Transport Equations for the Reynolds Stresses and Scalar Fluxes

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Convention:

$$f = \langle f \rangle_{z,t} + f'$$

Summation over repeated indices. All the equations are in non-dimensional form.
‡ Note that each number under each term is the corresponding entry in the statistics list.
• **Transport equations for $\frac{1}{2}u_i^2$, $\frac{1}{2}\langle u_i \rangle^2$, and $\frac{1}{2}\langle u_i' u_i' \rangle$**

$$\begin{aligned}
\frac{\partial u_i^2/2}{\partial t} + \underset{(-125)}{u_l \frac{\partial u_i^2/2}{\partial x_l}} &= \underset{(-133)}{-u_i \frac{\partial p}{\partial x_i}} + \frac{1}{Re}(\underset{(-128)}{\frac{\partial^2 u_i^2/2}{\partial x_l \partial x_l}} - \underset{(-136)}{\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}}) && (1)\\
\frac{\partial \langle u_i \rangle^2/2}{\partial t} + \underset{(-126)}{\langle u_l \rangle \frac{\partial \langle u_i \rangle^2/2}{\partial x_l}} &= \underset{(-89)}{\langle u_i' u_l' \rangle \frac{\partial \langle u_i \rangle}{\partial x_l}} - \underset{(-131)}{\frac{\partial \langle u_i' u_l' \rangle \langle u_i \rangle}{\partial x_l}} - \underset{(-134)}{\langle u_i \rangle \frac{\partial \langle p \rangle}{\partial x_i}}\\
&+ \frac{1}{Re}(\underset{(-129)}{\frac{\partial^2 \langle u_i \rangle^2/2}{\partial x_l \partial x_l}} - \underset{(-137)}{\frac{\partial \langle u_i \rangle}{\partial x_l}\frac{\partial \langle u_i \rangle}{\partial x_l}}) && (2)\\
\frac{\partial \langle u_i'^2 \rangle/2}{\partial t} + \underset{(-127)}{\langle u_l \rangle \frac{\partial \langle u_i'^2 \rangle/2}{\partial x_l}} &= \underset{(-89)}{-\langle u_i' u_l' \rangle \frac{\partial \langle u_i \rangle}{\partial x_l}} - \underset{(-132)}{\frac{1}{2}\frac{\partial \langle u_i'^2 u_l' \rangle}{\partial x_l}} - \underset{(-135)}{\langle u_i' \frac{\partial p'}{\partial x_i} \rangle}\\
&+ \frac{1}{Re}(\underset{(-130)}{\frac{\partial^2 \langle u_i'^2/2 \rangle}{\partial x_l \partial x_l}} - \underset{(-138)}{\langle \frac{\partial u_i'}{\partial x_l}\frac{\partial u_i'}{\partial x_l} \rangle}) && (3)
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{Re}(\underset{(-128)}{\frac{\partial^2 u_i^2/2}{\partial x_j \partial x_j}} - \underset{(-136)}{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}) &= \frac{1}{Re}(\underset{(-139)}{\frac{\partial 2 u_i S_{ij}}{\partial x_j}} - \underset{(-82)}{2 S_{ij} S_{ij}})\\
\frac{1}{Re}(\underset{(-129)}{\frac{\partial^2 \langle u_i \rangle^2/2}{\partial x_j \partial x_j}} - \underset{(-137)}{\frac{\partial \langle u_i \rangle}{\partial x_j}\frac{\partial \langle u_i \rangle}{\partial x_j}}) &= \frac{1}{Re}(\underset{(-140)}{\frac{\partial 2 \langle u_i \rangle \langle S_{ij} \rangle}{\partial x_j}} - \underset{(-83)}{2 \langle S_{ij} \rangle \langle S_{ij} \rangle})\\
\frac{1}{Re}(\underset{(-130)}{\frac{\partial^2 \langle u_i'^2/2 \rangle}{\partial x_j \partial x_j}} - \underset{(-138)}{\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j} \rangle}) &= \frac{1}{Re}(\underset{(-141)}{\frac{\partial \langle 2 u_i' S_{ij}' \rangle}{\partial x_j}} - \underset{(-84)}{\langle 2 S_{ij}' S_{ij}' \rangle})
\end{aligned}$$

and

$$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

- **Transport equations for $u_i u_j$, $\langle u_i u_j \rangle$, and $\langle u'_i u'_j \rangle$**

