CSE/ECE 848 Introduction to Evolutionary Computation

Module 4, Lecture 19, Part 1

Multi-Level Optimization

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Multi-Level Optimization

Min $F^1(x_1, x_2, \dots, x_n)$
$x \in X^1$

Fixed $\bar{x}_1$

Min $F^2(\bar{x}_1, x_2, \dots, x_n)$
$x \in X^2$

Fixed $\bar{x}_1$ & $\bar{x}_1$

Min $F^3(\bar{x}_1, \bar{x}_2, \dots, x_n)$
$x \in X^3$

- Nested optimization
- Multiple interconnected but hierarchical optimization problems
  - Bilevel optimization
- Usually computationally expensive
- But appears commonly in practice
- Single and multiple objectives considered

Reference: Dempe, Stephan. *Foundations of bilevel programming*. Springer Science & Business Media, 2002.

Bilevel Optimization

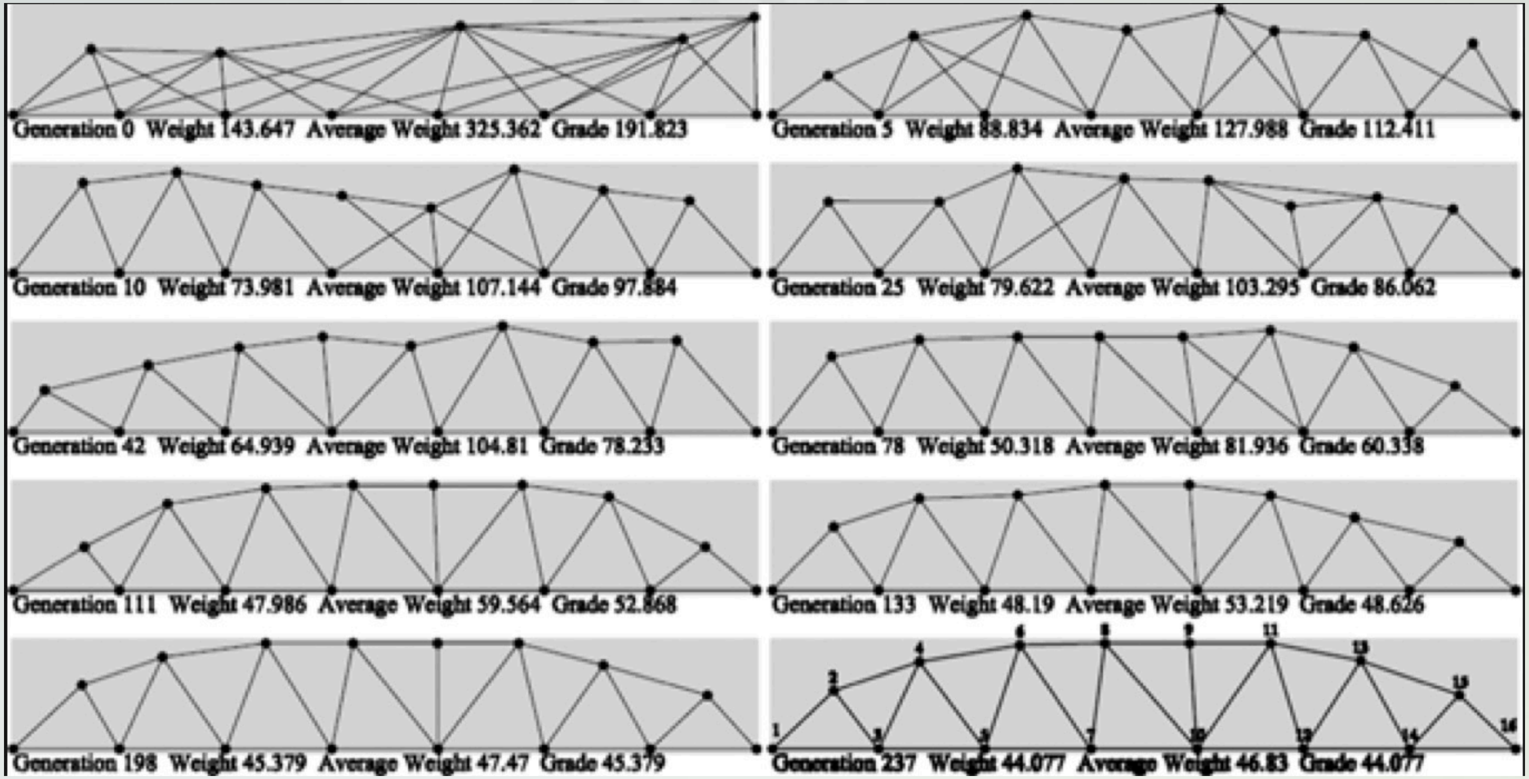
- Two levels of optimization tasks
  - Upper level: (x_u, x_l)
  - Lower level: (x_l), x_u is fixed
- An upper level feasible solution must be an optimal lower level solution: $(x_u, x_l^*(x_u))$

