Worksheet on "Decision Theory (incl. Bayes Classifiers)"

CS5691 PRML Jul–Nov 2025

August 13, 2025 (Version 2)

1. Consider a continuous random variable X and a discrete random variable Y. Let

   - $P_Y(Y = 1) = 0.5$ and $P_Y(Y = -1) = 0.5$, and
   - $(X|Y = 1) \sim \text{Unif}(-1, 2)$ and $(X|Y = -1) \sim \text{Unif}(-2, 1)$.

   a. What is the marginal distribution of X? Specifically, plot the pdf of X denoted $f_X(x)$.
   b. Write down the pdf $f_X(x)$ of X.
   c. Write down and plot the posterior $Y|X$.
   d. What is the optimal classifier for predicting Y from X, given the above assumptions?

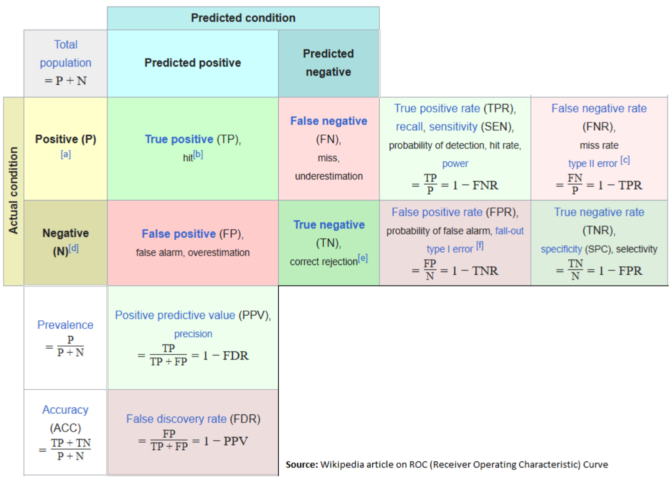
2. Derive the Bayes classifier for binary classification ($Y = \pm 1$) under the below assumptions:

$$P(Y = 1) = 0.7 \text{ and } P(Y = -1) = 0.3$$
$$X|Y = 1 \sim Unif(-1, 3)$$
$$X|Y = -1 \sim Unif(-2, 0)$$

3. [Link theory to practice] For a binary classifer h, let $L = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ be the loss matrix; and $C_{\text{train}} = \begin{bmatrix} 100 & 10 \\ 20 & 120 \end{bmatrix}$, and $C_{\text{test}} = \begin{bmatrix} 90 & 45 \\ 30 & 85 \end{bmatrix}$ be the confusion matrix when h is applied on the training and test data respectively. All three matrices have ground-truth classes t along the rows and predictions h along the columns in the same order for the two classes. Express your estimate of the risk (expected loss) of h in terms of p to s above.

4. [Link practice to theory] Besides expected loss, many other performance metrics can help evaluate the quality of a binary classifier h in practice (see figure below beside the confusion matrix).

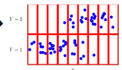
   Consider these performance/evaluation metrics of h: **Precision, Recall/Sensitivity, and Specificity**. The formula given in the figure for these metrics is actually a test-dataset-based estimate of a probability (defined over the probability space $(\mathbf{x}, t)$, where $\mathbf{x}$ is the input and $t \in \{-1, +1\}$ is the binary target). Write down this probability, i.e., **express each of these three metrics for a classifier** $h(\mathbf{x})$ **as a probability** over the joint probability space of $(\mathbf{x}, t)$.



Source: Wikipedia article on ROC (Receiver Operating Characteristic) Curve

5. Consider the four examples of two jointly distributed rvs (X, Y) from Slide 19 of "M0a. Background on Probabillity", a screenshot of which is shown below. For each of these examples, write down the optimal (Bayes) classifier for predicting Y given X (in case of discrete Y) and optimal regressor for predicting Y given X (in case of continuous Y). Assume that standard loss functions (0-1 loss function for classification and squared loss function for regression) need to be optimized.

- X discr., Y discr.: Already seen example →



- X cont., Y discr.: Let $p(X,Y) = \underbrace{p(Y)}_{\text{pmf}}\underbrace{p(X|Y)}_{\text{pdf}} = \underbrace{0.5}_{\text{assuming uniform prior}} \times \mathcal{N}(X \mid \underbrace{2Y}_{\mu \text{ can be any fn. of } Y \text{ (here it is } 2Y)}, \sigma^2)$

- X cont., Y cont.: Let W_1, W_2 be two indept. Gaussian rvs, i.e.,

$p(W_1) = \mathcal{N}(W_1 \mid \mu_1, \sigma^2)$, $p(W_2) = \mathcal{N}(W_2 \mid \mu_2, \sigma^2)$, & $p(W_1, W_2) = p(W_1)p(W_2)$.
  - Independent rvs: Let $X = W_1$, and $Y = W_2$
  - Dependent rvs: Let $X = W_1 + W_2$, and $Y = W_2$