$$
\begin{aligned}
\frac{\partial u_i u_j}{\partial t} + u_l \frac{\partial u_i u_j}{\partial x_l} &= -(u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i}) + \frac{1}{Re}(\frac{\partial^2 u_i u_j}{\partial x_l \partial x_l} - 2\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}) \qquad (4)\\
\frac{\partial \langle u_i \rangle \langle u_j \rangle}{\partial t} + \langle u_l \rangle \frac{\partial \langle u_i \rangle \langle u_j \rangle}{\partial x_l} &= \langle u'_j u'_l \rangle \frac{\partial \langle u_i \rangle}{\partial x_l} + \langle u'_i u'_l \rangle \frac{\partial \langle u_j \rangle}{\partial x_l} - (\frac{\partial \langle u'_j u'_l \rangle \langle u_i \rangle}{\partial x_l} + \frac{\partial \langle u'_i u'_l \rangle \langle u_j \rangle}{\partial x_l})\\
&- (\langle u_i \rangle \frac{\partial \langle p \rangle}{\partial x_j} + \langle u_j \rangle \frac{\partial \langle p \rangle}{\partial x_i}) + \frac{1}{Re}(\frac{\partial^2 \langle u_i \rangle \langle u_j \rangle}{\partial x_l \partial x_l} - 2\frac{\partial \langle u_i \rangle}{\partial x_l}\frac{\partial \langle u_j \rangle}{\partial x_l}) \qquad (5)\\
\frac{\partial \langle u'_i u'_j \rangle}{\partial t} + \langle u_l \rangle \frac{\partial \langle u'_i u'_j \rangle}{\partial x_l} &= -(\langle u'_j u'_l \rangle \frac{\partial \langle u_i \rangle}{\partial x_l} + \langle u'_i u'_l \rangle \frac{\partial \langle u_j \rangle}{\partial x_l}) - \frac{\partial \langle u'_i u'_j u'_l \rangle}{\partial x_l}\\
&- (\langle u'_i \frac{\partial p'}{\partial x_j} \rangle + \langle u'_j \frac{\partial\, p'}{\partial x_i} \rangle) + \frac{1}{Re}(\frac{\partial^2 \langle u'_i u'_j \rangle}{\partial x_l \partial x_l} - 2\langle \frac{\partial u'_i}{\partial x_l}\frac{\partial u'_i}{\partial x_l} \rangle) \qquad (6)
\end{aligned}
$$

- **Transport equations for uu, $\langle uu \rangle$, and $\langle u'u' \rangle$**

$$
\begin{aligned}
\frac{\partial u^2}{\partial t} + \underset{(-142)}{u_l \frac{\partial u^2}{\partial x_l}} &= \underset{(-196)}{-2u\frac{\partial p}{\partial x}} + \underset{(-160)}{\frac{1}{Re}(\frac{\partial^2 u^2}{\partial x_l \partial x_l}} \underset{(-247)}{- 2\frac{\partial u}{\partial x_l}\frac{\partial u}{\partial x_l})} \qquad (7)\\
\frac{\partial \langle u \rangle \langle u \rangle}{\partial t} + \underset{(-148)}{\langle u_l \rangle \frac{\partial \langle u \rangle \langle u \rangle}{\partial x_l}} &= \underset{(-190)}{2\langle u' u'_l \rangle \frac{\partial \langle u \rangle}{\partial x_l}} \underset{(-178)}{- 2\frac{\partial \langle u' u'_l \rangle \langle u \rangle}{\partial x_l}} \underset{(-202)}{- 2\langle u \rangle \frac{\partial \langle p \rangle}{\partial x}}\\
&+ \underset{(-166)}{\frac{1}{Re}(\frac{\partial^2 \langle u \rangle \langle u \rangle}{\partial x_l \partial x_l}} \underset{(-253)}{- 2\frac{\partial \langle u \rangle}{\partial x_l}\frac{\partial \langle u \rangle}{\partial x_l})} \qquad (8)\\
\frac{\partial \langle u'^2 \rangle}{\partial t} + \underset{(-154)}{\langle u_l \rangle \frac{\partial \langle u'^2 \rangle}{\partial x_l}} &= \underset{(-190)}{-2\langle u' u'_l \rangle \frac{\partial \langle u \rangle}{\partial x_l}} \underset{(-184)}{- \frac{\partial \langle u'^2 u'_l \rangle}{\partial x_l}} \underset{(-208)}{- 2\langle u' \frac{\partial p'}{\partial x} \rangle}\\
&+ \underset{(-172)}{\frac{1}{Re}(\frac{\partial^2 \langle u'^2 \rangle}{\partial x_l \partial x_l}} \underset{(-259)}{- 2\langle \frac{\partial u'}{\partial x_l}\frac{\partial u'}{\partial x_l} \rangle)} \qquad (9)
\end{aligned}
$$