Min is default, can be max in any of the levels

$$
\begin{aligned}
\mathrm{Min}_{(\mathbf{x}_u,\mathbf{x}_l)} \quad & F(\mathbf{x}_u,\mathbf{x}_l),\\
\mathrm{st} \quad & \mathbf{x}_l \in \mathrm{argmin}_{(\mathbf{x}_l)} \left\{ \begin{array}{c} f(\mathbf{x}_u,\mathbf{x}_l) \\ \mathbf{g}(\mathbf{x}_u,\mathbf{x}_l) \geq \mathbf{0}, \mathbf{h}(\mathbf{x}_u,\mathbf{x}_l) = \mathbf{0} \end{array} \right\},\\
& \mathbf{G}(\mathbf{x}_u,\mathbf{x}_l) \geq \mathbf{0}, \mathbf{H}(\mathbf{x}_u,\mathbf{x}_l) = \mathbf{0},\\
& (\mathbf{x}_u)_{min} \leq \mathbf{x}_u \leq (\mathbf{x}_u)_{max}, (\mathbf{x}_l)_{min} \leq \mathbf{x}_l \leq (\mathbf{x}_l)_{max}
\end{aligned}
$$

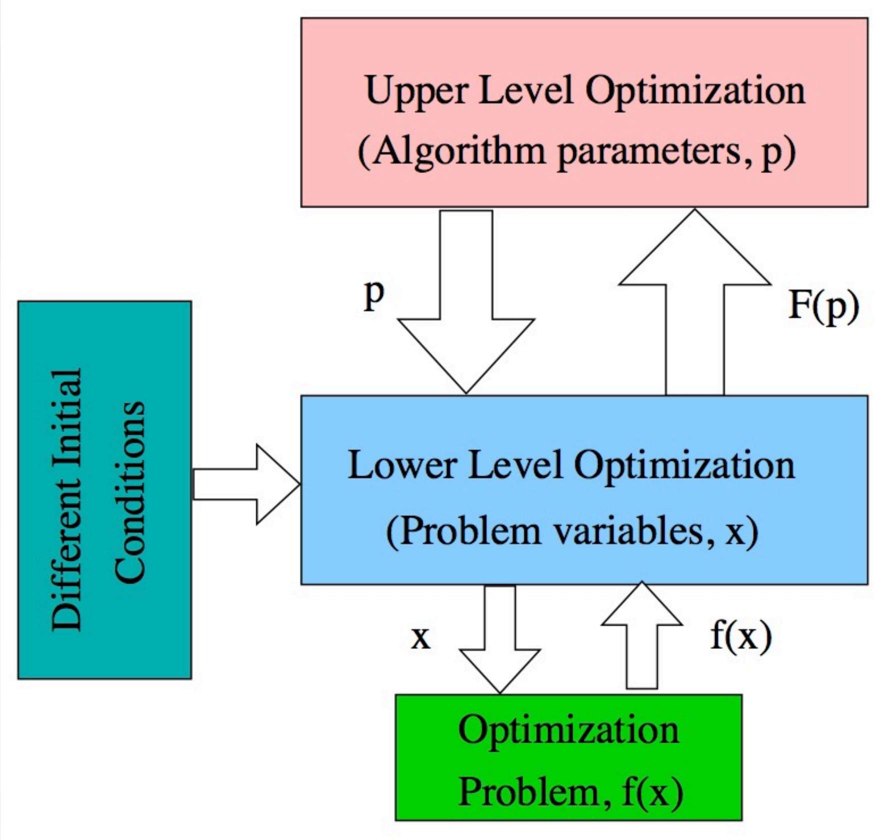
Structural Optimization

- Upper level: Topology (connectivity of members)
- Lower level: Sizes and coordinates



Parameter Tuning

- Upper Level: Find optimal parameters that maximize algorithm performance over a number of initial conditions
- Lower Level: Run the optimization algorithm to find optimized solution



Does it make sense to conclude an algorithm's performance without spelling out algorithm parameters?

