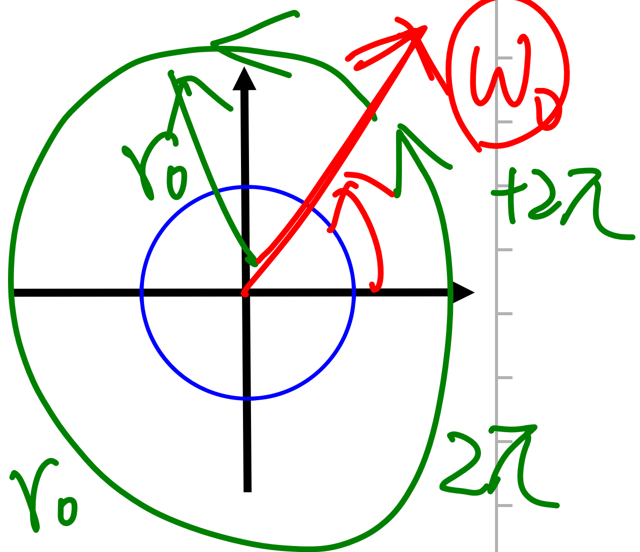
$$x[n] = r^{n} \mathcal{F}^{-1}\left\{ X(re^{jw}) \right\}$$

$$= r^{n} \frac{1}{2\pi} \int_{2\pi} X(re^{jw}) e^{jwn} dw$$

$$= \frac{1}{2\pi} \int_{2\pi} X(re^{jw}) (re^{jw})^{n} dw$$

z z $\frac{1}{jz} dz$

$$\Rightarrow \boxed{x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz}$$



$$z = re^{jw}$$

$$dz = jre^{jw} dw = jz dw$$

$$dw = \frac{1}{jz} dz$$

The Inverse z-Transform:

$$a^n u[n] \quad \overset{z}{\longleftrightarrow} \quad \frac{z}{z-a} = \frac{1}{1-az^{-1}} \qquad |z| > |a|$$

$$-a^n u[-n-1] \quad \overset{z}{\longleftrightarrow} \quad \frac{z}{z-a} = \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

- By the technique of partial fraction expansion

$$X(z) = \frac{A}{1-az^{-1}} + \frac{B}{1-bz^{-1}} + \cdots + \frac{M}{1-mz^{-1}}$$

$$x[n] = A\,a^n u[n] \quad - \quad B\,b^n u[-n-1] \quad + \cdots + \quad x_m[n]$$

(if ROC outside $z = a$) (if ROC inside $z = b$)

$\frac{5}{18} = \frac{10}{36}$

$\frac{1}{4} = \frac{9}{36}$ $\frac{1}{3} = \frac{12}{36}$

- **Example 10.9:**

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}$$

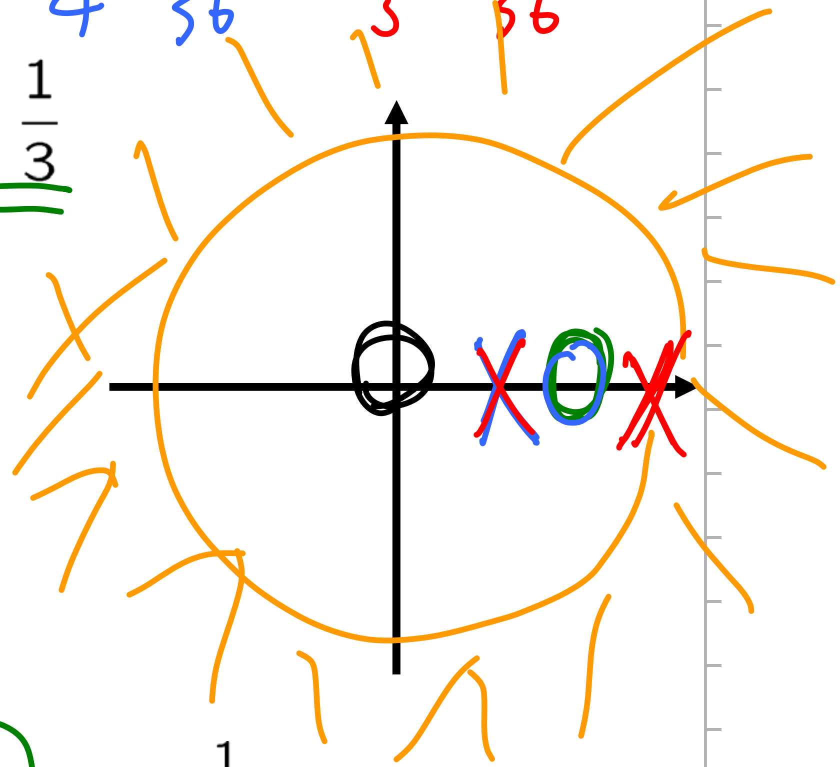
$= 0$

$= 0$

$$= \frac{3z(z - \frac{5}{18})}{(z - \frac{1}{4})(z - \frac{1}{3})}$$

$= 0$

$= 0$



$$X(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})} + \frac{2}{(1 - \frac{1}{3}z^{-1})}$$

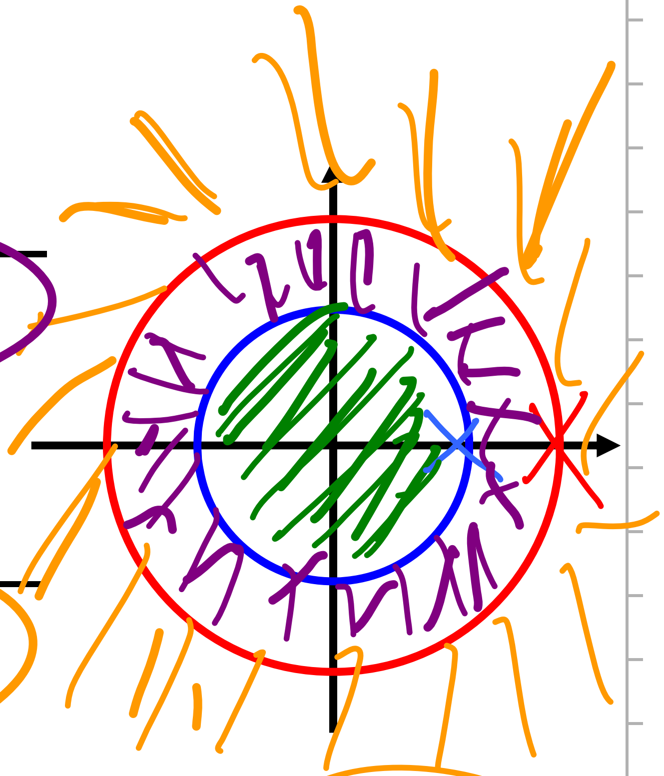
$$\left(\frac{1}{4}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{(1 - \frac{1}{4}z^{-1})}, \quad |z| > \frac{1}{4}$$

$$2\left(\frac{1}{3}\right)^n u[n] \overset{z}{\longleftrightarrow} \frac{2}{(1 - \frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{3}$$

$$\Rightarrow \quad x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

- **Examples 10.9, 10.10, 10.11:**

	$\lvert z\rvert < \frac{1}{4}$	$\frac{1}{4} < \lvert z\rvert$
$\frac{1}{(1-\frac{1}{4}z^{-1})}$	$-\left(\frac{1}{4}\right)^n u[-n-1]$	$\left(\frac{1}{4}\right)^n u[n]$



	$\lvert z\rvert < \frac{1}{3}$	$\frac{1}{3} < \lvert z\rvert$
$\frac{1}{(1-\frac{1}{3}z^{-1})}$	$-\left(\frac{1}{3}\right)^n u[-n-1]$	$\left(\frac{1}{3}\right)^n u[n]$

	$\lvert z\rvert < \frac{1}{4}$	$\frac{1}{4} < \lvert z\rvert < \frac{1}{3}$	$\frac{1}{3} < \lvert z\rvert$
$a\,\frac{1}{(1-\frac{1}{4}z^{-1})}$	$a\left(-\left(\frac{1}{4}\right)^n u[-n-1]\right)$	$a\left(\left(\frac{1}{4}\right)^n u[n]\right)$	$a\left(\left(\frac{1}{4}\right)^n u[n]\right)$
$\pm$	$\pm$	$\pm$	$\pm$
$b\,\frac{1}{(1-\frac{1}{3}z^{-1})}$	$b\left(-\left(\frac{1}{3}\right)^n u[-n-1]\right)$	$b\left(-\left(\frac{1}{3}\right)^n u[-n-1]\right)$	$b\left(\left(\frac{1}{3}\right)^n u[n]\right)$

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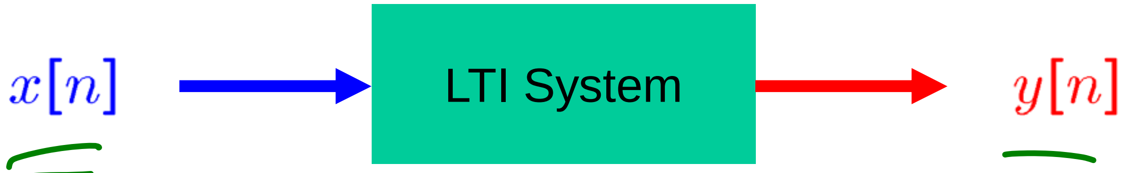
First-Order DT Systems: (p.461)

$$Y(e^{jw}) = X(e^{jw})\, H(e^{jw})$$



$$y[n] - a\, y[n-1] = x[n] \qquad |a| < 1$$

$$\Rightarrow \boxed{H(e^{jw}) = \frac{1}{1 - a e^{-jw}}}$$

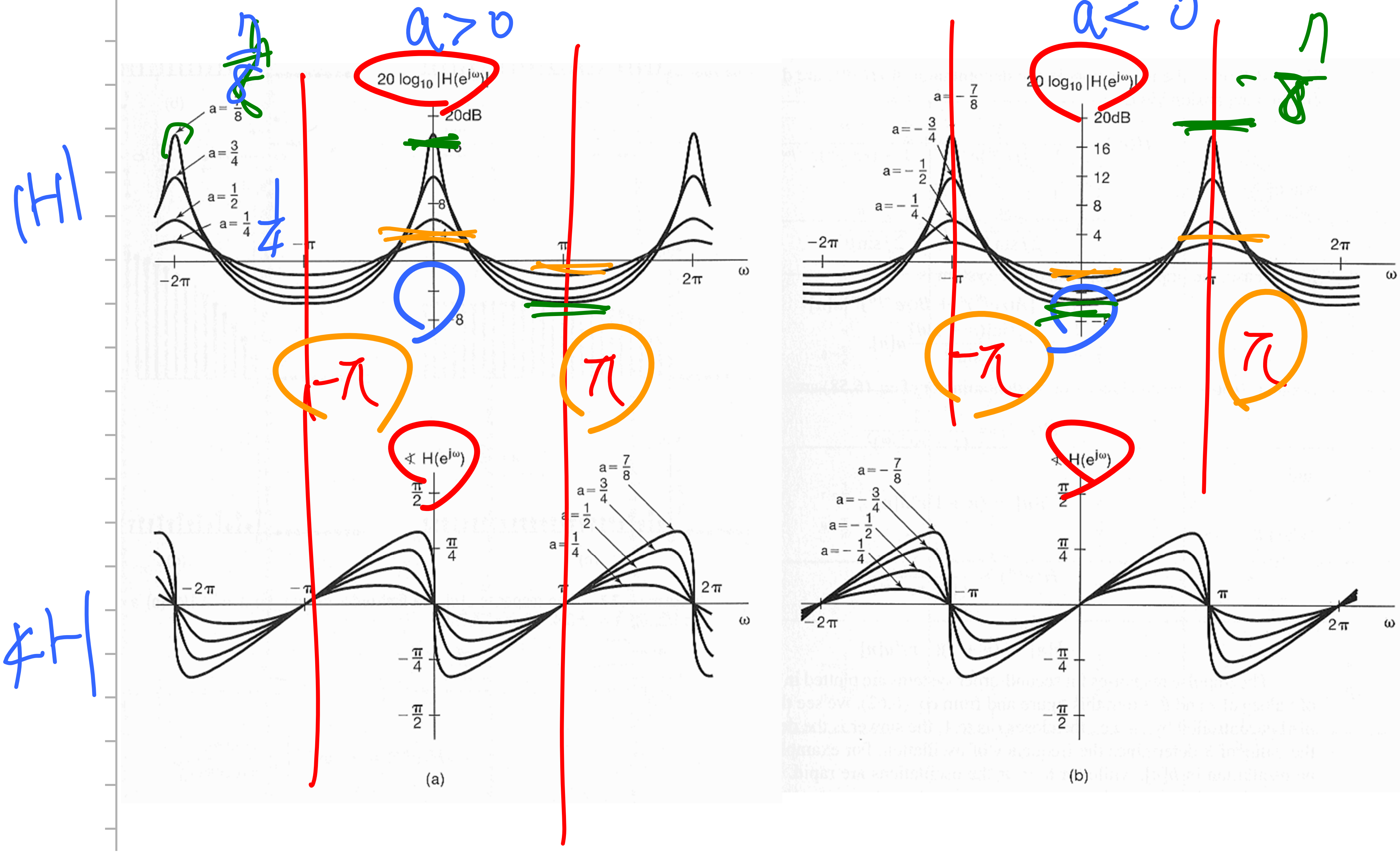
$|a| < 1$

$$\Rightarrow \quad h[n] = a^n\, u[n]$$

$$\Rightarrow \quad s[n] = h[n] * u[n] = \frac{1 - a^{n+1}}{1 - a}\, u[n]$$

$(-\frac{7}{8})^n = (-1)^n (\frac{7}{8})^n$

- Magnitude & Phase of Frequency Response:



First-Order Systems:

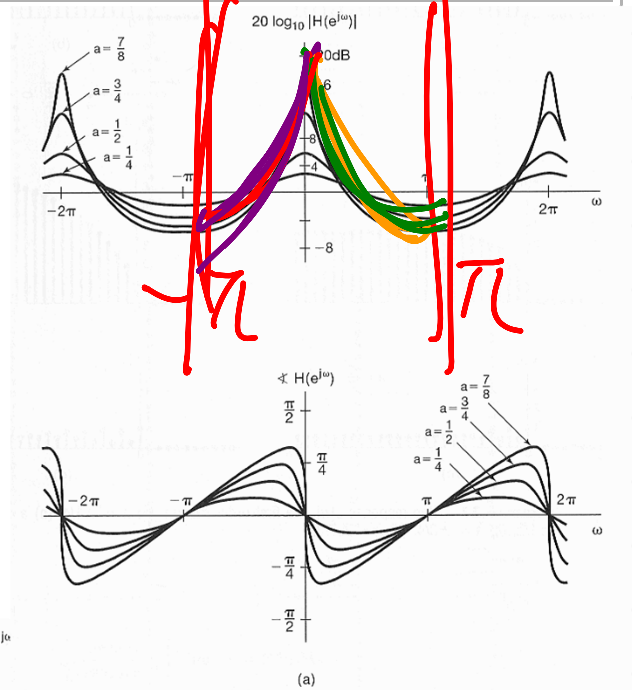
$$h[n] = \boxed{a^n u[n]}$$

$$H(z) = \frac{1}{1 - a z^{-1}} = \boxed{\frac{z}{z-a}}, \quad \boxed{|z| > |a|}$$