- **Transport equations for vv, $\langle vv \rangle$, and $\langle v'v' \rangle$**

$$
\begin{aligned}
\frac{\partial v^2}{\partial t} + \underset{(-143)}{u_l \frac{\partial v^2}{\partial x_l}} &= \underset{(-197)}{-2v\frac{\partial p}{\partial y}} + \underset{(-161)}{\frac{1}{Re}(\frac{\partial^2 v^2}{\partial x_l \partial x_l}} \underset{(-248)}{- 2\frac{\partial v}{\partial x_l}\frac{\partial v}{\partial x_l})} \qquad (10)\\
\frac{\partial \langle v \rangle \langle v \rangle}{\partial t} + \underset{(-149)}{\langle u_l \rangle \frac{\partial \langle v \rangle \langle v \rangle}{\partial x_l}} &= \underset{(-191)}{2\langle v' u'_l \rangle \frac{\partial \langle v \rangle}{\partial x_l}} \underset{(-179)}{- 2\frac{\partial \langle v' u'_l \rangle \langle v \rangle}{\partial x_l}} \underset{(-203)}{- 2\langle v \rangle \frac{\partial \langle p \rangle}{\partial y}}\\
&+ \underset{(-167)}{\frac{1}{Re}(\frac{\partial^2 \langle v \rangle \langle v \rangle}{\partial x_l \partial x_l}} \underset{(-254)}{- 2\frac{\partial \langle v \rangle}{\partial x_l}\frac{\partial \langle v \rangle}{\partial x_l})} \qquad (11)\\
\frac{\partial \langle v'^2 \rangle}{\partial t} + \underset{(-155)}{\langle u_l \rangle \frac{\partial \langle v'^2 \rangle}{\partial x_l}} &= \underset{(-191)}{-2\langle v' u'_l \rangle \frac{\partial \langle v \rangle}{\partial x_l}} \underset{(-185)}{- \frac{\partial \langle v'^2 u'_l \rangle}{\partial x_l}} \underset{(-209)}{- 2\langle v' \frac{\partial p'}{\partial y} \rangle}\\
&+ \underset{(-173)}{\frac{1}{Re}(\frac{\partial^2 \langle v'^2 \rangle}{\partial x_l \partial x_l}} \underset{(-260)}{- 2\langle \frac{\partial v'}{\partial x_l}\frac{\partial v'}{\partial x_l} \rangle)} \qquad (12)
\end{aligned}
$$

- **Transport equations for ww, $\langle ww\rangle$, and $\langle w'w'\rangle$**

$$
\underset{(-144)}{\frac{\partial w^2}{\partial t}+u_l\frac{\partial w^2}{\partial x_l}} = \underset{(-198)}{-2w\frac{\partial p}{\partial z}}+\frac{1}{Re}(\underset{(-162)}{\frac{\partial^2 w^2}{\partial x_l\partial x_l}}\underset{(-249)}{-2\frac{\partial w}{\partial x_l}\frac{\partial w}{\partial x_l}}) \tag{13}
$$

$$
\begin{aligned}
\underset{(-150)}{\frac{\partial\langle w\rangle\langle w\rangle}{\partial t}+\langle u_l\rangle\frac{\partial\langle w\rangle\langle w\rangle}{\partial x_l}} &= \underset{(-192)}{2\langle w'u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l}}\underset{(-180)}{-2\frac{\partial\langle w'u_l'\rangle\langle w\rangle}{\partial x_l}}\underset{(-204)}{-2\langle w\rangle\frac{\partial\langle p\rangle}{\partial z}} \\
&+ \frac{1}{Re}(\underset{(-168)}{\frac{\partial^2\langle w\rangle\langle w\rangle}{\partial x_l\partial x_l}}\underset{(-255)}{-2\frac{\partial\langle w\rangle}{\partial x_l}\frac{\partial\langle w\rangle}{\partial x_l}})
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\underset{(-156)}{\frac{\partial\langle w'^2\rangle}{\partial t}+\langle u_l\rangle\frac{\partial\langle w'^2\rangle}{\partial x_l}} &= \underset{(-192)}{-2\langle w'u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l}}\underset{(-186)}{-\frac{\partial\langle w'^2u_l'\rangle}{\partial x_l}}\underset{(-210)}{-2\langle w'\frac{\partial p'}{\partial z}\rangle} \\
&+ \frac{1}{Re}(\underset{(-174)}{\frac{\partial^2\langle w'^2\rangle}{\partial x_l\partial x_l}}\underset{(-261)}{-2\langle\frac{\partial w'}{\partial x_l}\frac{\partial w'}{\partial x_l}\rangle})
\end{aligned} \tag{15}
$$