Properties of Bilevel Problems

- Bilevel problems are typically non-convex, disconnected and strongly NP-hard
- An optimal solution for a lower level optimization problem produces a feasible solution at upper level
- Multiple globally optimal solutions at lower level can induce additional challenges
- Two levels can be cooperating or conflicting

Why Use Evolutionary Algorithms?

- First, no *implementable* mathematical optimality conditions exist (Dempe, Dutta, Mordokhovich, 2007)
  - LL problem is replaced with KKT conditions and constraint qualification (CQ) conditions of LL
  - UL problem requires KKT of LL-KKT conditions, but handling LL-CQ conditions in UL-KKT becomes difficult
  - Involves second-order differentials
- EA's flexible operators, direct use of objectives, and population approach should help solve BO problems better

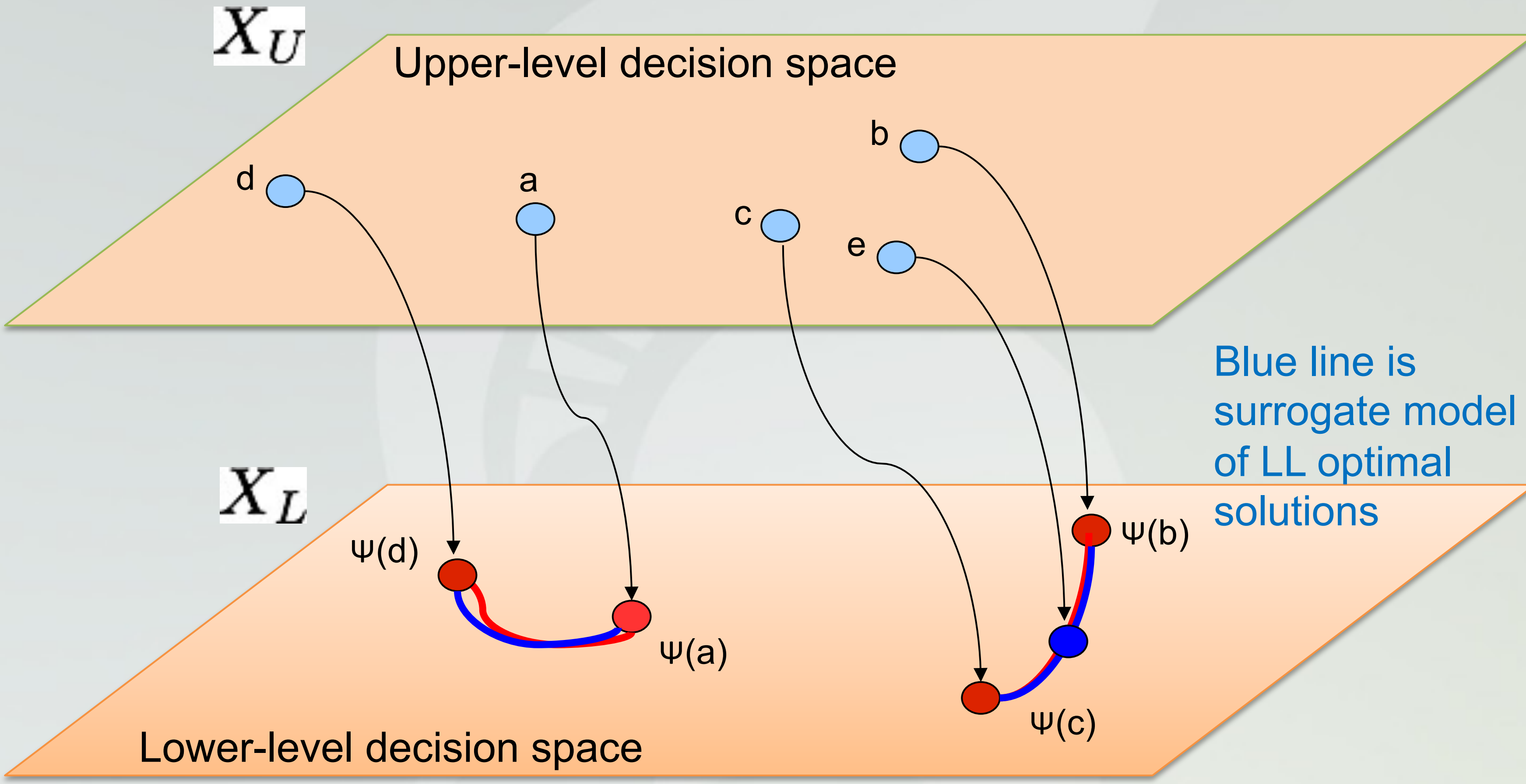
Approach 1 (Lower Level Reaction Set Mapping)

