Lab Session 1:

Law of Propagation of Uncertainty (LPU)

Exercise A1   Comparison of two gauge blocks

The *observation equation* associated with the measurement of a gauge block takes the form:

$$y = \emptyset(L, t, c) = L\big(1 + c(t - t_0)\big),$$
$$z = y + e$$

where

z = indication returned by a length measuring device;
L = length at temperature t_0;
t = temperature at time of measurement;
c = coefficient of thermal expansion;
ϕ(L, t, c) model the length of the gauge block as a function of L, t and c;
e = random effect.

Measurement equation:

$$L = \varphi(t, c, y) = \frac{y}{1 + c(t - t_0)} \approx y\big(1 - c(t - t_0)\big).$$

- Suppose two gauge blocks of the same material and nominal length L_0 are measured. Calculate the 2×2 variance matrix V_L associated with their lengths
  $L = (L_1, L_2)^T$ for data:
  z_1 = 100.000 090 mm, z_2 = 100.000 050 mm,
  $u(z_1) = u(z_2) = u(z)$ = 0.000 050 mm,
  t_1 = 20.5 °C, t_2 = 20.3 °C,
  $u(t_1) = u(t_2) = u(t)$ = 0.1 °C,
  $c = 10 \times 10^{-6}$ m K^{-1}
  $u(c) = 1 \times 10^{-6}$

- Use V_L to calculate the uncertainties associated with $L_1 \pm L_2$.

Calculations:

$$L_1 = \frac{y_1}{1 + c(t_1 - t_0)}, L_2 = \frac{y_2}{1 + c(t_2 - t_0)}.$$

Best estimate of y_1 is z_1,
Best estimate of y_1 is z_2,
$u(y_1) = u(y_2) = u(z)$ = 0.000 050 mm
Influence factors $\alpha = (y_1, y_2, t_1, t_2, c)^T$.
Variance matrix for influence factors:

$$V = \begin{bmatrix} u^2(y) & & & & \\ & u^2(y) & & & \\ & & u^2(t) & & \\ & & & u^2(t) & \\ & & & & u^2(c) \end{bmatrix}$$

Let C be the 2×5 sensitivity matrix

$$C_{kj} = \frac{\partial L_k}{\partial \alpha_j}$$

Then $V_L = CVC^T$.

- Study the Matlab code R_GAUGE_BLOCK_A.m to understand how the uncertainty evaluation is implemented.

```matlab
% ---------------------------------------------------------------------------
% R_GAUGE_BLOCK_A    Different temperatures.
%
% v1A 2010-03-08 A B Forbes NPL (c) Crown Copyright.
% ---------------------------------------------------------------------------
  format compact

  L0 =100;          % Nominal length in mm

  t0 = 20;          % Fixed temperature in degrees C

% Length indications
  z1 = L0 + 90e-6;
  z2 = L0 + 50e-6;
  uz = 50e-6;

% Temperature
  t1 = 20.5;
  t2 = 20.3;
  ut = 0.1;

% Coefficient of thermal expansion
  c = 10e-6;
  uc = 1e-6;

% Estimates of y1, y2
  y1 = z1;
  y2 = z2;
  uy = uz;

% alpha = (y1,y2,t1,t2,c)';

% Variance matrix for inputs
  ualpha = [uy uy ut ut uc]';

  V = diag(ualpha.*ualpha)

% Measurement equations
   L1 = y1*(1-c*(t1-t0));
   L2 = y2*(1-c*(t2-t0));

% Difference from nominal
   L1 - L0
   L2 - L0

% Sensitivity matrix
  C = zeros(2,5);

  C(1,1) = (1-c*(t1-t0));
  C(1,2) = 0;
  C(1,3) = -y1*c;
  C(1,4) = 0;
  C(1,5) = -y1*(t1-t0);

  C(2,2) = (1-c*(t2-t0));
  C(2,1) = 0;
  C(2,4) = -y2*c;
  C(2,3) = 0;
  C(2,5) = -y2*(t2-t0);

  C
% Apply LPU
  VL = C*V*C'
```

```
  uL = sqrt(diag(VL))

  uL1 = uL(1);
  uL2 = uL(2);

% Variance components
  v1 = C(1,:).*ualpha';

  vv1 = v1.*v1;
  v2 = C(2,:).*ualpha';

  vv2 = v2.*v2;

  Var_decom= [vv1;vv2]

% Uncertainty in A = L1+L2

   cA = [1 1]';
    A = L1 + L2;
   vA = cA'*VL*cA;
   uA = sqrt(vA)

% Uncertainty in B = L1-L2
   cB = [1 -1]';
   B = L1-L2;
   vB = cB'*VL*cB;
   uB = sqrt(vB)
```