- **Transport equations for uv, $\langle uv\rangle$, and $\langle u'v'\rangle$**

$$
\underset{(-145)}{\frac{\partial uv}{\partial t}+u_l\frac{\partial uv}{\partial x_l}} = \underset{(-199)}{-(u\frac{\partial p}{\partial y}+v\frac{\partial p}{\partial x})}+\frac{1}{Re}(\underset{(-163)}{\frac{\partial^2 uv}{\partial x_l\partial x_l}}\underset{(-250)}{-2\frac{\partial u}{\partial x_l}\frac{\partial v}{\partial x_l}}) \tag{16}
$$

$$
\begin{aligned}
\underset{(-151)}{\frac{\partial\langle u\rangle\langle v\rangle}{\partial t}+\langle u_l\rangle\frac{\partial\langle u\rangle\langle v\rangle}{\partial x_l}} &= \underset{(-193)}{(\langle v'u_l'\rangle\frac{\partial\langle u\rangle}{\partial x_l}+\langle u'u_l'\rangle\frac{\partial\langle v\rangle}{\partial x_l})}\underset{(-181)}{-(\frac{\partial\langle v'u_l'\rangle\langle u\rangle}{\partial x_l}+\frac{\partial\langle u'u_l'\rangle\langle v\rangle}{\partial x_l})} \\
&- \underset{(-205)}{(\langle u\rangle\frac{\partial\langle p\rangle}{\partial y}+\langle v\rangle\frac{\partial\langle p\rangle}{\partial x})}+\frac{1}{Re}(\underset{(-169)}{\frac{\partial^2\langle u\rangle\langle v\rangle}{\partial x_l\partial x_l}}\underset{(-256)}{-2\frac{\partial\langle u\rangle}{\partial x_l}\frac{\partial\langle v\rangle}{\partial x_l}})
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\underset{(-157)}{\frac{\partial\langle u'w'\rangle}{\partial t}+\langle u_l\rangle\frac{\partial\langle u'w'\rangle}{\partial x_l}} &= \underset{(-193)}{-(\langle w'u_l'\rangle\frac{\partial\langle u\rangle}{\partial x_l}+\langle u'u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l})}\underset{(-187)}{-\frac{\partial\langle u'w'u_l'\rangle}{\partial x_l}} \\
&- \underset{(-211)}{(\langle u'\frac{\partial p'}{\partial z}\rangle+\langle w'\frac{\partial p'}{\partial x}\rangle)}+\frac{1}{Re}(\underset{(-175)}{\frac{\partial^2\langle u'v'\rangle}{\partial x_l\partial x_l}}\underset{(-262)}{-2\langle\frac{\partial u'}{\partial x_l}\frac{\partial v'}{\partial x_l}\rangle})
\end{aligned} \tag{18}
$$

- **Transport equations for uw, $\langle uw\rangle$, and $\langle u'w'\rangle$**

$$
\underset{(-146)}{\frac{\partial uw}{\partial t}+u_l\frac{\partial uw}{\partial x_l}} = \underset{(-200)}{-(u\frac{\partial p}{\partial z}+w\frac{\partial p}{\partial x})}+\frac{1}{Re}(\underset{(-164)}{\frac{\partial^2 uw}{\partial x_l\partial x_l}}\underset{(-251)}{-2\frac{\partial u}{\partial x_l}\frac{\partial w}{\partial x_l}}) \tag{19}
$$

$$
\begin{aligned}
\underset{(-152)}{\frac{\partial\langle u\rangle\langle w\rangle}{\partial t}+\langle u_l\rangle\frac{\partial\langle u\rangle\langle w\rangle}{\partial x_l}} &= \underset{(-194)}{(\langle w'u_l'\rangle\frac{\partial\langle u\rangle}{\partial x_l}+\langle u'u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l})}\underset{(-182)}{-(\frac{\partial\langle w'u_l'\rangle\langle u\rangle}{\partial x_l}+\frac{\partial\langle u'u_l'\rangle\langle w\rangle}{\partial x_l})} \\
&- \underset{(-206)}{(\langle u\rangle\frac{\partial\langle p\rangle}{\partial z}+\langle w\rangle\frac{\partial\langle p\rangle}{\partial x})}+\frac{1}{Re}(\underset{(-170)}{\frac{\partial^2\langle u\rangle\langle w\rangle}{\partial x_l\partial x_l}}\underset{(-257)}{-2\frac{\partial\langle u\rangle}{\partial x_l}\frac{\partial\langle w\rangle}{\partial x_l}})
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\frac{\partial\langle u'w'\rangle}{\partial t}+\underset{(-158)}{\langle u_l\rangle\frac{\partial\langle u'w'\rangle}{\partial x_l}} &= \underset{(-194)}{-(\langle w'u_l'\rangle\frac{\partial\langle u\rangle}{\partial x_l}+\langle u'u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l})}-\underset{(-188)}{\frac{\partial\langle u'w'u_l'\rangle}{\partial x_l}} \\
&- \underset{(-212)}{(\langle u'\frac{\partial p'}{\partial z}\rangle+\langle w'\frac{\partial p'}{\partial x}\rangle)}+\frac{1}{Re}(\underset{(-176)}{\frac{\partial^2\langle u'w'\rangle}{\partial x_l\partial x_l}}-\underset{(-263)}{2\langle\frac{\partial u'}{\partial x_l}\frac{\partial w'}{\partial x_l}\rangle}) \qquad (21)
\end{aligned}
$$

