Regression II

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Solving the Least Squares using Normal Equation

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\mathbf{w} &= (V_r\Sigma_r U_r^T U_r\Sigma_r V_r^T)^{-1} V_r S_r U_r^T \mathbf{t} \\
&= V_r\Sigma_r^{-2} V_r^T V_r\Sigma_r U_r^T \mathbf{t} \\
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- How about when $\sigma_i \to 0$?

Regularization

- Adding regularization to control overfitting

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$$= V(\Sigma^2 + \lambda I)^{-1}SU^T\mathbf{t} = \sum_{i=1}^{M}\frac{\sigma_i u_i^T\mathbf{t}}{\sigma_i^2 + \lambda}v_i$$

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  - Using conjugate prior $p(\mathbf{w}) = \mathcal{N}(\mathbf{w}|\mathbf{m}_0,\mathbf{S}_0)$, we can show that Posterior $p(\mathbf{w}|\mathbf{t},\mathbf{X},\beta,\lambda) = \mathcal{N}(\mathbf{w}|\mathbf{m}_N,\mathbf{S}_N)$ where $\mathbf{m}_N = \mathbf{S}_N(\mathbf{S}_0^{-1}\mathbf{m}_0 + \beta\boldsymbol{\Phi}^T\mathbf{t})$, $\mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \beta\boldsymbol{\Phi}^T\boldsymbol{\Phi}$.

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  - This is your home work.

- When $p(\mathbf{w}|\alpha) = \mathcal{N}(\mathbf{w}|\mathbf{0}, \alpha^{-1}\mathbf{I})$, we have $\mathbf{m}_N = \beta \mathbf{S}_N \mathbf{\Phi}^T \mathbf{t}$, $\mathbf{S}_N^{-1} = \alpha \mathbf{I} + \beta \mathbf{\Phi}^T \mathbf{\Phi}$

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- The maximum a posteriori (MAP) estimation is given by

$$\ln p(\mathbf{w}|\mathbf{t}, \mathbf{X}, \beta, \lambda) = -\frac{\beta}{2} \sum_{n=1}^{N} \{t_n - \mathbf{w}^T \boldsymbol{\phi}(\mathbf{x}_n)\}^2 - \frac{\alpha}{2} \mathbf{w}^T \mathbf{w} + \text{consts.}$$

$$\Rightarrow \arg\max_{\mathbf{w}} \ln p(\mathbf{w}|\mathbf{t}) = \arg\min_{\mathbf{w}} \frac{1}{2} \|\mathbf{\Phi}\mathbf{w} - \mathbf{t}\|_2^2 + \frac{\alpha}{2\beta} \|\mathbf{w}\|_2^2$$

Regularization Framework

$$\min_{\mathbf{w}} \frac{1}{2}\|\mathbf{\Phi}\mathbf{w} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}\sum_{j=1}^{M} |\mathbf{w}_j|^q = \min_{\mathbf{w}} \frac{1}{2}\|\mathbf{\Phi}\mathbf{w} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}\|\mathbf{w}\|_q^q$$

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- Ridge regression ($q = 2$)

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- When $q < 1$ the problem is non-convex (more sparsity).

Solutions in simple cases

- Analysis in 1D, when $x = 1$.
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- How about 2D case?

Notes on Regularization

- From regularized problems to constrained problems. For convex problems, a regularized problem has an equivalent constrained problem.

$$\min_x f(x) + r(x) \Leftrightarrow \min_x f(x) \text{ s.t. } x \in \mathcal{S}$$

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- Solving

$$\min_{\mathbf{w}} \frac{1}{2}\|\mathbf{\Phi w} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}\|\mathbf{w}\|_q^q$$

is equivalent to

$$\min_{\mathbf{w}} \frac{1}{2}\|\mathbf{\Phi w} - \mathbf{t}\|_2^2 \text{ s.t. } \|\mathbf{w}\|_q^q \leq z$$

Geometry Interpretation of Regularization

Ridge regression $(q = 2)$

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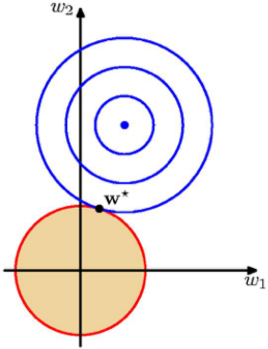