- For $|a| < 1$, ROC includes $|z| = 1$

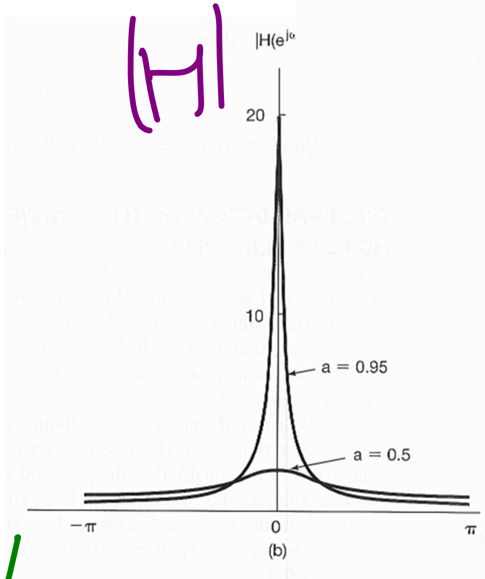
$$\Rightarrow \left| H(e^{jw}) \right| = \boxed{\frac{1}{1 - a e^{-jw}}}$$

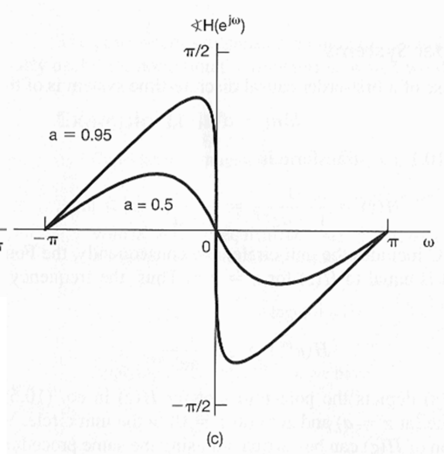
log

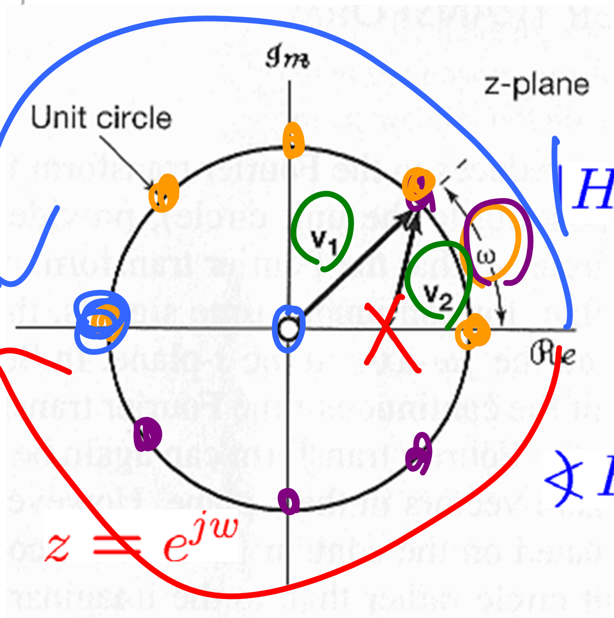


$-\pi$ $\quad$ π

|H|







$z = e^{jw}$

$-\pi$

$$|H| = \frac{|V_1|}{|V_2|}$$

$$\angle H = \angle V_1 - \angle V_2$$

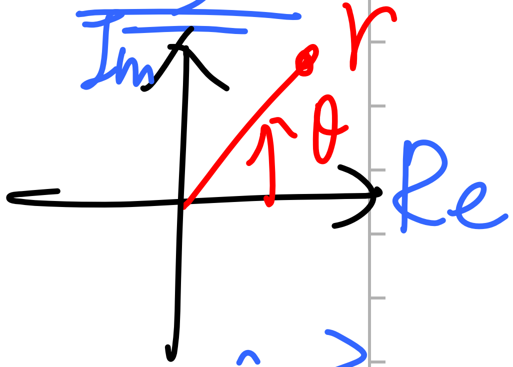
- Second-Order DT Systems: (p.465)

r, θ

$$0 < r < 1 \text{ and } 0 \le \theta \le \pi$$

$$y[n] - 2r\cos(\theta)\, y[n-1] + r^2 y[n-2] = x[n]$$

e^{-jw} e^{-j2w}



$$\Rightarrow H(e^{jw}) = \frac{1}{1 - 2r\cos(\theta)\, e^{-jw} + r^2 e^{-j2w}}$$

$(re^{-jw})^2$

$$\boxed{\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)}$$

$$= \frac{1}{[1 - (re^{j\theta})e^{-jw}]\ [1 - (re^{-j\theta})e^{-jw}]}$$

▪ Magnitude & Phase of Frequency Response:

$\theta = \pi \quad (-1)^n$

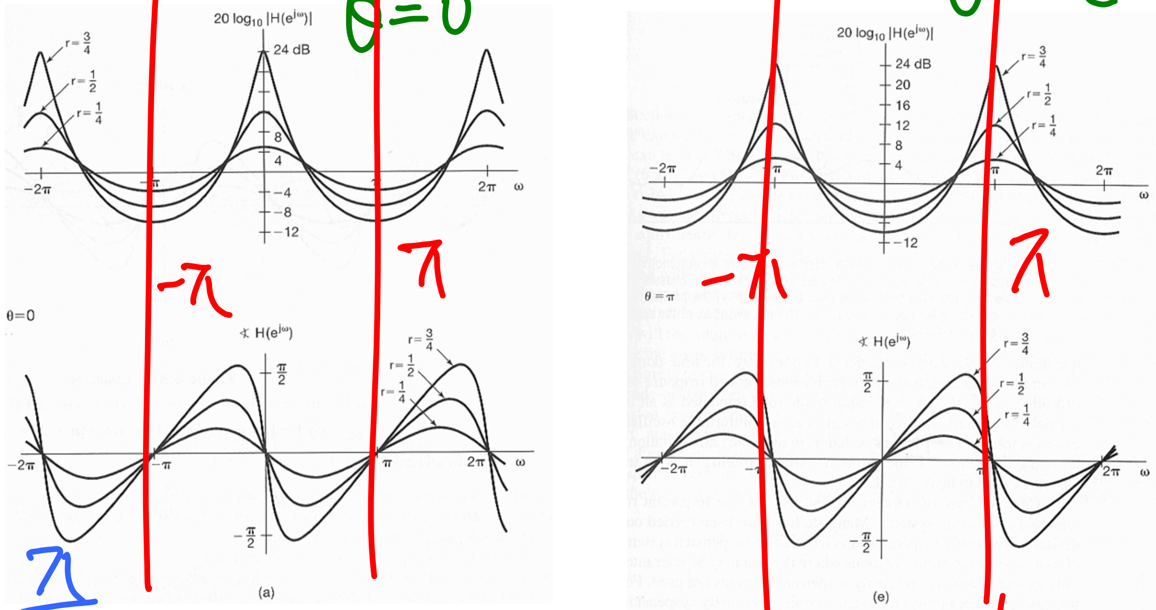
$|H| \quad \measuredangle$

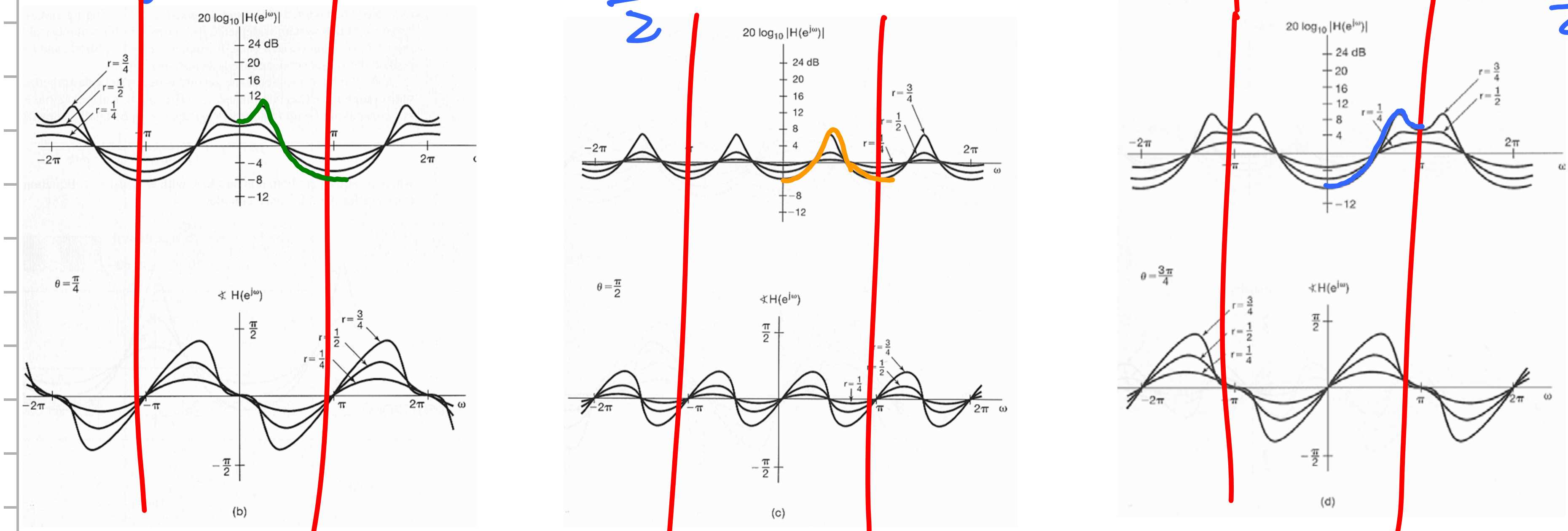
$$H(e^{jw}) = \frac{1}{1 - 2r\cos(\theta)\, e^{-jw} + r^2\, e^{-j2w}}$$

$\theta = 0$

$\theta = \frac{\pi}{4}$ $\quad \frac{\pi}{2}$ $\quad \frac{3}{4}\pi$

$-\pi \quad \pi \quad -\pi \quad \pi$





Geometric Evaluation of the Fourier Transform

- **Second-Order Systems:**

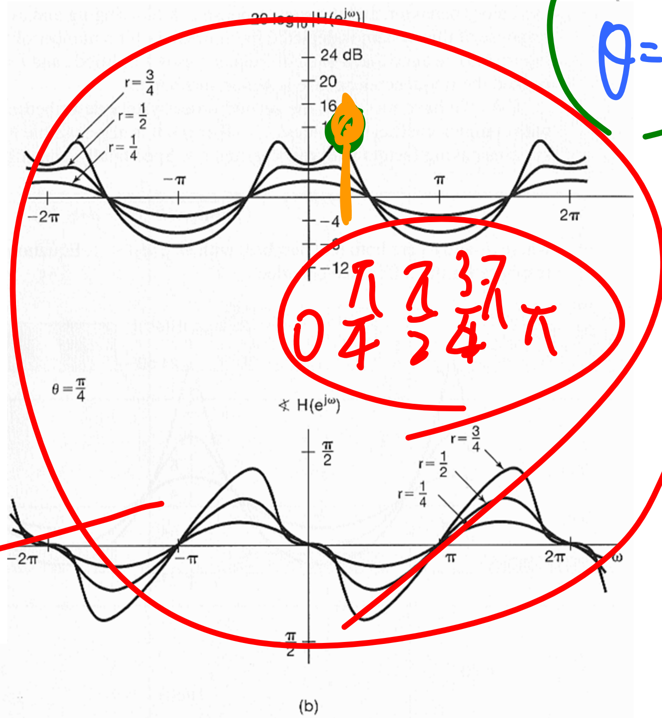
$$H(z) = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$= \frac{z^2}{(z-p_1)(z-p_2)}, \quad |z| > |r|$$

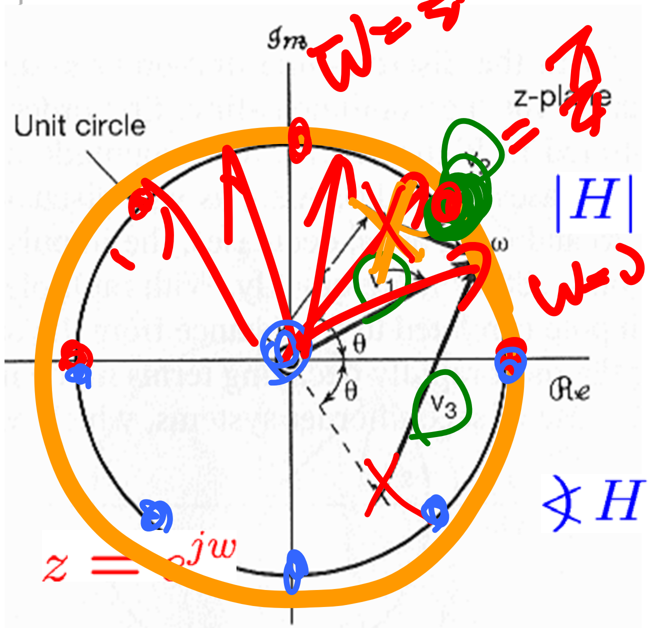
= 0

poles : $\begin{cases} p_1 = re^{j\theta} \\ p_2 = re^{-j\theta} \end{cases}$ zeros : $z_1 = z_2 = 0$