$$\Psi(x_u) = \underset{x_l}{\operatorname{argmin}}\{f(x_u, x_l) : g_j(x_u, x_l) \leq 0, j = 1, \ldots, J\}$$

$$\begin{aligned} \min_{x_u, x_l} \quad & F(x_u, x_l) \\ \text{s.t.} \quad & \\ & x_l \in \Psi(x_u) \\ & G_k(x_u, x_l) \leq 0, k = 1, \ldots, K \end{aligned}$$

Step 0: Solve the lower level problem completely for the initial population
Step 1: Use the population members to approximate the Ψ-mapping locally
Step 2: Solve the reduced single level problem for a few iterations
Step 3: Update the local Ψ-mappings and continue
Step 4: If termination criteria not met, go to Step 2

Using Approximate Ψ-mapping



Blue line is surrogate model (Ψ) of LL optimal solutions

For new upper level point 'e', we need not solve the lower level problem, as an approximate estimate is available

Reference: Sinha, A., Malo, P., and Deb, K. (2017). Evolutionary Algorithm for Bilevel Optimization using Approximations of the Lower Level Optimal Solution Mapping. European Journal of Operational Research, 257(2), 395–411.

Approach 2 (Optimal Value Function Mapping)

$$\varphi(x_u) = \min_{x_l}\{f(x_u, x_l) : x_l \in \Omega(x_u)\}$$

$$\begin{aligned}
\min_{x_u, x_l} \quad & F(x_u, x_l) \\
\text{s.t.} \quad & \\
& f(x_u, x_l) \leq \varphi(x_u) \\
& g_j(x_u, x_l) \leq 0, j = 1, \ldots, J \\
& G_k(x_u, x_l) \leq 0, k = 1, \ldots, K
\end{aligned}$$

Reference: Sinha, A., Malo, P., and Deb, K. (2018). A Review on Bilevel Optimization: From Classical to Evolutionary Approaches and Applications. *IEEE Transactions on Evolutionary Computation, 22* (2), 276–295.