Geometry Interpretation of Regularization

Lasso regression $(q = 1)$

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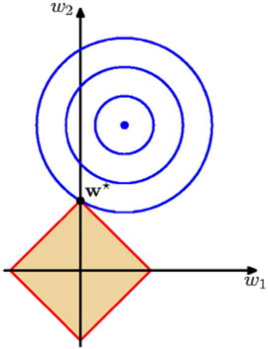

Geometry Interpretation of Regularization

The more general ℓ_p regularizer:

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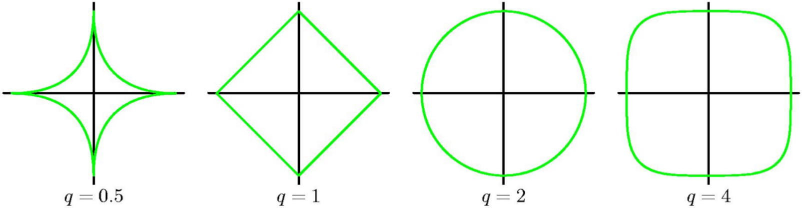

Sparsity-Inducing Norms

- Lasso is commonly used for feature selection.
- Theory
  - `http://www-stat.stanford.edu/~tibs/lasso.html`
- Software
  - SLEP `http://www.public.asu.edu/~jye02/Software/SLEP`
  - FPC `http://www-stat.stanford.edu/~tibs/lasso.html`
  - L1_Ls `http://www-stat.stanford.edu/~tibs/lasso.html`

From Online Learning to Bayesian Learning

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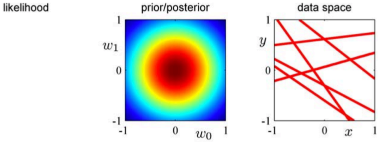
\begin{aligned}
p(\mathbf{w}|\mathcal{D};\alpha,\beta) =& p(\mathbf{w}|\mathbf{t},\mathbf{X};\alpha,\beta) \propto p(\mathbf{t}|\mathbf{X};\mathbf{w},\beta)p(\mathbf{w}|\alpha) \\
&= \prod_{i=1}^{n} p(\mathbf{t}_i|\mathbf{x}_i;\mathbf{w},\beta)p(\mathbf{w}|\alpha) \\
&= \underbrace{p(\mathbf{t}_n|\mathbf{x}_n;\mathbf{w},\beta)}_{\text{Likelihood}} \underbrace{\left[\prod_{i=1}^{n-1} p(\mathbf{t}_i|\mathbf{x}_i;\mathbf{w},\beta)p(\mathbf{w}|\alpha)\right]}_{\text{Prior}}
\end{aligned}
$$

From Online Learning to Bayesian Learning



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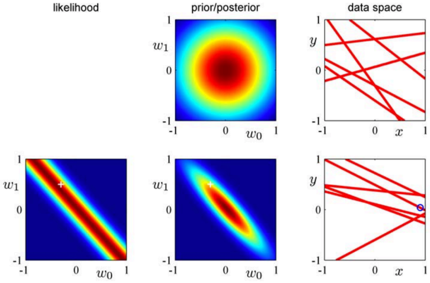

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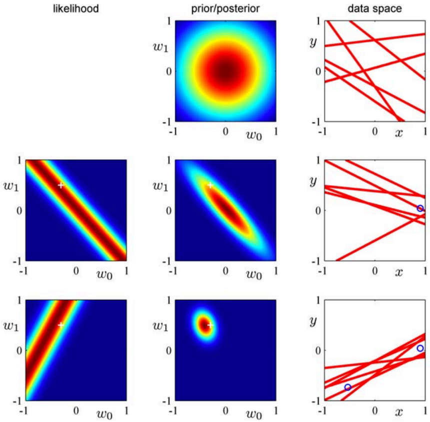