$\theta = \frac{\pi}{4}$



$0 \quad \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4} \quad \pi$



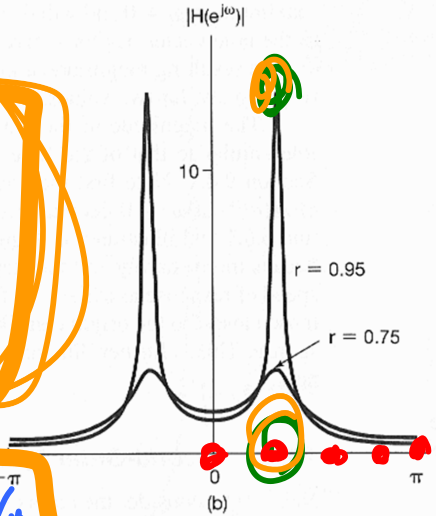
$\omega = \frac{\pi}{4}$

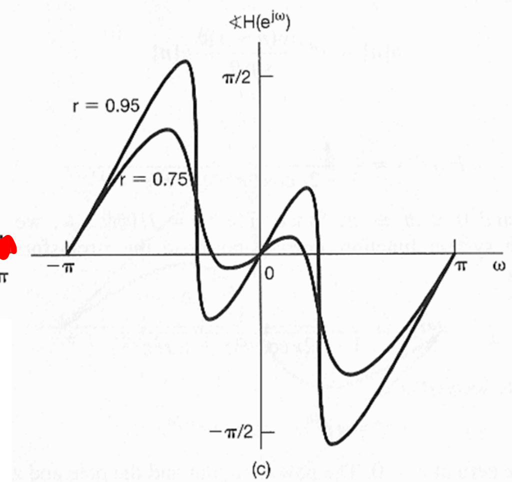
$\omega = 0$

$z = e^{jw}$

$$|H| = \frac{|V_1||V_1|}{|V_2||V_3|}$$

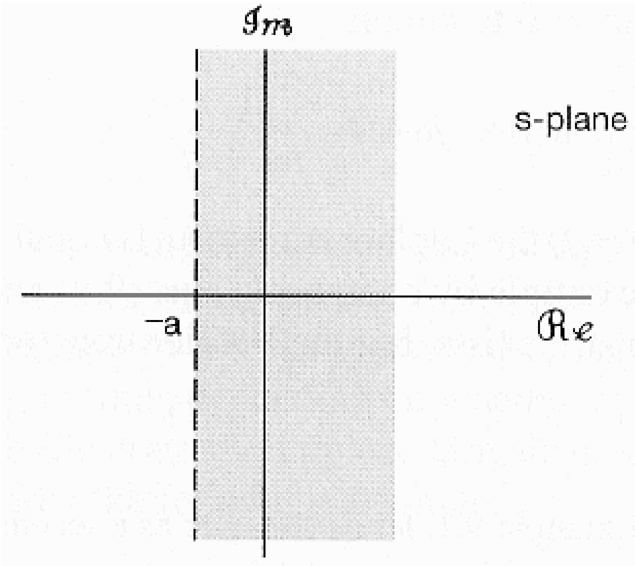
$$\measuredangle H = \measuredangle V_1 + \measuredangle V_1 - \measuredangle V_2 - \measuredangle V_3$$

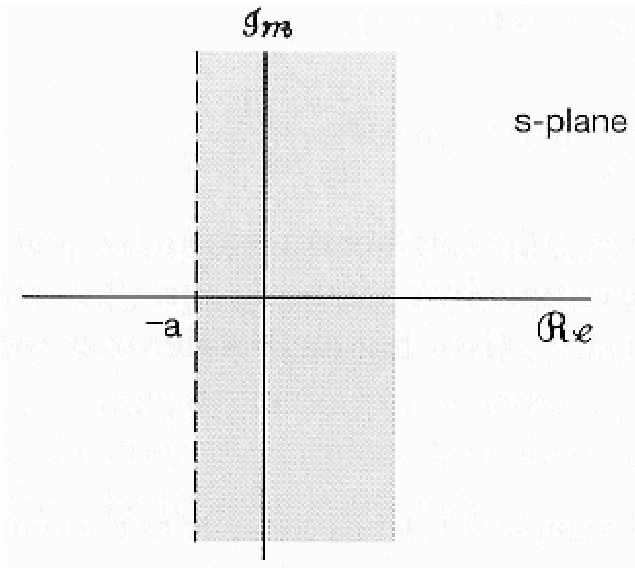




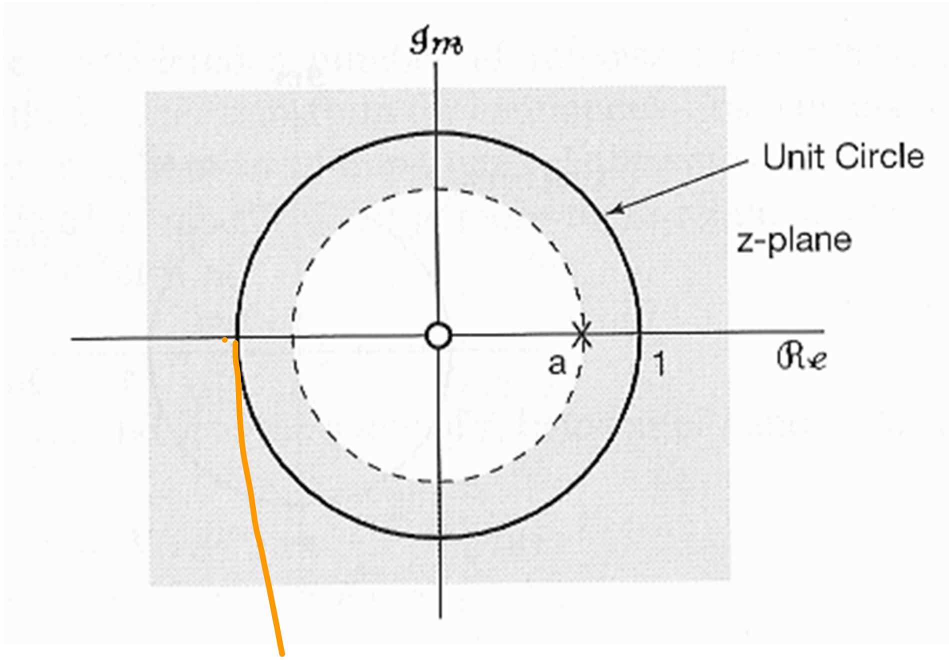
Laplace Transform and The z-Transform

CT $s = \sigma + jw$





DT $z = re^{jw}$



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- Some Common z-Transform Pairs
- Analysis & Characterization of LTI Systems
  Using the z-Transforms
- System Function Algebra and
  Block Diagram Representations
- The Unilateral z-Transform

Outline

Property	CTFS	DTFS	CTFT	DTFT	LT	zT
Linearity	3.5.1		4.3.1	5.3.2	9.5.1	10.5.1
Time Shifting	3.5.2		4.3.2	5.3.3	9.5.2	10.5.2
Frequency Shifting (in s, z)			4.3.6	5.3.3	9.5.3	10.5.3
Conjugation	3.5.6		4.3.3	5.3.4	9.5.5	10.5.6
Time Reversal	3.5.3		4.3.5	5.3.6		10.5.4
Time & Frequency Scaling	3.5.4		4.3.5	5.3.7	9.5.4	10.5.5
(Periodic) Convolution			4.4	5.4	9.5.6	10.5.7
Multiplication	3.5.5	3.7.2	4.5	5.5		
Differentiation/First Difference		3.7.2	4.3.4, 4.3.6	5.3.5, 5.3.8	9.5.7, 9.5.8	10.5.7, 10.5.8
Integration/Running Sum (Accumulation)			4.3.4	5.3.5	9.5.9	10.5.7
Conjugate Symmetry for Real Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Even Signals	3.5.6		4.3.3	5.3.4		
Symmetry for Real and Odd Signals	3.5.6		4.3.3	5.3.4		
Even-Odd Decomposition for Real Signals			4.3.3	5.3.4		
Parseval's Relation for (A)Periodic Signals	3.5.7	3.7.3	4.3.7	5.3.9		
Initial- and Final-Value Theorems					9.5.10	10.5.9

- **Linearity of the z-Transform:**

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x_1[n] \xleftrightarrow{z} X_1(z), \quad ROC = R_1$$

$$x_2[n] \xleftrightarrow{z} X_2(z), \quad ROC = R_2$$

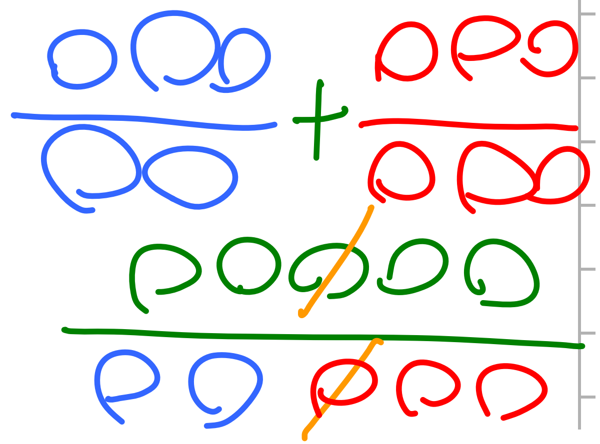
$$\sum_{n=-\infty}^{+\infty} \left(a x_1[n] + b x_2[n] \right) z^{-n} \xleftrightarrow{z} a X_1(z) + b X_2(z),$$

with ROC containing $R_1 \cap R_2$

$$= \sum_{n=-\infty}^{+\infty} a x_1[n] z^{-n} + \sum_{n=-\infty}^{+\infty} b x_2[n] z^{-n}$$

$$= a \left(\sum_{n=-\infty}^{+\infty} x_1[n] z^{-n} \right) + b \left(\sum_{n=-\infty}^{+\infty} x_2[n] z^{-n} \right)$$

$$= a X_1(z) + b X_2(z)$$



- <u>Time Shifting:</u>

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$\sum_{n=-\infty}^{+\infty} x[n-n_0] \, z^{-n} \overset{z}{\longleftrightarrow} z^{-n_0} X(z), \quad ROC = R$$

$m = n - n_0 \qquad n = m + n_0$

except for the possible addition or deletion of the origin or infinity

$z = 0 \qquad z \to \infty$

$$= \sum_{m=-\infty}^{+\infty} x[m] \, z^{-(m+n_0)}$$

$$= \left(\sum_{m=-\infty}^{+\infty} x[m] \, z^{-(m)} \right) z^{-n_0}$$

$$= z^{-n_0} X(z)$$

$z^{-n_0} \; X(z)$

$z^{-5} \cdot \cancel{z^{3}} \qquad \frac{OOO}{\cancel{O}OO} z^{5}$

$\frac{1}{z^5} \qquad z^3 \cdot \qquad \cancel{z}$

- **Scaling in the z-Domain:**

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$\sum_{n=-\infty}^{+\infty} z_0^n x[n] \; z^{-n} \stackrel{z}{\longleftrightarrow} X\left(\frac{z}{z_0}\right), \quad ROC = |z_0| R$$

$$= \sum_{n=-\infty}^{+\infty} z_0^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \left(\frac{z}{z_0}\right)^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \left(y\right)^{-n}$$

$$= X(y) = X\left(\frac{z}{z_0}\right)$$

$$= \frac{1}{2\pi j} \oint X\left(\frac{z}{z_0}\right) z^{n-1} dz$$

$$\frac{z}{z_0} = y = \frac{z}{z_0} \qquad dz = z_0 dy \qquad z = y z_0$$

$$= \frac{1}{2\pi j} \oint X(y) (y z_0)^{n-1} z_0 dy$$

$$= \frac{1}{2\pi j} \oint X(y) (y)^{n-1} \boxed{(z_0)^{n-1} z_0} dy$$

$$= z_0^n \boxed{\frac{1}{2\pi j} \oint X(y) (y)^{n-1} dy}$$

$$= z_0^n \; x[n]$$

- **Scaling in the z-Domain:**

$|z| > \frac{1}{2}$

$(\frac{1}{2})^n$ $\quad a^n u[n] \quad \xleftrightarrow{z} \quad X(z) = \frac{z}{z-a}, \quad |z| > |a|$

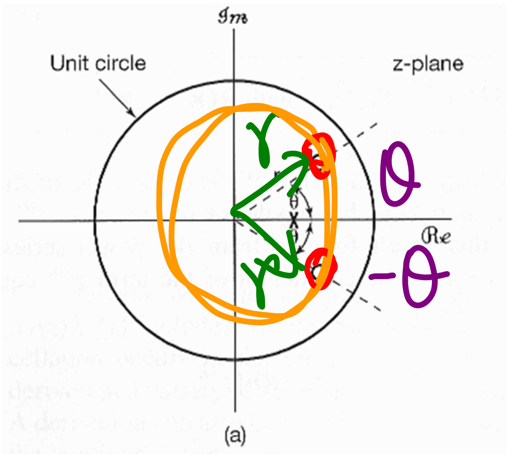
$(4\cdot\frac{1}{2})^n$ $\quad (ba)^n u[n] \quad \xleftrightarrow{z} \quad X(\frac{z}{b}) = \frac{\frac{z}{b}}{\frac{z}{b} - a}, \quad |\frac{z}{b}| > |a|$ $\quad |z| > |b||a|$

$= 2^n$ $\quad |z| > 4\cdot\frac{1}{2} = 2$

$e^{j\omega_0 n} x[n] \quad \xleftrightarrow{z} \quad X\left(e^{-j\omega_0} z\right), \quad ROC = R$

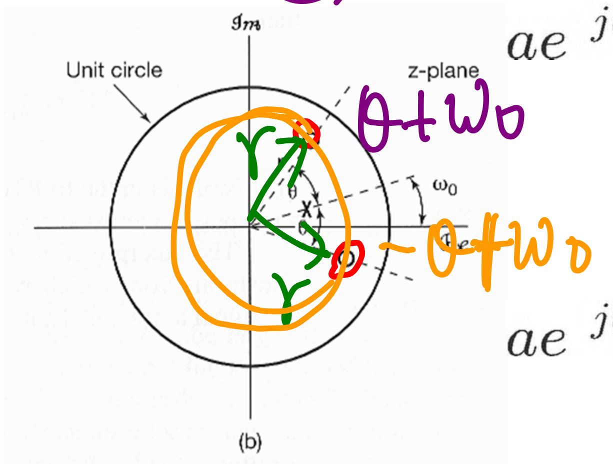
$(e^{j\omega_0})^n$

$|\ \ | = 1$



ae^{jb}

ae^{jb}



$ae^{jb} \quad e^{j\omega_0}$

$ae^{jb} \quad e^{j\omega_0}$

- Time Reversal:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$\sum_{n=-\infty}^{+\infty} x[-n] z^{n} \stackrel{z}{\longleftrightarrow} X\left(\frac{1}{z}\right), \quad ROC = \frac{1}{R}$$

$-n = m$

$$= \sum_{m=+\infty}^{-\infty} x[m] \, (z)^{+m}$$

$m: +\infty \to -\infty$

$$= \sum_{m=-\infty}^{+\infty} x[m] \, \left(\frac{1}{z}\right)^{-m}$$

$m: -\infty \to +\infty$

$$= X\left(\frac{1}{z}\right)$$

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▪ Time Expansion:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n = k \cdot m \\ 0, & \text{otherwise} \end{cases}$$

k is a constant

m is a new time variable

$z = 5$

$z = \sqrt[3]{5}$

$$\sum_{n=-\infty}^{+\infty} x_{(k)}[n] z^{-n} \overset{z}{\longleftrightarrow} X(z^k), \quad ROC = \boxed{R^{1/k}}$$

$$= \sum_{n=-\infty}^{+\infty} x\left[\frac{n}{k}\right] z^{-n}$$

$n = km$

$$= \sum_{m=-\infty}^{+\infty} x\left[\frac{km}{k}\right] z^{-(km)}$$

$$= \sum_{m=-\infty}^{+\infty} x[m] (z^k)^{-(m)} = X(z^k)$$

$X(z)$

$\Downarrow$

$X(z^3)$

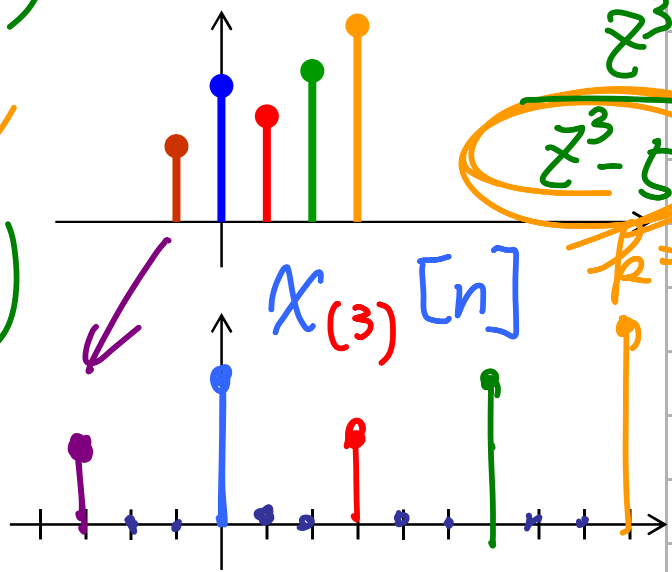
$\frac{z}{z-5}$

$\frac{z^3}{z^3-5}$

$k=3$

$x[n]$

$x_{(3)}[n]$



- Conjugation:

$$\boxed{X(z)} = \sum_{n=-\infty}^{+\infty} \boxed{x[n]} z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

$$x[n] \stackrel{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$x^*[n] \stackrel{z}{\longleftrightarrow} X^*(z^*), \quad ROC = R$$

$$\left(X(z) \right)^* = \left(\sum_{n=-\infty}^{+\infty} x[n] \, z^{-n} \right)^*$$

$$= \sum_{n=-\infty}^{+\infty} x^*[n] \left(z^{-n} \right)^*$$

$$= \sum_{n=-\infty}^{+\infty} x^*[n] \, (z^*)^{-n}$$

$$\boxed{X^*(z^*)} = \sum_{n=-\infty}^{+\infty} \boxed{x^*[n]} \, (z^*)^{-n}$$

■ Convolution Property:

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

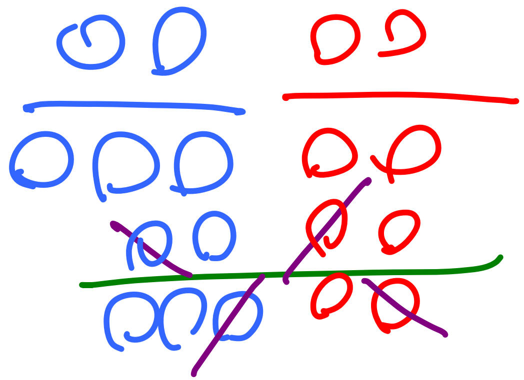
$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \quad ROC = R_1$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \quad ROC = R_2$$

$$\sum_{n=-\infty}^{+\infty} \left(x_1[n] * x_2[n] \right) z^{-n} \qquad x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z),$$

with ROC containing $\boxed{R_1 \cap R_2}$

$R_1 \cap R_2$ may be larger if pole-zero cancellation occurs in the product



$$= \sum_{n=-\infty}^{+\infty} \left(\sum_{m=-\infty}^{+\infty} x_1[m] \; x_2[n-m] \right) z^{-n}$$

$$= \sum_{m=-\infty}^{+\infty} x_1[m] \left(\sum_{n=-\infty}^{+\infty} x_2[\underbrace{n-m}_{k}] z^{-n} \right)$$

$$= \sum_{m=-\infty}^{+\infty} x_1[m] \left(\sum_{=-\infty}^{+\infty} x_2[k] z^{-(k+m)} \right)$$

$$= \left(\sum_{m=-\infty}^{+\infty} x_1[m] \, z^{-m} \right) \left(\sum_{=-\infty}^{+\infty} x_2[k] z^{-(k)} \right) = X_1(z) \cdot X_2(z)$$

- Differentiation in the z-Domain:

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad ROC = R$$

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z), \quad ROC = R$$

$$\frac{d}{dz} X(z) = \frac{d}{dz}\left(\sum_{n=-\infty}^{+\infty} x[n] \, z^{-n} \right)$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \frac{d}{dz} z^{-n}$$

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{+\infty} x[n] \, (-n) \, z^{-n-1}$$

$$(-z) \frac{d}{dz} X(z) = \sum_{n=-\infty}^{+\infty} x[n] \, n \, z^{-n}$$

$$\boxed{(-z) \frac{d}{dz} X(z)} = \sum_{n=-\infty}^{+\infty} \boxed{n \, x[n]} \, z^{-n}$$

$$\frac{d}{dz}\left(\quad\right)(-z)$$

lim sX(s)
$s \to 0$

If $x[n] = 0$ for $n < 0$

- The Initial-Value Theorem:

$$\Rightarrow x[0] = \lim_{z \to \infty} X(z)$$

lim
$z \to \infty$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} = x[0] + x[1] z^{-1} + x[2] z^{-2}$$

0 0 0

$\frac{z-1}{z}$

- The Final-Value Theorem:

$$\Rightarrow x[\infty] = \lim_{z \to 1} (1 - z^{-1}) X(z) \qquad X(z) - (z^{-1}) X(z)$$

$z \to 1$

$$X(z) = x[0] + x[1] z^{-1} + x[2] z^{-2}$$

$$-(z^{-1}) X(z) = \quad -x[0] z^{-1} - x[1] z^{-2} - x[2] z^{-3}$$

$\boxed{x[\infty]}$

Properties of the z-Transform

TABLE 10.1 PROPERTIES OF THE z-TRANSFORM

Section	Property	Signal	z-Transform	ROC
		$x[n]$	$X(z)$	R
		$x_1[n]$	$X_1(z)$	R_1
		$x_2[n]$	$X_2(z)$	R_2
10.5.1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
10.5.2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
10.5.3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
10.5.4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
10.5.5	Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
10.5.6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
10.5.7	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
10.5.7	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
10.5.7	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
10.5.8	Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

10.5.9

Initial Value Theorem

If $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z)$$

- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
- Geometric Evaluation of the Fourier Transform
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Some z-Transform Pairs

TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z, except 0 (if $m>0$) or ∞ (if $m<0$)
5. $\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
6. $-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$
7. $n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
8. $-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \|\alpha\|$
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$

signals

- The z-Transform
- The Region of Convergence for z-Transforms
- The Inverse z-Transform
- Geometric Evaluation of the Fourier Transform
- Properties of the z-Transform
- Some Common z-Transform Pairs

systems

- Analysis & Characterization of LTI Systems Using the z-Transforms
- System Function Algebra and Block Diagram Representations
- The Unilateral z-Transform

Analysis & Characterization of LTI Systems:

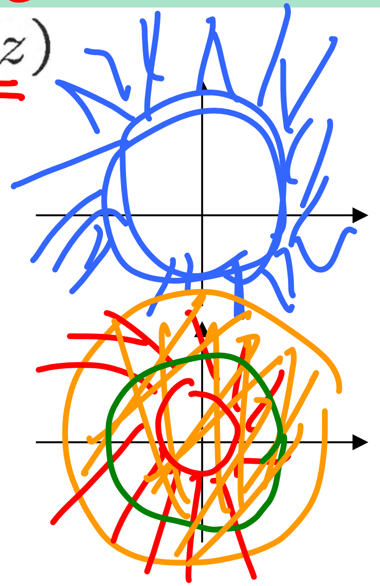
$$x[n] \quad X(z) \longrightarrow \boxed{\text{LTI System}} \longrightarrow \quad y[n] = h[n] * x[n], \quad Y(z) = H(z)\,X(z)$$

$h[n]$

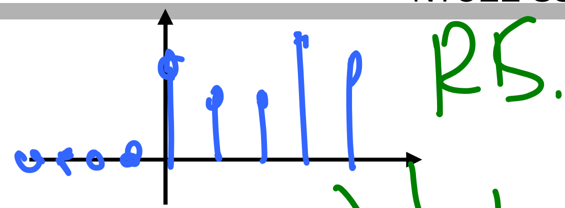
$$H(z) = \mathcal{Z}\{h[n]\}$$

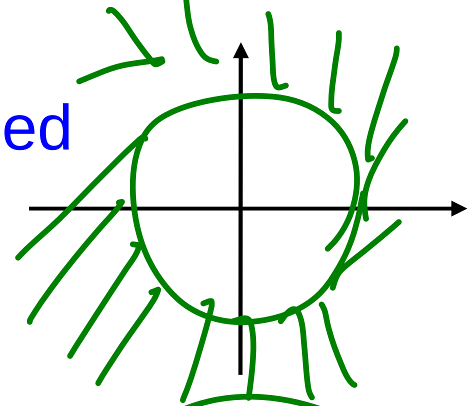
$H(z)$: system function or transfer function

① def ② conv LTI ⑩ z-T

	$h[n]$	$H(z)$
▪ Causality: $x[m] \rightarrow y[n]$, $m \le n$	$h[n]=0 \quad \forall n<0$	
▪ Stability: $\lvert x[n] \rvert < A \Rightarrow \lvert y[n] \rvert < B \quad \forall n$	$\sum \lvert h[n] \rvert < C$	$\lvert z \rvert = 1$

- **Causality:** $\sum_{n=0}^{+\infty} h[n] z^{-n} = H(z)$





  - For a causal LTI system, h[n] = 0 for n < 0, and thus is right-sided
  - A DT LTI system is causal if and only if the ROC of the system function H(z) is the exterior of a circle in the z-plane, including infinity

    $z = \infty$

  - A DT LTI system with a rational H(z) is causal if and only if
    (a) ROC is exterior of a circle outside outermost pole; and infinity must be in the ROC; and
    (b) order of numerator <= order of denominator

$$H(z) = \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N} = \frac{Y(z)}{X(z)} \qquad y[n+N] + \cdots = x[n+M] + \cdots$$

4 3

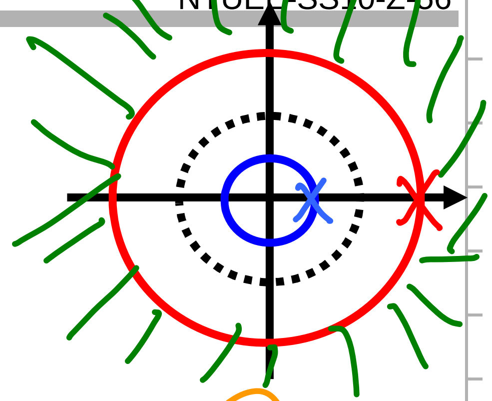
3 4

$M \le N$

4 3

3 4

- Example 10.21:



$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$$H(z) = \frac{2-\frac{5}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{2z(z-\frac{5}{4})}{(z-\frac{1}{2})(z-2)} = \frac{2z^2-\frac{5}{2}z}{z^2-\frac{5}{2}z+1}$$

$\Rightarrow$ ROC: the exterior of a circle of outside the outermost pole

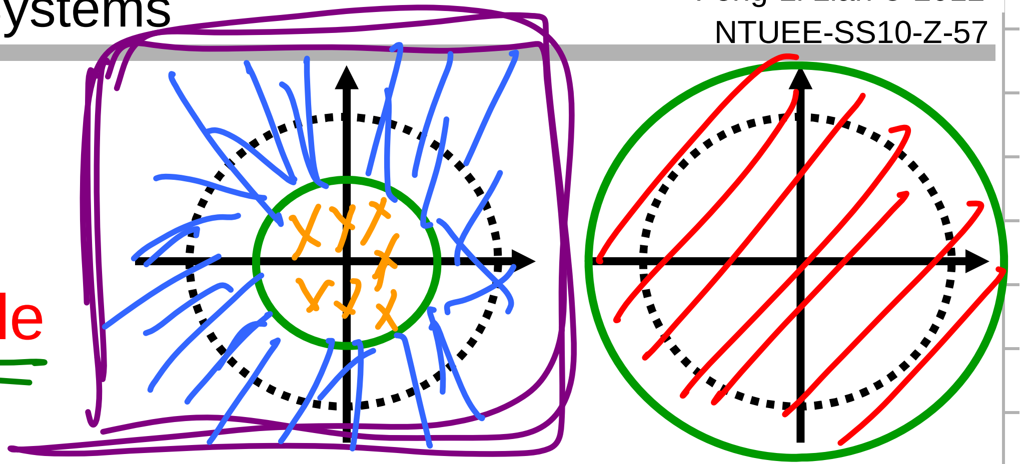
$\Rightarrow$ the impulse response is right-sided

$\Rightarrow$ deg of num of $H(z)$ = deg of den of $H(z)$

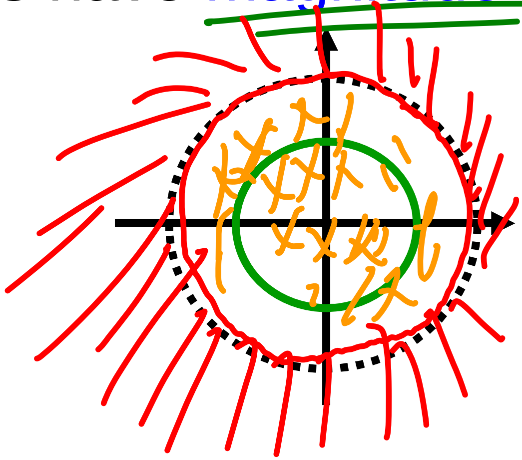
$\Rightarrow$ the system is causal

$\Rightarrow$ $h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right] u[n]$ $\qquad \Rightarrow$ $h[n] = 0, n < 0$

- Stability:



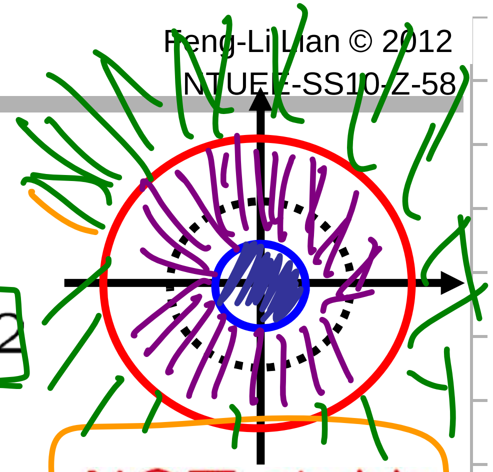
  - An DT LTI system is stable
    if and only if
    the ROC of H(z) includes the unit circle [i.e., |z| = 1]
  - A causal LTI system with rational H(z) is stable
    if and only if
    all of the poles of H(z) lie in the inside the unit circle,
    i.e., all of the poles have magnitude < 1



ROC

- Example 10.22:

Causal



$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2$$

$\Rightarrow$ ROC does not include the unit circle $\Rightarrow$ NOT stable

$\Rightarrow$ i.e., $h[n] = \left[\left(\frac{1}{2}\right)^n + 2^n\right] u[n] \rightarrow \infty$, as $n \rightarrow \infty$

- If $ROC = |z| < 1/2$ $\Rightarrow h[n] = -\left[\left(\frac{1}{2}\right)^n + 2^n\right] u[-n-1]$

  $\Rightarrow$ the system is neigher causal nor stable

- If $ROC = 1/2 < |z| < 2$ $\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$