Step 0: Solve the lower level problem completely for the initial population
Step 1: Use the population members to approximate the φ-mapping locally
Step 2: Solve the reduced single level problem for a few iterations
Step 3: Update the local φ-mappings and continue
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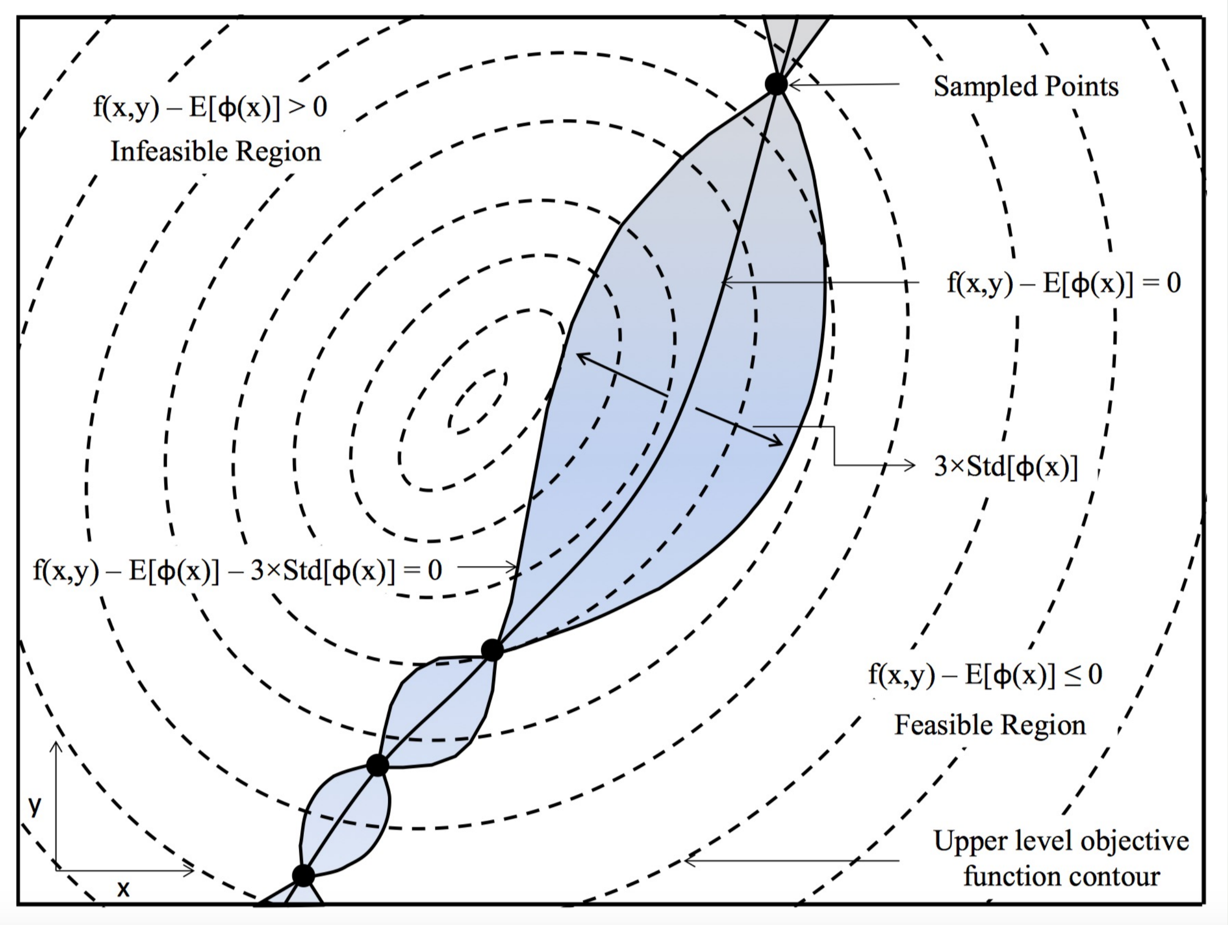
Approximation of Φ-mapping Through Kriging



Kriging provides both mean and standard deviation

$$f(x_u, x_l) \leq \varphi(x_u) + 3\times\text{Std}[\phi(x)]$$

Addition of the standard deviation term ensures feasibility of the auxiliary problem

Comparison with other approaches

Approach 1: Ψ- Mapping (Approach 1)
Approach 2: φ –Mapping (Approach 2)