- **Transport equations for vw, $\langle vw\rangle$, and $\langle v'w'\rangle$**

$$
\frac{\partial vw}{\partial t}+\underset{(-147)}{u_l\frac{\partial vw}{\partial x_l}} = \underset{(-201)}{-(v\frac{\partial p}{\partial z}+w\frac{\partial p}{\partial y})}+\frac{1}{Re}(\underset{(-165)}{\frac{\partial^2 vw}{\partial x_l\partial x_l}}-\underset{(-252)}{2\frac{\partial v}{\partial x_l}\frac{\partial w}{\partial x_l}}) \qquad (22)
$$

$$
\begin{aligned}
\frac{\partial\langle v\rangle\langle w\rangle}{\partial t}+\underset{(-153)}{\langle u_l\rangle\frac{\partial\langle v\rangle\langle w\rangle}{\partial x_l}} &= \underset{(-195)}{\langle w'u_l'\rangle\frac{\partial\langle v\rangle}{\partial x_l}+\langle v'u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l}}-\underset{(-183)}{(\frac{\partial\langle w'u_l'\rangle\langle v\rangle}{\partial x_l}+\frac{\partial\langle v'u_l'\rangle\langle w\rangle}{\partial x_l})} \\
&- \underset{(-207)}{(\langle v\rangle\frac{\partial\langle p\rangle}{\partial z}+\langle w\rangle\frac{\partial\langle p\rangle}{\partial y})}+\frac{1}{Re}(\underset{(-171)}{\frac{\partial^2\langle v\rangle\langle w\rangle}{\partial x_l\partial x_l}}-\underset{(-258)}{2\frac{\partial\langle v\rangle}{\partial x_l}\frac{\partial\langle w\rangle}{\partial x_l}}) \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\langle v'w'\rangle}{\partial t}+\underset{(-159)}{\langle u_l\rangle\frac{\partial\langle v'w'\rangle}{\partial x_l}} &= \underset{(-195)}{-(\langle w'u_l'\rangle\frac{\partial\langle v\rangle}{\partial x_l}+\langle v'u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l})}-\underset{(-189)}{\frac{\partial\langle v'w'u_l'\rangle}{\partial x_l}} \\
&- \underset{(-213)}{(\langle v'\frac{\partial p'}{\partial z}\rangle+\langle w'\frac{\partial p'}{\partial y}\rangle)}+\frac{1}{Re}(\underset{(-177)}{\frac{\partial^2\langle v'w'\rangle}{\partial x_l\partial x_l}}-\underset{(-264)}{2\langle\frac{\partial v'}{\partial x_l}\frac{\partial w'}{\partial x_l}\rangle}) \qquad (24)
\end{aligned}
$$

- **Transport equations for $\frac{1}{2}\theta^2$, $\frac{1}{2}\langle\theta\rangle^2$, and $\frac{1}{2}\langle\theta'\theta'\rangle$**

$$
\frac{\partial\theta^2/2}{\partial t}+\underset{(-26)}{u_l\frac{\partial\theta^2/2}{\partial x_l}} = \frac{1}{Pe}(\underset{(-29)}{\frac{\partial^2\theta^2/2}{\partial x_l\partial x_l}}-\underset{(-35)}{\frac{\partial\theta}{\partial x_l}\frac{\partial\theta}{\partial x_l}}) \qquad (25)
$$