  $\Rightarrow$ the system is NOT causal, but stable

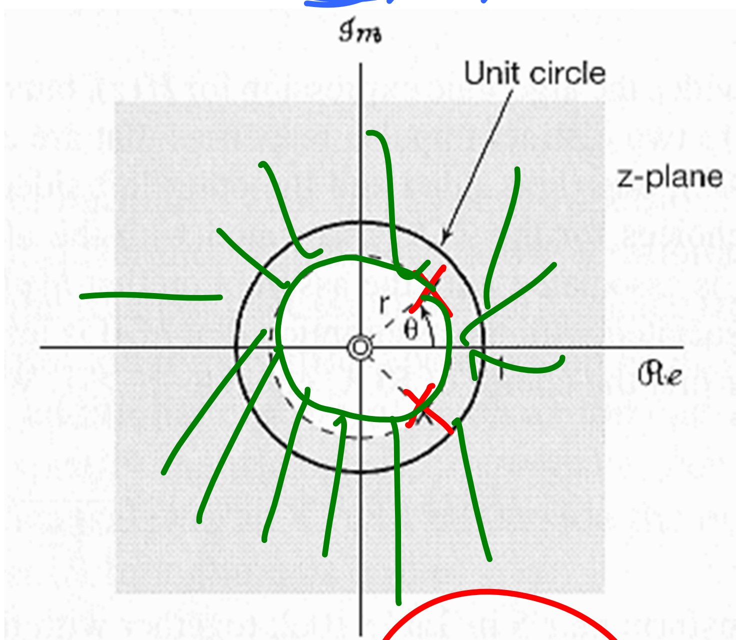
- Example 10.24:

$\Rightarrow h[n] = \left((re^{j\theta})^n + (re^{-j\theta})^n \right) u[n]$

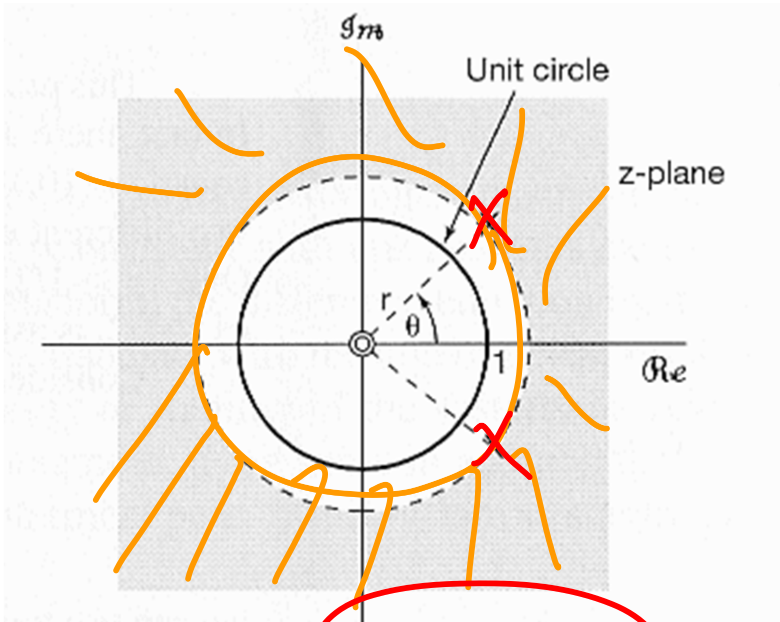
$$H(z) = \frac{1}{1-(2r\cos\theta)z^{-1}+r^2z^{-2}} = \frac{z^2}{z^2-(2r\cos\theta)z+r^2}$$

$$\Rightarrow \; z_1 = re^{j\theta}, \quad z_2 = re^{-j\theta}$$

If it is causal, $|z| > |r|$



$r < 1$, stable



$r > 1$, unstable

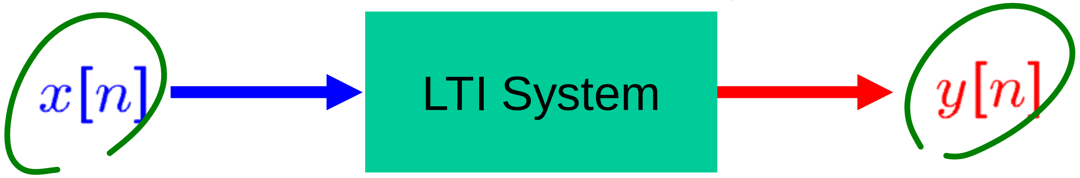
- LTI Systems by Linear Constant-Coef Difference Equations:

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N]$$

$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

→ h[n], H(z)



$$Y(z) = X(z)H(z) \qquad H(z) = \frac{Y(z)}{X(z)}$$

$$x[n-n_0] \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z)$$

$$\mathcal{Z}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{Z}\left\{y[n-k]\right\} = \sum_{k=0}^{M} b_k \mathcal{Z}\left\{x[n-k]\right\}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

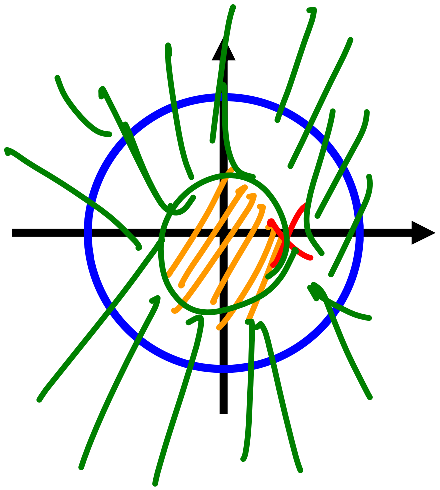
$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0 z^{N} + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{a_0 z^{N} + a_1 z^{N-1} + \cdots + a_N}$$

zeros

poles

- Example 10.25: $\mathcal{Z}\left\{ y[n] - \frac{1}{2} y[n-1] \right\} = \mathcal{Z}\left\{ x[n] + \frac{1}{3} x[n-1] \right\}$



$$\Rightarrow Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z) + \frac{1}{3} z^{-1} X(z)$$

$$\Rightarrow H(z) = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}} = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{\frac{1}{3} \cancel{z^{-1}}}{1 - \frac{1}{2} z^{-1}} \quad z^{-1}$$

$$= \frac{z + \frac{1}{3}}{z - \frac{1}{2}} = \frac{z}{z - \frac{1}{2}} + \frac{\frac{1}{3} z}{z - \frac{1}{2}} \quad z^{-1}$$

- If $ROC = \{|z| > 1/2\}$, $\Rightarrow$ $h[n]$ is right-sided

$$\Rightarrow h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$n \rightarrow n-1$

- If $ROC = \{|z| < 1/2\}$, $\Rightarrow$ $h[n]$ is left-sided

$$\Rightarrow h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[-n]$$

$u[-n-1]$

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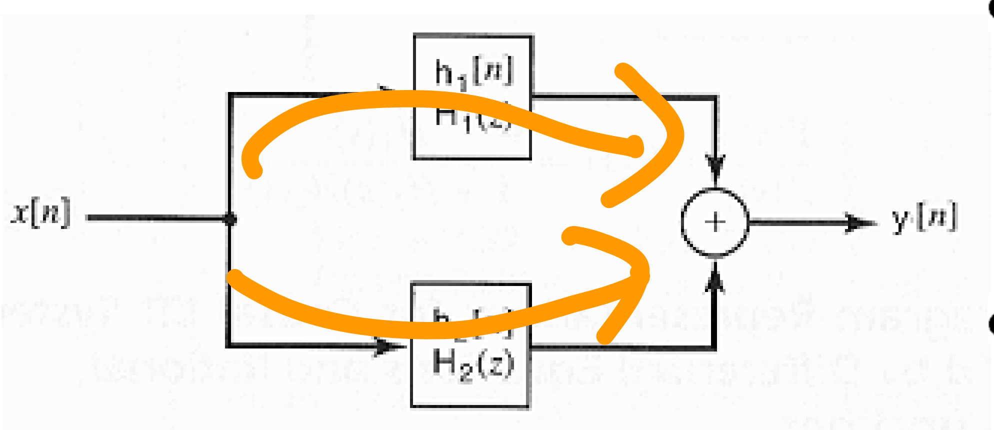
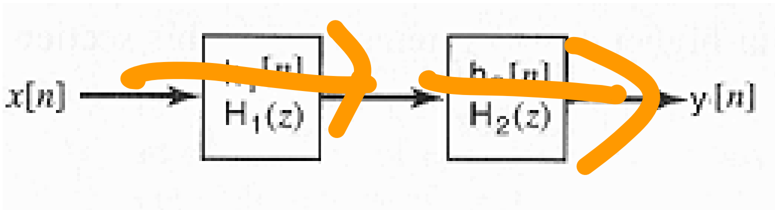
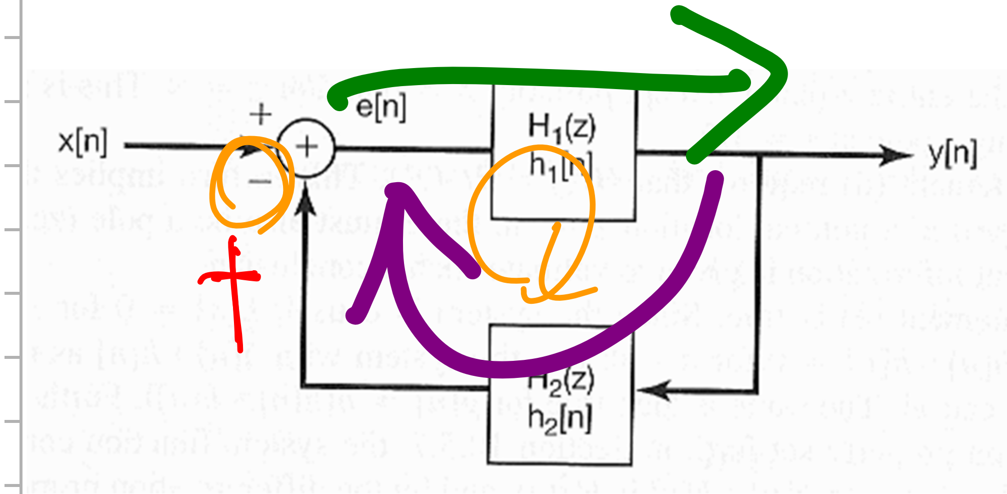
System Function Blocks:







- parallel interconnection

$$h[n] = h_1[n] + h_2[n]$$

$$H(z) = H_1(z) + H_2(z)$$

- series interconnection

$$h[n] = h_1[n] * h_2[n]$$

$$H(z) = H_1(z)\ H_2(z)$$

- feedback interconnection

$$Y = H_1\ E$$

$$Z = H_2\ Y$$

$$E = X - Z$$

$$H(z) = \frac{H_1(z)}{1 + H_1(z) H_2(z)}$$

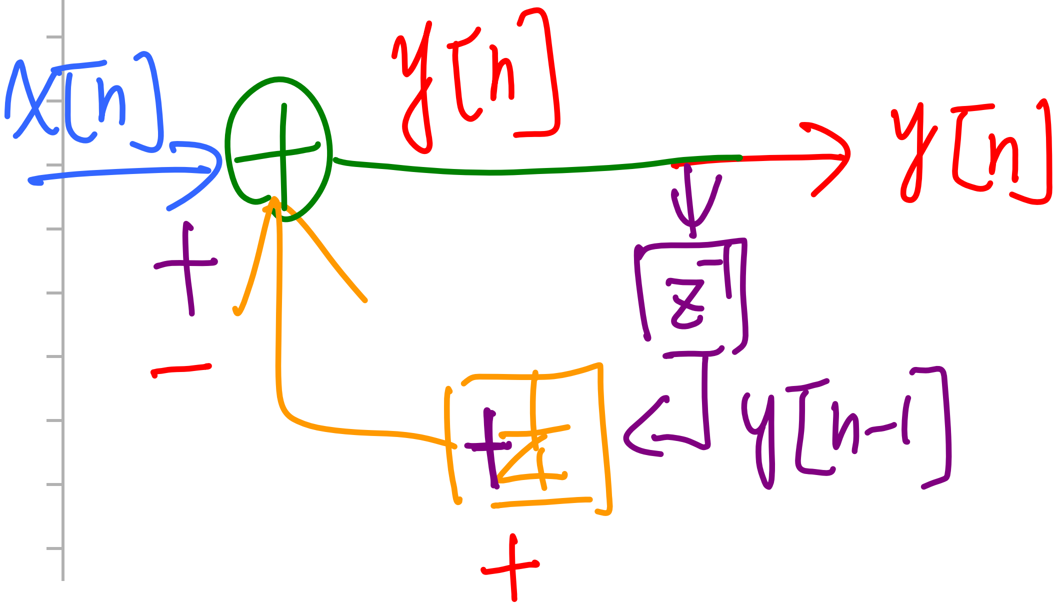
- **Example 10.28:**
  - Consider a causal LTI system with system function

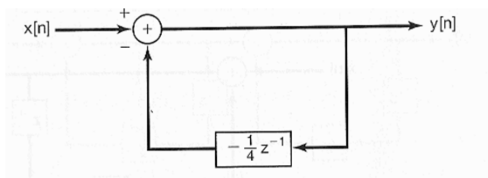
$$\frac{1}{s+a} \qquad \frac{1}{1-az^{-1}}$$

$$H(z) = \boxed{\frac{1}{1-\frac{1}{4}z^{-1}}} \qquad z^{-1}y[n] = y[n-1]$$

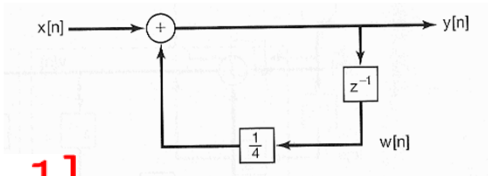
$$\Rightarrow y[n] = \frac{1}{1-\frac{1}{4}z^{-1}}x[n] \qquad \Rightarrow y[n] - \frac{1}{4}y[n-1] = x[n]$$

$$\Rightarrow y[n] = \frac{1}{4}y[n-1] + x[n]$$



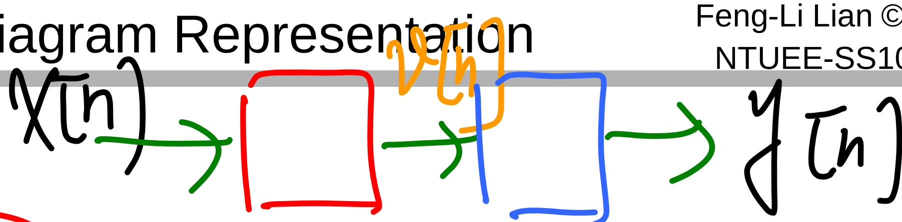


(b)



(a)