Exercise A2 Mass calculations

Modify the file R_GAUGE_BLOCK_A.m to evaluate the following exercise.

- The mass of a cylindrical artefact is given by

$$M = \pi\rho h r^2,$$

where ρ is the density at 20° C, r is the radius, h is the height.

- Suppose *r* and *h* are measured at temperature *t* and that the coefficient of thermal expansion is *c*. Express the uncertainty associated with *M* in terms of the uncertainties associated with *ρ*, *r*, *h*, *t* and *c*.

- Determine the 2×2 variance matrix V_M for the masses M_1 and M_2 of two cylindrical artefacts made from the same material from the following data:
r_1 = 25.005 mm is the radius measured at temperature t_1 = 20.5 °C,
h_1 = 63.510 mm is the height measured at temperature t_1.
r_2 = 25.020 mm is the radius measured at temperature t_2 = 20.8 °C,
h_2 = 63.515 mm is the height measured at temperature t_2.
ρ = 8000 kg m^{-3}; $c = 10 \times 10^{-6}$ m K^{-1}; $u(r_1) = u(r_2) = u(h_1) = u(h_2)$ = 0.001 mm,
$u(t_1) = u(t_2)$ = 0.1 °C; $u(\rho)$ = 0.05 kg m^{-3}; $u(c) = 1 \times 10^{-6}$ K^{-1}

- Use V_M to evaluate the uncertainties associated with M1 ± M2.

Notes:

- The mass of a cylindrical artefact is given by

$$M = \pi\rho h r^2,$$

- Taking into account temperature, we have:

$$M = \pi\rho h r^2\left(1 + c(t - t_0)\right)^3$$

- Sensitivity coefficients:

$$c_h = \frac{\partial M}{\partial h} = \pi\rho r^2\left(1 + c(t - t_0)\right)^3$$
$$c_r = \frac{\partial M}{\partial r} = 2\pi\rho h r\left(1 + c(t - t_0)\right)^3$$
$$c_t = \frac{\partial M}{\partial t} = 3\pi\rho h r^2\left(1 + c(t - t_0)\right)^2 c$$
$$c_c = \frac{\partial M}{\partial c} = 3\pi\rho h r^2\left(1 + c(t - t_0)\right)^2 (t - t_0)$$
$$c_\rho = \frac{\partial M}{\partial \rho} = \pi h r^2\left(1 + c(t - t_0)\right)^3$$

- For two masses M1 and M2, we have:

$$M_1 = \pi\rho h_1 r_1^2\left(1 + c(t_1 - t_0)\right)^3$$
$$M_2 = \pi\rho h_2 r_2^2\left(1 + c(t_2 - t_0)\right)^3$$

- **X** = (h1,h2,r1,r2,t1,t2,c,rho)‘;

$$C_1 = [c_h\ 0\ c_r\ 0\ c_t\ 0\ c_c\ c_{rho}]$$
$$C_2 = [0\ \ c_h\ 0\ c_r\ 0\ c_t\ c_c\ c_{rho}]$$
$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$
$$Y = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$
$$V_{\boldsymbol{X}} = diag[u_h^2,\ u_h^2,\ u_r^2,\ \ u_r^2,\ u_t^2,\ u_t^2,\ u_c^2,\ u_{rho}^2]$$
$$V_{\boldsymbol{Y}} = V(\boldsymbol{Y}) \approx C V_{\boldsymbol{X}} C^T$$

- $S = M_1 + M_2 = [1\ 1]\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$

$$u^2(S) = V(S) \approx \boldsymbol{c}^T V_{\boldsymbol{Y}} \boldsymbol{c} = [1\ 1] V_{\boldsymbol{Y}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$