$$
\frac{\partial\langle\theta\rangle^2/2}{\partial t}+\underset{(-27)}{\langle u_l\rangle\frac{\partial\langle\theta\rangle^2/2}{\partial x_l}} = \underset{(-34)}{\langle\theta'u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l}}-\underset{(-32)}{\frac{\partial\langle\theta'u_l'\rangle\langle\theta\rangle}{\partial x_l}}+\frac{1}{Pe}(\underset{(-30)}{\frac{\partial^2\langle\theta\rangle^2/2}{\partial x_l\partial x_l}}-\underset{(-36)}{\frac{\partial\langle\theta\rangle}{\partial x_l}\frac{\partial\langle\theta\rangle}{\partial x_l}}) \qquad (26)
$$

$$
\frac{\partial\langle\theta'^2\rangle/2}{\partial t}+\underset{(-28)}{\langle\theta_l\rangle\frac{\partial\langle\theta'^2\rangle/2}{\partial x_l}} = \underset{(-34)}{-\langle\theta'u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l}}-\underset{(-33)}{\frac{1}{2}\frac{\partial\langle\theta'^2u_l'\rangle}{\partial x_l}}+\frac{1}{Pe}(\underset{(-31)}{\frac{\partial^2\langle\theta'^2/2\rangle}{\partial x_l\partial x_l}}-\underset{(-37)}{\langle\frac{\partial\theta'}{\partial x_l}\frac{\partial\theta'}{\partial x_l}\rangle}) \qquad (27)
$$

- **Transport equations for $u_i\theta$, $\langle u_i\theta\rangle$, and $\langle u_i'\theta'\rangle$**

$$
\begin{aligned}
\frac{\partial u_i\theta}{\partial t}+u_l\frac{\partial u_i\theta}{\partial x_l} &= -\theta\frac{\partial p}{\partial x_i}+(\frac{1}{Re}\frac{\partial}{\partial x_l}(\theta\frac{\partial u_i}{\partial x_l})+\frac{1}{Pe}\frac{\partial}{\partial x_l}(u_i\frac{\partial\theta}{\partial x_l})) \\
&- (\frac{1}{Re}+\frac{1}{Pe})\frac{\partial u_i}{\partial x_l}\frac{\partial\theta}{\partial x_l} \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\langle u_i\rangle\langle\theta\rangle}{\partial t}+\langle u_l\rangle\frac{\partial\langle u_i\rangle\langle\theta\rangle}{\partial x_l} &= (\langle\theta'u_l'\rangle\frac{\partial\langle u_i\rangle}{\partial x_l}+\langle u_i'u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l})-(\frac{\partial\langle\theta'u_l'\rangle\langle u_i\rangle}{\partial x_l}+\frac{\partial\langle u_i'u_l'\rangle\langle\theta\rangle}{\partial x_l}) \\
&- \langle\theta\rangle\frac{\partial\langle p\rangle}{\partial x_i}+(\frac{1}{Re}\frac{\partial}{\partial x_l}(\langle\theta\rangle\frac{\partial\langle u_i\rangle}{\partial x_l})+\frac{1}{Pe}\frac{\partial}{\partial x_l}(\langle u_i\rangle\frac{\partial\langle\theta\rangle}{\partial x_l})) \\
&- (\frac{1}{Re}+\frac{1}{Pe})\frac{\partial\langle u_i\rangle}{\partial x_l}\frac{\partial\langle\theta\rangle}{\partial x_l} \qquad (29)
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial\langle u_i'\theta'\rangle}{\partial t} + \langle u_l\rangle\frac{\partial\langle u_i'\theta'\rangle}{\partial x_l} &= -(\langle\theta' u_l'\rangle\frac{\partial\langle u_i\rangle}{\partial x_l} + \langle u_i'u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l}) - \frac{\partial\langle u_i'\theta'u_l'\rangle}{\partial x_l} - \langle\theta'\frac{\partial p'}{\partial x_i}\rangle \\
&+ (\frac{1}{Re}\frac{\partial}{\partial x_l}\langle\theta'\frac{\partial u_i'}{\partial x_l}\rangle + \frac{1}{Pe}\frac{\partial}{\partial x_l}\langle u_i'\frac{\partial\theta'}{\partial x_l}\rangle) - (\frac{1}{Re}+\frac{1}{Pe})\langle\frac{\partial u_i'}{\partial x_l}\frac{\partial\theta'}{\partial x_l}\rangle
\end{aligned} \tag{30}$$