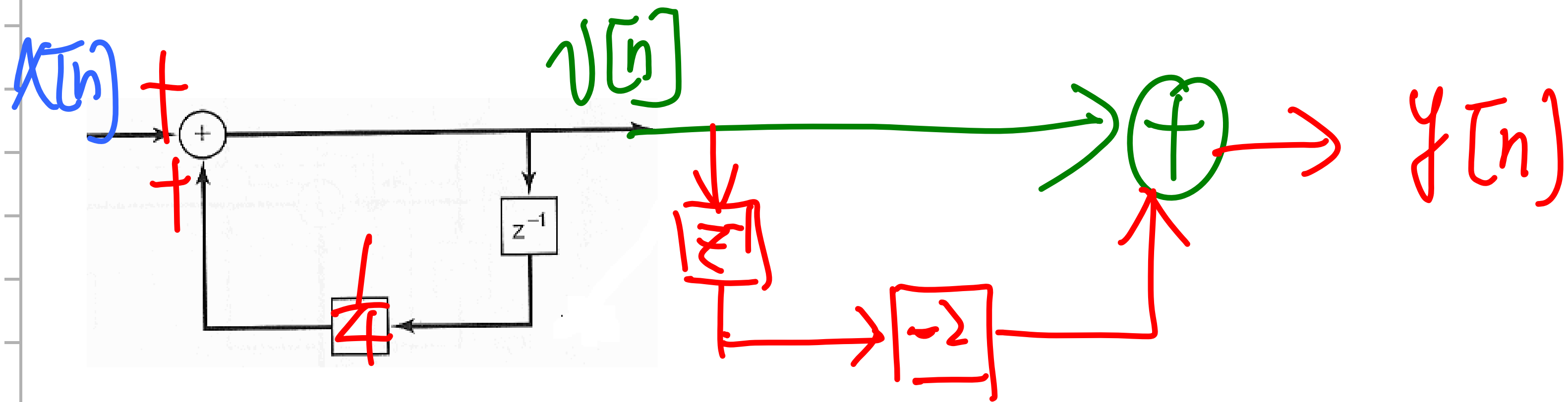
$$w[n] = y[n-1]$$

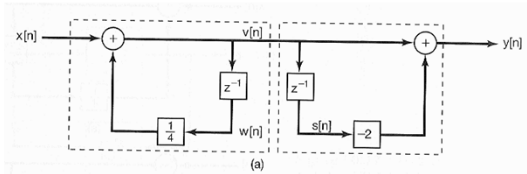


- Example 10.29:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\left(1-2z^{-1}\right)$$

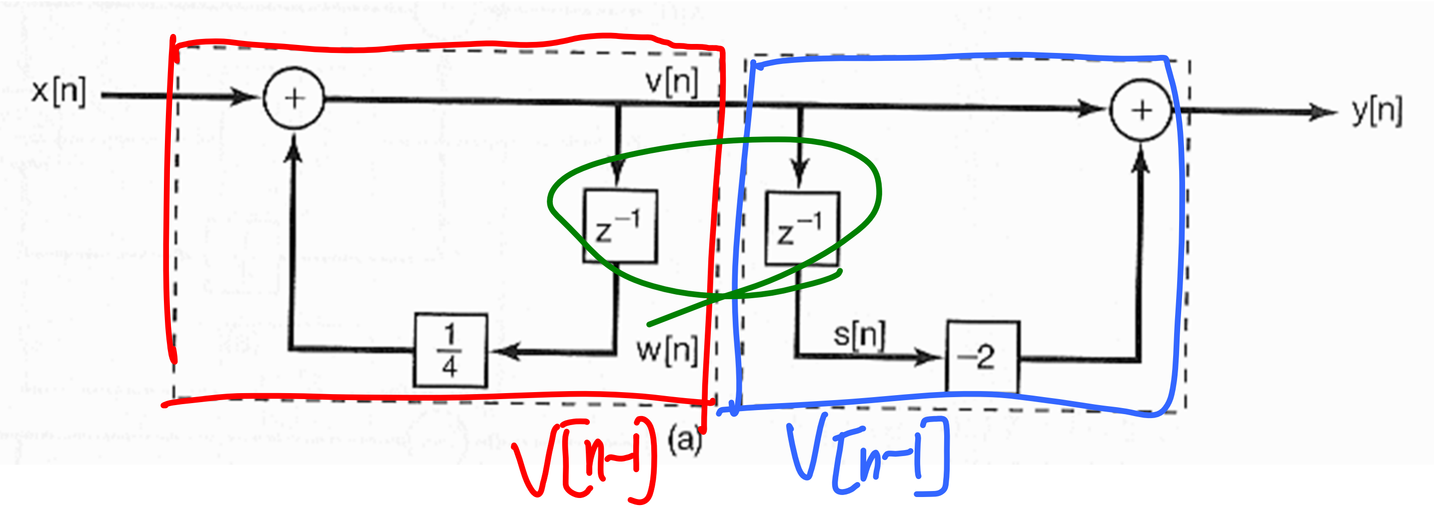
$$Y(z) = \left(1-2z^{-1}\right)V(z) \qquad V(z) = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)X(z)$$



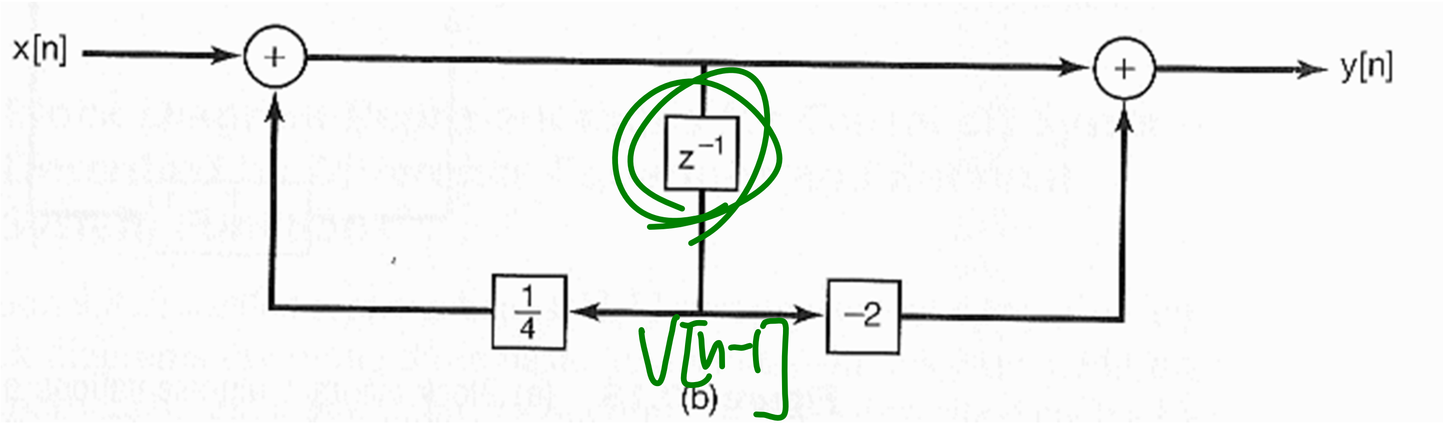


- Example 10.29:

$$H(z) = \frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}} = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)(1-2z^{-1})$$



(a)



(b)

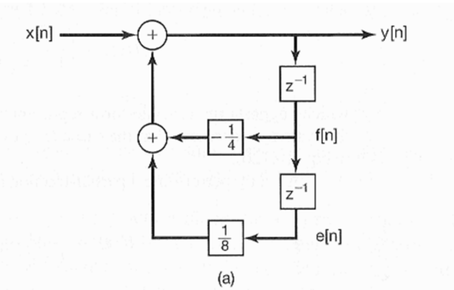
- Example 10.30:

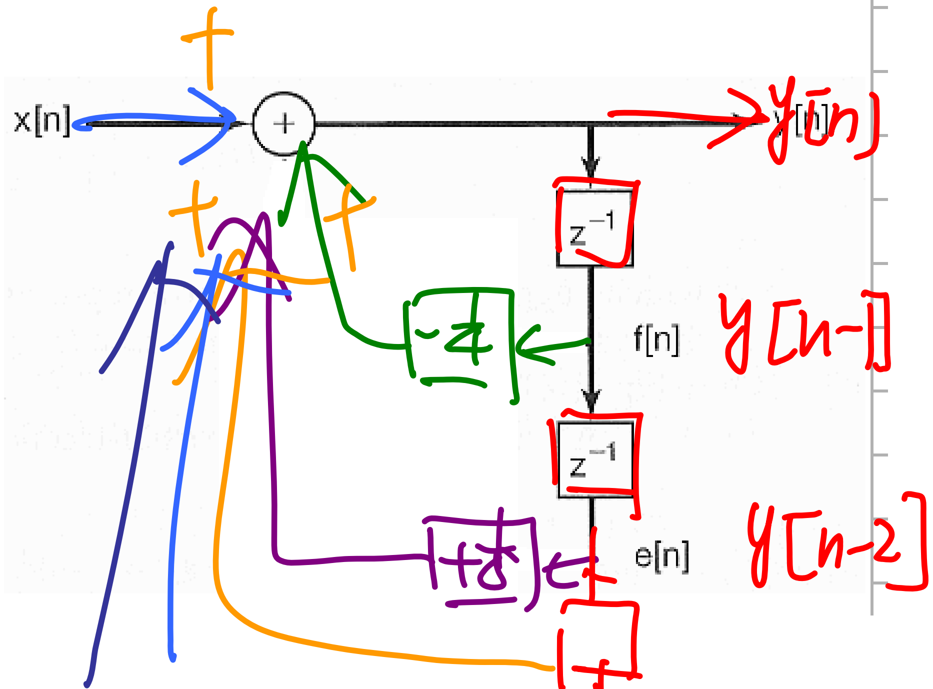
$$H(z) = \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$\Rightarrow \quad y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$

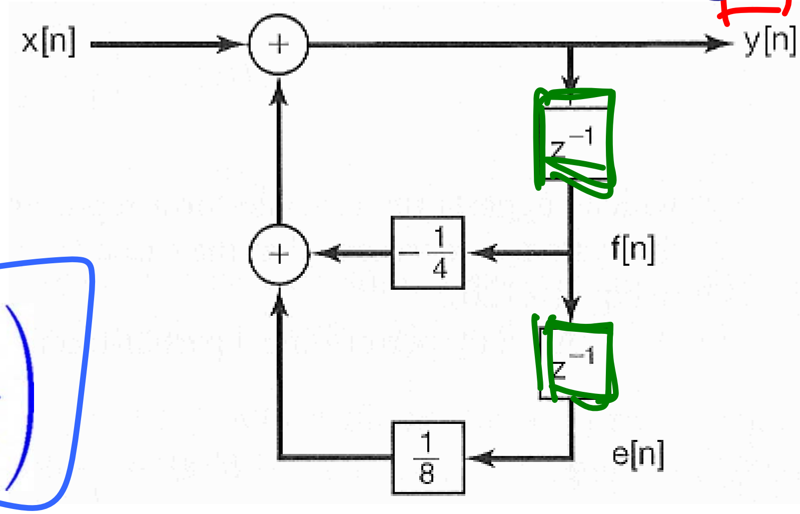
$$\Rightarrow \quad y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n]$$

$$\Rightarrow \quad \begin{cases} y[n-1] = f[n] \\ y[n-2] = e[n] = f[n-1] \end{cases}$$





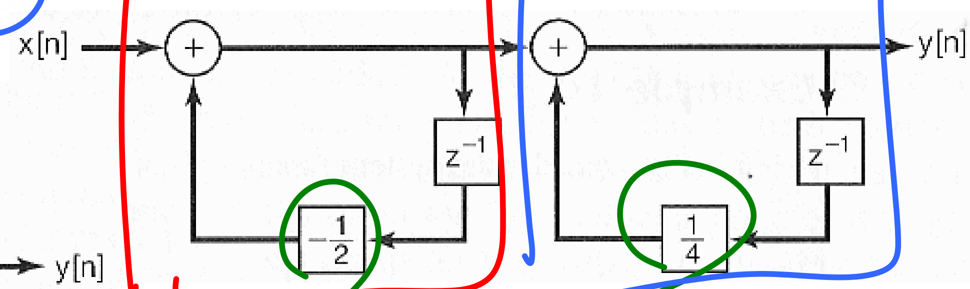
- **Example 10.30:**

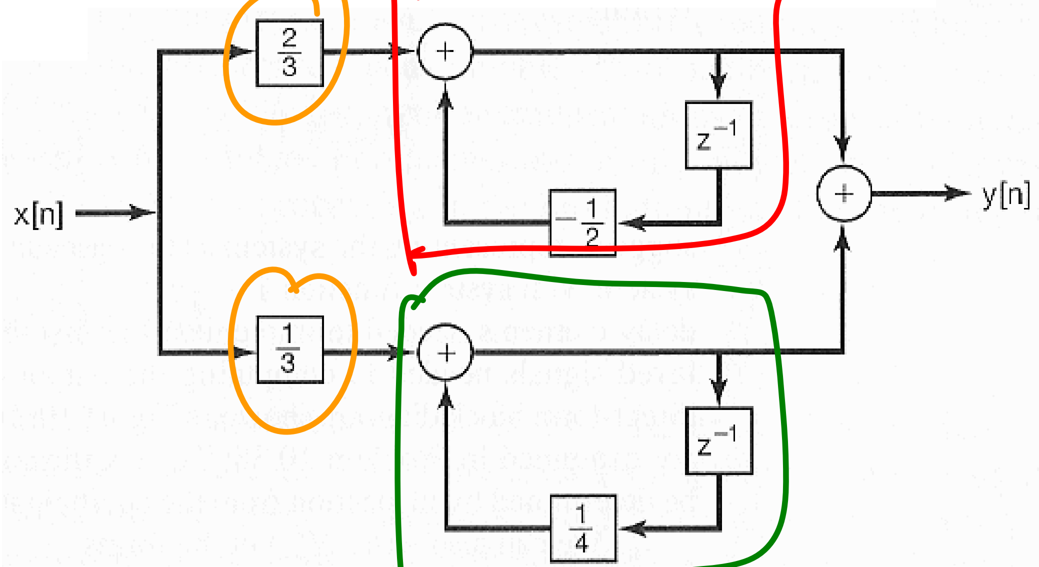


$$H(z) = \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$H(z) = \left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

$$H(z) = \frac{\frac{2}{3}}{(1+\frac{1}{2}z^{-1})} + \frac{\frac{1}{3}}{(1-\frac{1}{4}z^{-1})}$$





- The z-Transform
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The Unilateral z-Transform of x(t):

bilateral zT

unilateral zT
for causal system &
with nonzero init. cond.

$$X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{-1} x[n] z^{-n} + \sum_{n=0}^{+\infty} x[n] z^{-n}$$

f[0]

$$\mathcal{X}(z) \triangleq \sum_{n=0}^{+\infty} x[n] z^{-n}$$

n ≥ 0

n ≥ 0

n < 0
x[n] = 0

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \qquad x[n] \overset{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}(z)$$

$$X(z) = \mathcal{Z}\{x[n]\} \qquad \mathcal{X}(z) = \mathcal{UZ}\{x[n]\}$$

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} \qquad x[n] = \mathcal{UZ}^{-1}\{\mathcal{X}(z)\}$$