21 runs

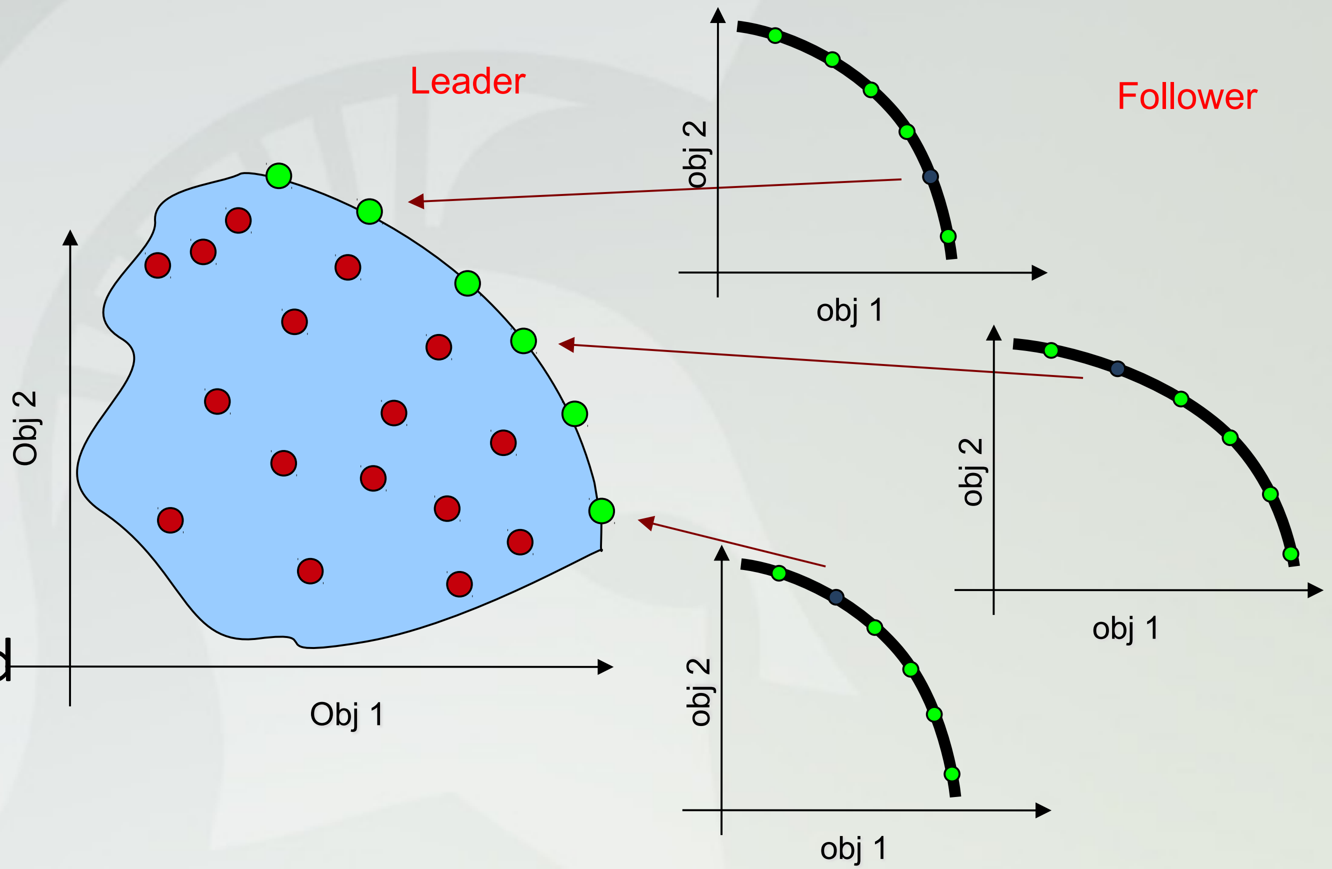
Mean Func. Evals. (UL+LL)					
	φ-appx.	Ψ-appx.	No-appx.	WJL	WLD
TP1	1595	2381	35896	85499	86067
TP2	1716	3284	5832	256227	171346
TP3	2902	1489	7469	92526	95851
TP4	3773	6806	21745	291817	211937
TP5	2941	3451	7559	77302	69471
TP6	1689	1162	1485	163701	65942
TP7	2126	1597	2389	1074742	944105
TP8	2699	4892	5215	213522	182121

In general, φ-Mapping approach is better

WJL – Wang et al. (2005), WLD – Wang et al. (2011)

Multi-objective Bilevel Optimization

- Multiple objectives involved at both the levels
- Two levels of decision making involved
- More complex and challenging optimization tasks



Reference: Deb, K. and Sinha, A. (2010). An Efficient and Accurate Solution Methodology for Bilevel Multi-Objective Programming Problems Using a Hybrid Evolutionary-Local-Search Algorithm. *Evolutionary Computation Journal, 18* (3). 403–449.

End of Module 4, Lecture 19, Part 1

- Multi-level optimization involves more than one intertwined optimization problems in a nested manner
- Computationally expensive
- Approximate methods proposed with faster approaches
- Extended to multi-objective bilevel problems
- Ideal problems for EAs

- Part 2:
  - Surrogate-assisted EAs