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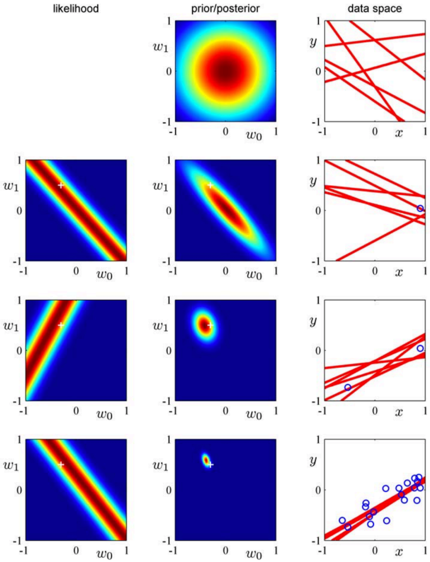

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where

$$p(\hat{t}|\hat{\mathbf{x}}, \mathbf{w}; \beta) = \mathcal{N}(\hat{t}|\hat{\boldsymbol{\phi}}^T \mathbf{w}, \beta^{-1}), \quad p(\mathbf{w}|\mathcal{D}; \alpha, \beta) = \mathcal{N}(\mathbf{w}|\mathbf{m}_N, \mathbf{S}_N)$$

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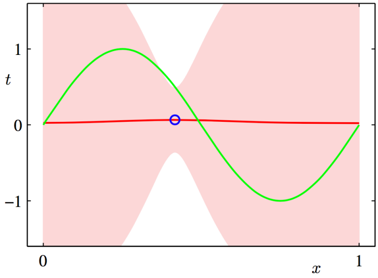
- You will show that (in homework):

$$p(\hat{t}|\hat{\mathbf{x}}; \mathcal{D}, \alpha, \beta) = \mathcal{N}(\hat{t}|\mathbf{m}_N^T \boldsymbol{\phi}(\mathbf{x}), \sigma_N^2(\mathbf{x}))$$

and variance $\sigma_N^2(\mathbf{x}) = \frac{1}{\beta} + \boldsymbol{\phi}(\mathbf{x})^T \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x})$.

Predictive Distribution



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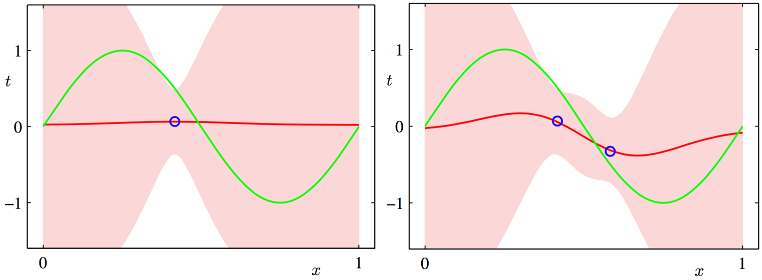

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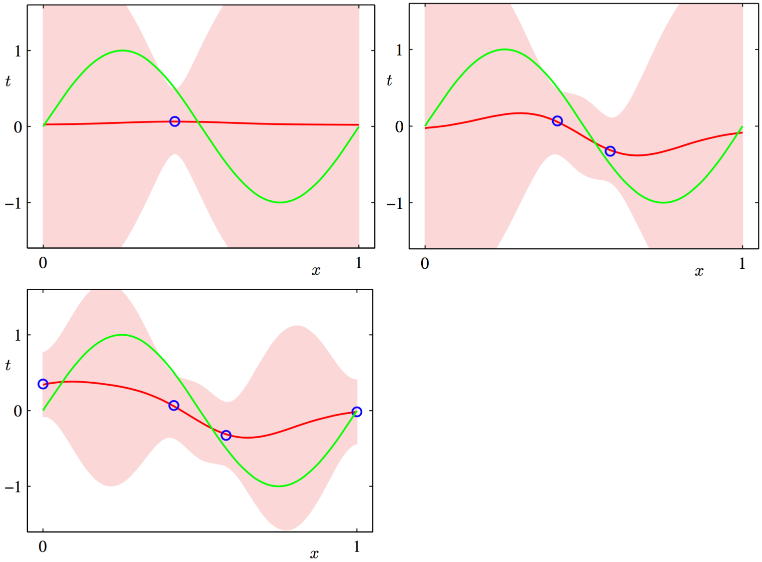

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