ROC: exterior of a circle

- **Time-Shifting Property**

$$x[n] \overset{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}(z) \qquad z^{-1} \qquad z^{-2} \qquad z^{-3}$$

$$\mathcal{X}(z) = \sum_{n=0}^{+\infty} x[n] z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$x[n-1] \overset{\mathcal{UZ}}{\longleftrightarrow} z^{-1}\mathcal{X}(z) + x[-1]$$

$$\sum_{n=0}^{+\infty} x[n-1] z^{-n} = x[-1] + x[0]z^{-1} + x[1]z^{-2} + x[2]z^{-3} + \cdots$$

$$z^{-1}\mathcal{X}(z) = x[0]z^{-1} + x[1]z^{-2} + x[2]z^{-3} + \cdots$$

$$x[n-2] \overset{\mathcal{UZ}}{\longleftrightarrow} z^{-2}\mathcal{X}(z) + x[-1]z^{-1} + x[-2]$$

$$x[n+1] \overset{\mathcal{UZ}}{\longleftrightarrow} z\mathcal{X}(z) - zx[0]$$

Example 10.33:

$$x[n] = a^{n+1}u[n+1]$$

$a^n u[n]$

$n=-1$
$n=0$
$n=1$

BZT UZT

- since $x[-1] = 1 \neq 0$ $\quad X(z) \neq \mathcal{X}(z)$

$\Rightarrow$ bilateral transform : $\quad X(z) = \dfrac{z}{1-az^{-1}}, \quad |z| > |a|$

$\Rightarrow$ unilateral transform :

$$\mathcal{X}(z) = \sum_{n=0}^{+\infty} x[n]z^{-n}$$

$$= \sum_{n=0}^{+\infty} a^{n+1}z^{-n}$$

$$= \frac{a}{1-az^{-1}}, \quad |z| > |a|$$

$$x_1[n] = a^n u[n] \qquad X_1(z) = \frac{z}{z-a} = \frac{1}{1-z^{-1}a}$$

$$x_2[n] = x_1[n+1]$$
$$= a^{n+1} u[n+1]$$

$$X_2(z) = z\frac{z}{z-a}\boxed{-z\,x_1[0]}$$

$$= \frac{z^2}{z-a} - z\cdot 1$$

$$= \frac{z^2 - z^2 + az}{z-a}$$

$$= \frac{az}{z-a} =$$

$$= \frac{a}{1-z^{-1}a}$$

The Unilateral z-Transform

TABLE 10.3 PROPERTIES OF THE UNILATERAL z-TRANSFORM

Property	Signal	Unilateral z-Transform
—	$x[n]$	$\mathcal{X}(z)$
—	$x_1[n]$	$\mathcal{X}_1(z)$
—	$x_2[n]$	$\mathcal{X}_2(z)$
Linearity	$ax_1[n] + bx_2[n]$	$a\mathcal{X}_1(z) + b\mathcal{X}_2(z)$
Time delay	$x[n-1]$	$\boxed{z^{-1}\mathcal{X}(z) + x[-1]}$
Time advance	$x[n+1]$	$\boxed{z\mathcal{X}(z) - zx[0]}$
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$\mathcal{X}(e^{-j\omega_0}z)$
	$z_0^n x[n]$	$\mathcal{X}(z/z_0)$
	$a^n x[n]$	$\mathcal{X}(a^{-1}z)$
Time expansion	$x_k[n] = \begin{cases} x[m], & n = mk \\ 0, & n \neq mk \text{ for any } m \end{cases}$	$\mathcal{X}(z^k)$
Conjugation	$x^*[n]$	$\mathcal{X}^*(z^*)$
Convolution (assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$)	$x_1[n] * x_2[n]$	$\mathcal{X}_1(z)\mathcal{X}_2(z)$
First difference	$x[n] - x[n-1]$	$\boxed{(1 - z^{-1})\mathcal{X}(z) - x[-1]}$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{1}{1 - z^{-1}}\mathcal{X}(z)$
Differentiation in the z-domain	$nx[n]$	$-z\frac{d\mathcal{X}(z)}{dz}$

Initial Value Theorem

$$x[0] = \lim_{z\to\infty} \mathcal{X}(z)$$

$$z^{-1}\mathcal{X}(z) + x[-1]$$

$$z\mathcal{X}(z) - zx[0]$$

$$x_1[n] = x_2[n] \equiv 0, \quad n < 0$$

$$(1 - z^{-1})\mathcal{X}(z) - x[-1]$$

Summary of Fourier Transform and z Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) e^{jwn} dw$$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

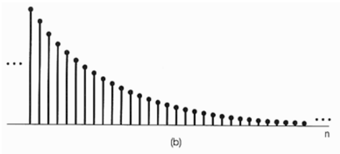
$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

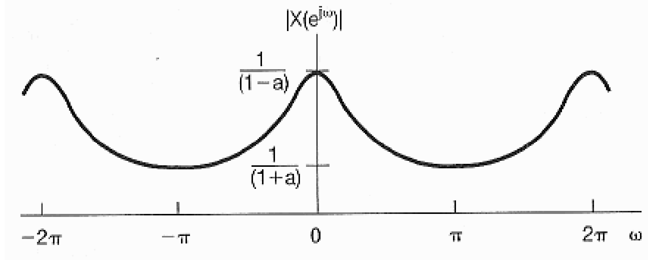
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

$$X(z) = X(re^{jw}) = \mathcal{Z}\left\{x[n]\right\} = \mathcal{F}\left\{x[n] r^{-n}\right\}$$

$$X(e^{jw}) = \mathcal{F}\left\{x[n]\right\} = \mathcal{Z}\left\{x[n]\right\}\Big|_{z=e^{jw}} = X(z)\Big|_{z=e^{jw}}$$

Summary of Fourier Transform and z Transform

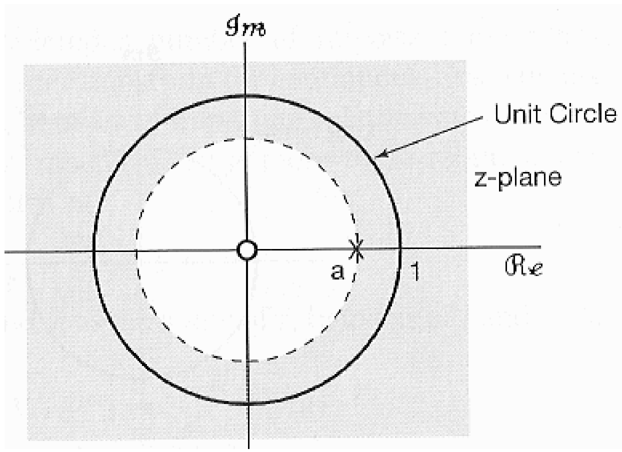




$$x[n] = a^n \, u[n], \quad |a| < 1 \qquad \stackrel{\mathcal{F}}{\longleftrightarrow} \qquad X(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$x[n] = a^n u[n] \qquad \stackrel{\mathcal{L}}{\longleftrightarrow} \qquad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

$$|z| > |a|$$



definition
theorem
property

Causality

Stability

ROC

Problem 10.12 (p. 799)

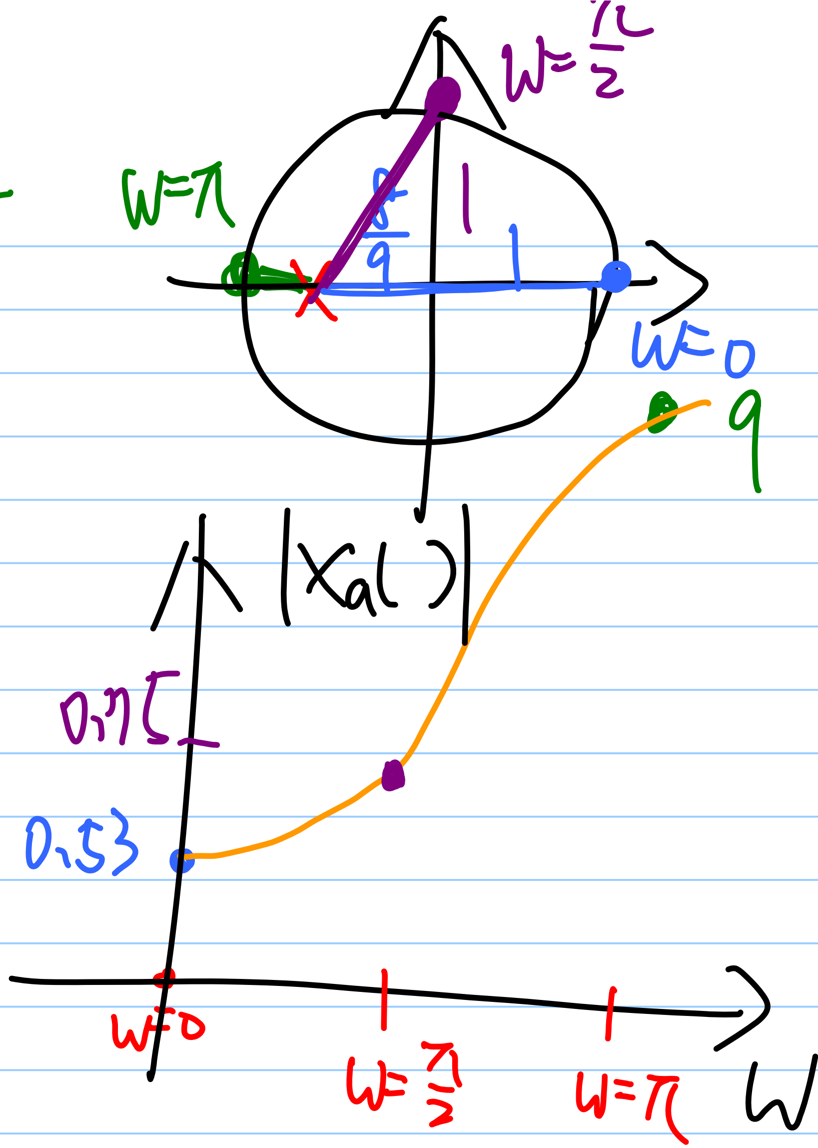
10.12. By considering the geometric interpretation of the magnitude of the Fourier transform from the pole-zero plot, determine, for each of the following z-transforms, whether the corresponding signal has an approximately lowpass, bandpass, or highpass characteristic:

(a) $X(z) = \dfrac{z^{-1}}{1 + \frac{8}{9}z^{-1}}, \quad |z| > \frac{8}{9}$

(b) $X(z) = \dfrac{1 + \frac{8}{9}z^{-1}}{1 - \frac{16}{9}z^{-1} + \frac{64}{81}z^{-2}}, \quad |z| > \frac{8}{9}$

(c) $X(z) = \dfrac{1}{1 + \frac{64}{81}z^{-2}}, \quad |z| > \frac{8}{9}$

(a) $X_a(z) = \dfrac{z^{-1}}{1+\frac{8}{9}z^{-1}} = \dfrac{1}{z+\frac{8}{9}}$



$|X_a(e^{jw})|$

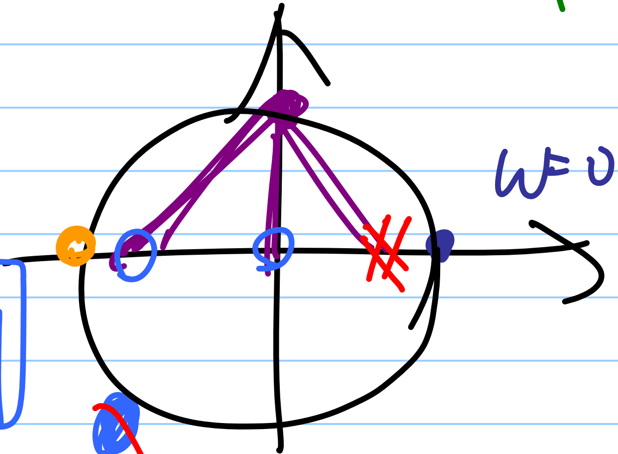
$\underline{\underline{w=0}}$ $\dfrac{1}{1+\frac{8}{9}} = \dfrac{9}{17} = \boxed{0.5294}$

$\underline{\underline{w=\frac{\pi}{2}}}$ $\dfrac{1}{\sqrt{(\frac{8}{9})^2+1^2}} = \dfrac{1}{1.338} = \boxed{0.7474}$

$\underline{\underline{w=\pi}}$ $\dfrac{1}{\frac{1}{9}} = \boxed{9}$

high pass

(b) $X_b(z) = \dfrac{1+\frac{8}{9}z^{-1}}{1-\frac{16}{9}z^{-1}+\frac{64}{81}z^{-2}} = \dfrac{z\left(z+\frac{8}{9}\right)}{z^2-\frac{16}{9}z+\frac{64}{81}} = \dfrac{z\left(z+\frac{8}{9}\right)}{\left(z-\frac{8}{9}\right)^2}$

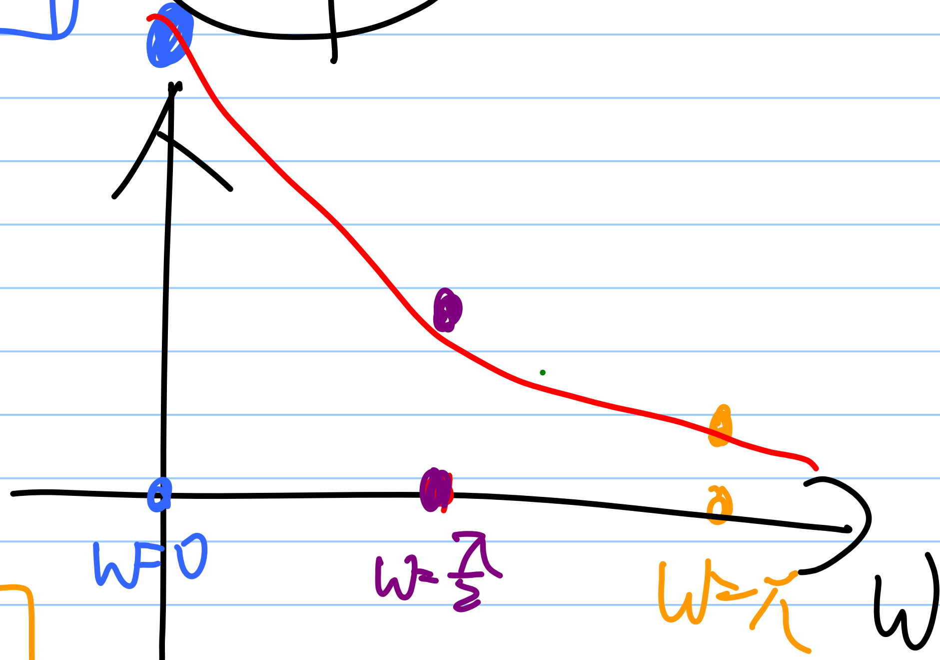


$\left|X_b(e^{jw})\right|$

$\underline{\underline{w=0}}$ $\quad \dfrac{1\left(1+\frac{8}{9}\right)}{\frac{1}{9}\cdot\frac{1}{9}} = \dfrac{\frac{17}{\cancel{9}}}{\frac{1}{81\cancel{}\,9}} = \boxed{17\times 9}$

$\underline{\underline{w=\frac{\pi}{2}}}$ $\quad \dfrac{1 \quad \cancel{1{,}338}}{(1{,}338)\cancel{(1{,}338)}} = \boxed{0{,}7474}$