- **Transport equations for $u\theta$, $\langle u\theta\rangle$, and $\langle u'\theta'\rangle$**

$$\begin{aligned}
\frac{\partial u\theta}{\partial t} + \underset{(-38)}{u_l\frac{\partial u\theta}{\partial x_l}} &= \underset{(-77)}{-\theta\frac{\partial p}{\partial x}} + \underset{(-47)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}(\theta\frac{\partial u}{\partial x_l}) + \frac{1}{Pe}\frac{\partial}{\partial x_l}(u\frac{\partial\theta}{\partial x_l}))} \\
&- \underset{(-86)}{(\frac{1}{Re}+\frac{1}{Pe})\frac{\partial u}{\partial x_l}\frac{\partial\theta}{\partial x_l}}
\end{aligned} \tag{31}$$

$$\begin{aligned}
\frac{\partial\langle u\rangle\langle\theta\rangle}{\partial t} + \underset{(-41)}{\langle u_l\rangle\frac{\partial\langle u\rangle\langle\theta\rangle}{\partial x_l}} &= \underset{(-62)}{(\langle\theta' u_l'\rangle\frac{\partial\langle u\rangle}{\partial x_l} + \langle u'u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l})} - \underset{(-56)}{(\frac{\partial\langle\theta'u_l'\rangle\langle u\rangle}{\partial x_l} + \frac{\partial\langle u'u_l'\rangle\langle\theta\rangle}{\partial x_l})} \\
&- \underset{(-80)}{\langle\theta\rangle\frac{\partial\langle p\rangle}{\partial x}} + \underset{(-50)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}(\langle\theta\rangle\frac{\partial\langle u\rangle}{\partial x_l}) + \frac{1}{Pe}\frac{\partial}{\partial x_l}(\langle u\rangle\frac{\partial\langle\theta\rangle}{\partial x_l}))} \\
&- \underset{(-89)}{(\frac{1}{Re}+\frac{1}{Pe})\frac{\partial\langle u\rangle}{\partial x_l}\frac{\partial\langle\theta\rangle}{\partial x_l}}
\end{aligned} \tag{32}$$

$$\begin{aligned}
\frac{\partial\langle u'\theta'\rangle}{\partial t} + \underset{(-44)}{\langle u_l\rangle\frac{\partial\langle u'\theta'\rangle}{\partial x_l}} &= \underset{(-62)}{-(\langle\theta' u_l'\rangle\frac{\partial\langle u\rangle}{\partial x_l} + \langle u'u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l})} - \underset{(-59)}{\frac{\partial\langle u'\theta'u_l'\rangle}{\partial x_l}} - \underset{(-83)}{\langle\theta'\frac{\partial p'}{\partial x}\rangle} \\
&+ \underset{(-53)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}\langle\theta'\frac{\partial u'}{\partial x_l}\rangle + \frac{1}{Pe}\frac{\partial}{\partial x_l}\langle u'\frac{\partial\theta'}{\partial x_l}\rangle)} - \underset{(-92)}{(\frac{1}{Re}+\frac{1}{Pe})\langle\frac{\partial u'}{\partial x_l}\frac{\partial\theta'}{\partial x_l}\rangle}
\end{aligned} \tag{33}$$

- **Transport equations for $v\theta$, $\langle v\theta\rangle$, and $\langle v'\theta'\rangle$**

$$\begin{aligned}
\frac{\partial v\theta}{\partial t} + \underset{(-39)}{u_l\frac{\partial v\theta}{\partial x_l}} &= \underset{(-78)}{-\theta\frac{\partial p}{\partial y}} + \underset{(-48)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}(\theta\frac{\partial v}{\partial x_l}) + \frac{1}{Pe}\frac{\partial}{\partial x_l}(v\frac{\partial\theta}{\partial x_l}))} \\
&- \underset{(-87)}{(\frac{1}{Re}+\frac{1}{Pe})\frac{\partial v}{\partial x_l}\frac{\partial\theta}{\partial x_l}}
\end{aligned} \tag{34}$$

$$\begin{aligned}
\frac{\partial\langle v\rangle\langle\theta\rangle}{\partial t} + \underset{(-42)}{\langle u_l\rangle\frac{\partial\langle v\rangle\langle\theta\rangle}{\partial x_l}} &= \underset{(-63)}{\langle\theta' u_l'\rangle\frac{\partial\langle v\rangle}{\partial x_l} + \langle v'u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l}} - \underset{(-57)}{(\frac{\partial\langle\theta'u_l'\rangle\langle v\rangle}{\partial x_l} + \frac{\partial\langle v'u_l'\rangle\langle\theta\rangle}{\partial x_l})} \\
&- \underset{(-81)}{\langle\theta\rangle\frac{\partial\langle p\rangle}{\partial y}} + \underset{(-51)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}(\langle\theta\rangle\frac{\partial\langle v\rangle}{\partial x_l}) + \frac{1}{Pe}\frac{\partial}{\partial x_l}(\langle v\rangle\frac{\partial\langle\theta\rangle}{\partial x_l}))} \\
&- \underset{(-90)}{(\frac{1}{Re}+\frac{1}{Pe})\frac{\partial\langle v\rangle}{\partial x_l}\frac{\partial\langle\theta\rangle}{\partial x_l}}
\end{aligned} \tag{35}$$

