Tutorial/Worksheet on "M0a. Probability Background"

CS5691 PRML Jul–Nov 2025

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1. [Probability Basics] Do all the tutorial/exercise problems in the slides on "M0a. Probability Background". The specific slides in this slide deck include:
   (a) Slides 10-11 on conditional probability and Bayes' theorem examples.
   (b) Slide 19 on conditional distribution and conditional expectation questions.
   (c) Slides 32 and 35 on questions related to pmf (probability mass function) and pdf (probability density function) respectively.
   (d) Slide 43 on covariance between two rvs (random variables) X, Y whose joint distribution is given.

2. [Deriving Posteriors]
   (a) Consider a continuous random variable X and a discrete random variable Y. Let
      - $P_Y(Y = 1) = 0.5$ and $P_Y(Y = -1) = 0.5$, and
      - $(X|Y = 1) \sim \text{Unif}(1, 2)$ and $(X|Y = -1) \sim \text{Unif}(-2, -1)$.

      Draw the plots for $P(Y = 1|X = x)$ and $P(Y = -1|X = x)$ given the above assumptions.
   (b) Consider the following setting:
      - $P_Y(Y = 1) = 0.7$ and $P_Y(Y = -1) = 0.3$
      - $(X|Y = 1) \sim Unif(-1, 3)$ and $(X|Y = -1) \sim Unif(-2, 0)$

      1. Derive $P(Y = 1|X = x)$ for different possible values of x.
      2. Plot the derived $P(Y = 1|X = x)$.

3. [Bayes' Theorem on Events] It is estimated that 50% of emails are spam emails. Some software has been applied to filter these spam emails before they reach your inbox. A certain brand of software claims that it can detect 99% of spam emails, and the probability for a false positive (a non-spam email detected as spam) is 5%. Now if an email is detected as spam, then what is the probability that it is in fact a non-spam email?

4. [Random Variable Basics] A discrete random variable X is said to follow/have a Poisson distribution with parameter $\lambda > 0$ over the support $S = \{0, 1, 2, \dots\}$ if it has the following pmf:

   $$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

   In this question, you are required to verify if the above pmf of X is indeed a valid pmf by verifying the following properties:

   1. $P(X = x) \geq 0 \ \ \forall x \in S$, and
   2. $\sum_{x \in S} P(X = x) = 1$.

   In addition, derive $\mathbb{E}(X)$ and $Var(X)$.

   [Hint: Use the Maclaurin's series of exponential function: $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$]

5. [RV Basics Contd.] A fair coin is tossed twice, and $\mathbf{X}$ is defined as the number of heads that are observed. Find the range of $\mathbf{X}$ (R_X) and the probability mass function (P_X).

6. [CHANGE OF VARIABLES VS. THE LAW OF THE UNCONSCIOUS STATISTICAN] Consider a continuous random variable X and a discrete random variable Y. Let

- $P_Y(Y = 1) = 0.5$ and $P_Y(Y = -1) = 0.5$, and
- $(X|Y = 1) \sim \text{Unif}(1, 2)$ and $(X|Y = -1) \sim \text{Unif}(-2, -1)$.

a. What is the marginal distribution of X? Specifically, plot the pdf of X denoted $f_X(x)$.

b. Write down the pdf $f_X(x)$ of X.

c. Let $Z = X^2$. What is the pdf of Z? Use it to compute $E[Z]$. (Hint: To obtain pdf of Z, you could simply derive the cdf of Z and differentiate it (or you could also use the change-of-variables formula).)

d. Now, use the pdf of X directly to compute $E[X^2]$ (using the law of the unconscious statistician). Does this give the same answer as the previous question? Which of these two methods do you prefer to compute the expectation of a function of a rv?