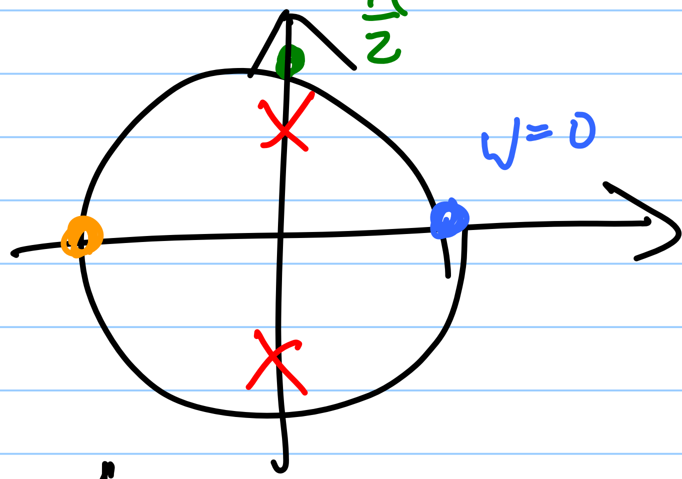
$\underline{\underline{w=\pi}}$ $\quad \dfrac{\frac{1}{9}\cdot 1}{\left(1+\frac{8}{9}\right)^2} = \dfrac{9}{17^2} = \boxed{0{,}0311}$



low pass

(c) $$X_c(z) = \frac{1}{1+\frac{64}{81}z^{-2}} = \frac{1}{z^2+\frac{64}{81}} = \frac{1}{(z+\frac{8}{9}i)(z-\frac{8}{9}i)}$$

$$|X_c(e^{jw})|$$



$w=0$

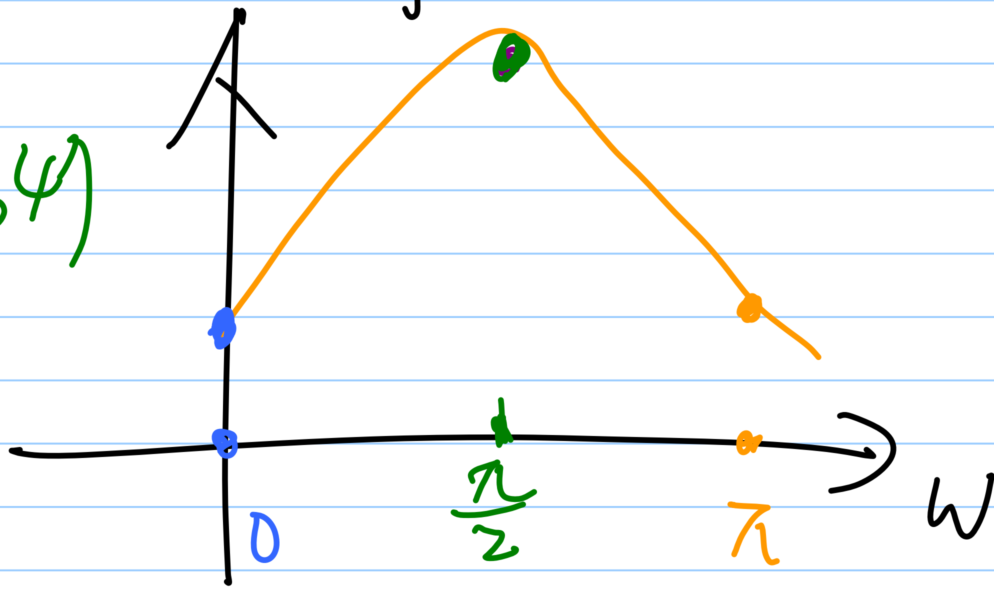
$$= \frac{1}{(1.338)(1.338)} = 0.5586$$

$w=\frac{\pi}{2}$

$$= \frac{1}{\frac{1}{9}\cdot\frac{17}{9}} = \frac{81}{17} = 4.7647$$

$w=\pi$

$$= \frac{1}{(1.338)(1.338)} = 0.5586$$



band pass

Problem 10.48 (p. 809)

10.48. Suppose a second-order causal LTI system has been designed with a real impulse response $h_1[n]$ and a rational system function $H_1(z)$. The pole-zero plot for $H_1(z)$ is shown in Figure P10.48(a). Now consider another causal second-order system with impulse response $h_2[n]$ and rational system function $H_2(z)$. The pole-zero plot for $H_2(z)$ is shown in Figure P10.48(b). Determine a sequence $g[n]$ such that the following three conditions hold:

(1) $h_2[n] = g[n]h_1[n]$

(2) $g[n] = 0$ for $n < 0$

(3) $\sum_{k=0}^{\infty} |g[k]| = 3$

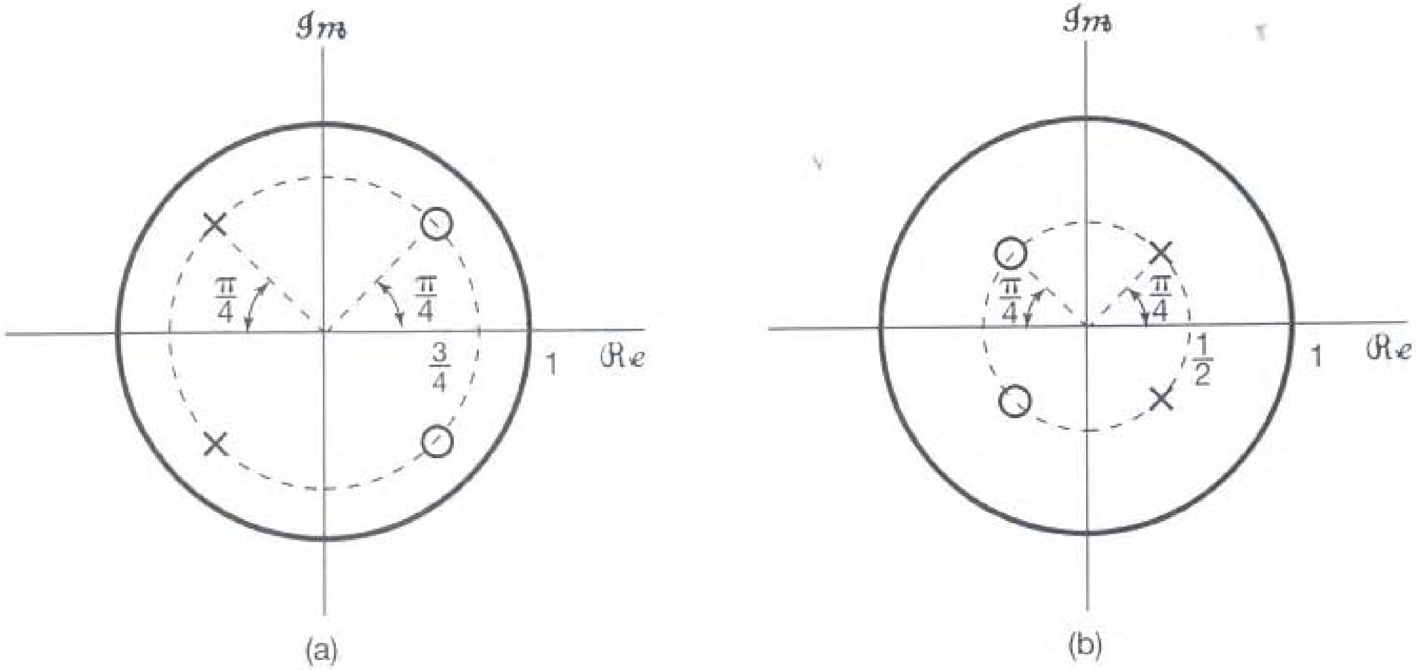


Figure P10.48

$$H_1(z) = A\frac{\left(z-\frac{3}{4}e^{j\frac{\pi}{4}}\right)\left(z-\frac{3}{4}e^{j\left(-\frac{\pi}{4}\right)}\right)}{\left(z-\frac{3}{4}e^{j\frac{3}{4}\pi}\right)\left(z-\frac{3}{4}e^{j\left(-\frac{3}{4}\pi\right)}\right)} \leftarrow \frac{3}{2}ze^{j\pi}$$

$$H_2(z) = B\frac{\left(z-\frac{1}{2}e^{j\frac{3}{4}\pi}\right)\left(z-\frac{1}{2}e^{j\left(-\frac{3}{4}\pi\right)}\right)}{\left(z-\frac{1}{2}e^{j\frac{\pi}{4}}\right)\left(z-\frac{1}{2}e^{j\left(-\frac{\pi}{4}\right)}\right)}$$

$$H_2(z) = \frac{B}{A}H_1\left(\underbrace{z=\frac{3}{2}ze^{j\pi}}\right)$$

$$= \frac{B}{A}H_1\left(-\frac{3}{2}z\right)$$

$$= \frac{B}{A}H_1\left(\frac{z}{\left(-\frac{3}{2}\right)}\right)$$

$$z_0^n x[n] \xleftrightarrow{\mathcal{Z}} X\left(\frac{z}{z_0}\right)$$

Tab 10.1
10.5.3

$$h_2[n] = \frac{B}{A}\left(-\frac{2}{3}\right)^n h_1[n]$$

$$h_2[n] = g[n]\, h_1[n]$$

$h_1[n], h_2[n]$: causal $\Rightarrow$ $g[n]$: causal

$$g[n] = \frac{B}{A}\left(-\frac{2}{3}\right)^n u[n]$$

$$\sum_{k=0}^{\infty} |g[k]| = 3 \Rightarrow \sum_{k=0}^{\infty} \left|\frac{B}{A}\left(-\frac{2}{3}\right)^k\right| = 3$$

$$\frac{1}{1-\left(-\frac{2}{3}\right)} = 3\cdot\left|\frac{A}{B}\right|$$

$$3 = 3\left|\frac{A}{B}\right| \Rightarrow \left|\frac{A}{B}\right| = 1$$

$$\frac{A}{B} = \pm 1$$

$$g[n] = \pm\left(-\frac{2}{3}\right)^n u[n]$$

2. (10%) Using the appropriate properties of the z-transform, determine the sequence $x[n]$ for which the z-transform is the following respectively:

(a) $X(z) = \ln(1-2z)$, $\boxed{|z| < \frac{1}{2}}$

(b) $X(z) = \ln(1-\frac{1}{2}z^{-1})$, $|z| > \frac{1}{2}$

Hint: $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

$$X(z) = \ln(1-2z)$$

$$z\frac{d}{dz}X(z) = \frac{1}{1-2z}(-2) = \frac{-2}{1-2z}$$

$$\mathcal{Z}^{-1}\left\{z\frac{1}{z-\frac{1}{2}}\right\} \Rightarrow \frac{z}{z-\frac{1}{2}}$$

$$\mathcal{Z}^{-1}\left\{z\frac{1}{z-\frac{1}{2}}\right\} = \left(\frac{1}{2}\right)^n u[n]$$

$$\boxed{= -\left(\frac{1}{2}\right)^n u[-n-1]}$$

$$\mathcal{Z}^{-1}\left\{-z\frac{d}{dz}X(z)\right\} = +\left(\frac{1}{2}\right)^n u[-n-1]$$

$$n\,x[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

$$n\,x[n] = \left(\frac{1}{2}\right)^n u[-n-1]$$

$$x[n] = \frac{1}{n}\left(\frac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \ln\left(1 - \frac{1}{2} z^{-1}\right) \qquad |z| > \frac{1}{2}$$

$$\boxed{(-z)\frac{d}{dz}X(z) =} \frac{\left(-\frac{1}{2}\right) z^{-2}(-1)}{1 - \frac{1}{2} z^{-1}} = \frac{\frac{1}{2} z^{-2}}{1 - \frac{1}{2} z^{-1}} = \frac{\frac{1}{2}}{z^2 - \frac{1}{2} z}$$

$$= \frac{\frac{1}{2}}{z\left(z - \frac{1}{2}\right)}(-z) = \left\{\left(-\frac{1}{2}\right) \frac{1}{z - \frac{1}{2}}\right\}$$

$$\mathcal{Z}^{-1}\left\{-\frac{1}{2} \frac{1}{z - \frac{1}{2}}\right\} = -\frac{1}{2} \mathcal{Z}^{-1}\left\{\frac{1}{z - \frac{1}{2}}\right\} = -\frac{1}{2} \mathcal{Z}^{-1}\left\{z^{-1} \boxed{\frac{z}{z - \frac{1}{2}}}\right\}$$

$$\mathcal{Z}^{-1}\left\{\frac{z}{z - \frac{1}{2}}\right\} = \left(\frac{1}{2}\right)^n u[n] \Leftarrow n \to n-1$$

$$\mathcal{Z}^{-1}\left\{z^{-1} \frac{z}{z - \frac{1}{2}}\right\} = \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$\Rightarrow \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} u[n-1] = \boxed{-\left(\frac{1}{2}\right)^{n} u[n-1]}$$

$n\,x[n]$

$$\Rightarrow x[n] = \frac{-1}{n}\left(\frac{1}{2}\right)^{n} u[n-1]$$

6. [18] Consider that a causal linear time-invariant (LTI) system has its input $x[n]$ and output $y[n]$ given by the following difference equation:

$$y[n] - y[n-1] - y[n-2] = x[n-1] \quad \text{(I)}$$

$\Rightarrow z$

$H(z) = \frac{\quad}{\quad}$

(a) Find the system function $H(z)$. Justify your answer. [3]
(b) Sketch the poles and zeros of $H(z)$. [3]
(c) Determine the region of convergence of $H(z)$. [3]
(d) Find the unit impulse response of the LTI system. Justify your answer. [3]
(e) Is the system stable? Using Part (c) to justify your answer. [3]
(f) Find a stable but noncausal LTI system whose unit impulse response satisfies the same difference equation given by (I). Justify your answer. [3]

$h[n]$

ROC

10.34. A causal LTI system is described by the difference equation

$$y[n] = y[n-1] + y[n-2] + x[n-1].$$

(a) Find the system function $H(z) = Y(z)/X(z)$ for this system. Plot the poles and zeros of $H(z)$ and indicate the region of convergence.
(b) Find the unit sample response of the system.
(c) You should have found the system to be unstable. Find a stable (noncausal) unit sample response that satisfies the difference equation.

- The z-Transform
- The ROC for z-T
- The Inverse z-T
- Geometric Evaluation of the FT
- Properties of the z-T
  - Linearity — Time Shifting — Shifting in the z-Domain
  - Time Reversal — Time Expansion — Conjugation
  - Convolution — First Difference — Accumulation
  - Differentiation in the z-Domain — Initial-Value Theorems
- Some Common z-T Pairs
- Analysis & Charac. of LTI Systems Using the z-T
- System Function Algebra, Block Diagram Repre.
- The Unilateral z-T

Flowchart

Introduction (Chap 1)

LTI & Convolution (Chap 2)

Bounded/Convergent

Periodic

FS — CT, DT (Chap 3)

Aperiodic

FT — CT (Chap 4), DT (Chap 5)

Unbounded/Non-convergent

LT — CT (Chap 9)

zT — DT (Chap 10)

Time-Frequency (Chap 6)

CT-DT (Chap 7)

Communication (Chap 8)

Control (Chap 11)

Digital Signal Processing (dsp-8)