$$
\begin{aligned}
\underset{(-45)}{\frac{\partial\langle v'\theta'\rangle}{\partial t} + \langle u_l\rangle\frac{\partial\langle v'\theta'\rangle}{\partial x_l}} &= -\underset{(-63)}{(\langle\theta' u_l'\rangle\frac{\partial\langle v\rangle}{\partial x_l} + \langle v' u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l})} - \underset{(-60)}{\frac{\partial\langle v'\theta' u_l'\rangle}{\partial x_l}} - \underset{(-84)}{\langle\theta'\frac{\partial p'}{\partial y}\rangle} \\
&+ \underset{(-54)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}\langle\theta'\frac{\partial v'}{\partial x_l}\rangle + \frac{1}{Pe}\frac{\partial}{\partial x_l}\langle v'\frac{\partial\theta'}{\partial x_l}\rangle)} - \underset{(-93)}{(\frac{1}{Re}+\frac{1}{Pe})\langle\frac{\partial v'}{\partial x_l}\frac{\partial\theta'}{\partial x_l}\rangle} \qquad (36)
\end{aligned}
$$

- **Transport equations for $w\theta$, $\langle w\theta\rangle$, and $\langle w'\theta'\rangle$**

$$
\begin{aligned}
\underset{(-40)}{\frac{\partial w\theta}{\partial t} + u_l\frac{\partial w\theta}{\partial x_l}} &= \underset{(-79)}{-\theta\frac{\partial p}{\partial z}} + \underset{(-49)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}(\theta\frac{\partial w}{\partial x_l}) + \frac{1}{Pe}\frac{\partial}{\partial x_l}(w\frac{\partial\theta}{\partial x_l}))} \\
&- \underset{(-88)}{(\frac{1}{Re}+\frac{1}{Pe})\frac{\partial w}{\partial x_l}\frac{\partial\theta}{\partial x_l}} \qquad (37) \\
\underset{(-43)}{\frac{\partial\langle w\rangle\langle\theta\rangle}{\partial t} + \langle u_l\rangle\frac{\partial\langle w\rangle\langle\theta\rangle}{\partial x_l}} &= \underset{(-64)}{\langle\theta' u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l} + \langle w' u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l}} - \underset{(-58)}{(\frac{\partial\langle\theta' u_l'\rangle\langle w\rangle}{\partial x_l} + \frac{\partial\langle w' u_l'\rangle\langle\theta\rangle}{\partial x_l})} \\
&- \underset{(-82)}{\langle\theta\rangle\frac{\partial\langle p\rangle}{\partial z}} + \underset{(-52)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}(\langle\theta\rangle\frac{\partial\langle w\rangle}{\partial x_l}) + \frac{1}{Pe}\frac{\partial}{\partial x_l}(\langle w\rangle\frac{\partial\langle\theta\rangle}{\partial x_l}))} \\
&- \underset{(-91)}{(\frac{1}{Re}+\frac{1}{Pe})\frac{\partial\langle w\rangle}{\partial x_l}\frac{\partial\langle\theta\rangle}{\partial x_l}} \qquad (38) \\
\underset{(-46)}{\frac{\partial\langle w'\theta'\rangle}{\partial t} + \langle u_l\rangle\frac{\partial\langle w'\theta'\rangle}{\partial x_l}} &= -\underset{(-64)}{(\langle\theta' u_l'\rangle\frac{\partial\langle w\rangle}{\partial x_l} + \langle w' u_l'\rangle\frac{\partial\langle\theta\rangle}{\partial x_l})} - \underset{(-61)}{\frac{\partial\langle w'\theta' u_l'\rangle}{\partial x_l}} - \underset{(-85)}{\langle\theta'\frac{\partial p'}{\partial z}\rangle} \\
&+ \underset{(-55)}{(\frac{1}{Re}\frac{\partial}{\partial x_l}\langle\theta'\frac{\partial w'}{\partial x_l}\rangle + \frac{1}{Pe}\frac{\partial}{\partial x_l}\langle w'\frac{\partial\theta'}{\partial x_l}\rangle)} - \underset{(-94)}{(\frac{1}{Re}+\frac{1}{Pe})\langle\frac{\partial w'}{\partial x_l}\frac{\partial\theta'}{\partial x_l}\rangle} \qquad (39)
\end{aligned}
$$