Probability Basics

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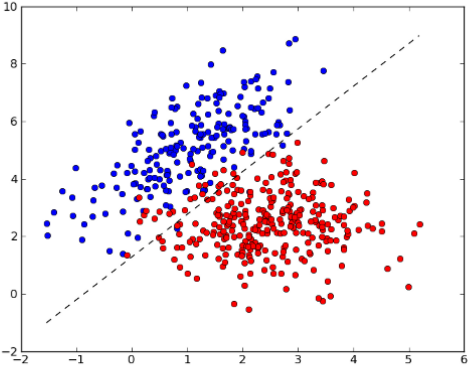
Noise



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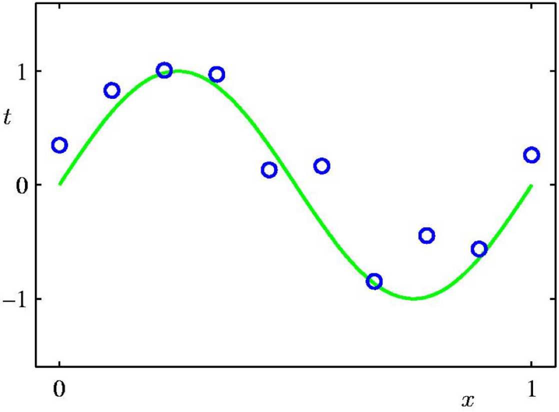

Noise in Sensors

Probability Theory

- Uncertainty arises both through noise on measurements, as well as through the finite size of data sets.
- Probability theory provides a consistent framework for the quantification and manipulation of uncertainty and forms one of the central foundations for machine learning.
- When combined with decision theory, it allows us to make optimal predictions given all the information available to us, even though that information may be incomplete or ambiguous.

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- $p(x)$: generative models, e.g., Generative Adversarial Networks
- $p(x|y)$: class conditioned density, e.g., conditional generative models

Definition of Probability

- **Experiment**: toss a coin twice
- **Sample space**: possible outcomes of an experiemnt
  S = {HH, HT, TH, TT}
- **Event**: a subset of possible outcomes
  A = {HH}, B = {HT, TH}
- **Probability of an event**: an number assigned to an event $p(A)$
  - Axiom 1: $p(A) \geq 0$
  - Axiom 2: $p(S) = 1$
  - Axiom 3: For every sequence of *disjoint* events

$$p(\cup_i A_i) = \sum_i p(A_i)$$

  - Example: $p(A) = n(A)/N$ (frequentist statistics)

Joint Probability

- For events A and B, **joint probability** $p(AB)$ stands for the probability that *both* events happen.
  - AB (or $A \cap B$) $\Rightarrow$ simultaneous occur. of events A and B

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- Example
  - $A = \{HH, HT\}$, $B = \{HH, TH\}$. What is $p(AB)$?
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Independence

- Two events A and B are **independent** in case

$$p(AB) = p(A)p(B)$$

- Can be extended to multiple events

$$p(\cap_i A_i) = \prod_i p(A_i)$$

Independence (cont.)

- Consider the experiment of tossing a coin twice
  - Example I: $A = \{HT, HH\}, B = \{HT\}$. Will event A independent from event B?

Independence (cont.)

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Independence (cont.)

- Consider the experiment of tossing a coin twice
  - Example I: $A = \{HT, HH\}, B = \{HT\}$. Will event A independent from event B?
  - Example II: $A = \{HT\}, B = \{TH\}$. Will event A independent from event B?
- Disjoint $\neq$ Independence.

Conditioning

- If A and B are events with $p(A) > 0$, the **conditional probability** of B given A is

$$p(B|A) = \frac{p(AB)}{p(A)}.$$

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- Example

	Women	Men
Success	200	1800
Failure	1800	200

A = Patient is a Woman, B = Drug fails, what are $p(B|A)$ and $p(A|B)$?

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	Women	Men
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A = Patient is a Woman, B = Drug fails, what are $p(B|A)$ and $p(A|B)$?

- Given A is independent from B, what is the relationship between $p(A|B)$ and $p(A)$?

Which Drug is Better?

	Women		**Men**	
	Drug I	Drug II	Drug I	Drug II
Success	200	10	19	1000
Failure	1800	190	1	1000

Which Drug is Better?

	Women		**Men**	
	Drug I	Drug II	Drug I	Drug II
Success	200	10	19	1000
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- View I: Comparing $p(C|A)$ and $p(C|B)$, where C={Drug succeed}, and A={Using Drug I}, and B={Using Drug II}.

	Drug I	Drug II
Success	219	1010
Failure	1801	1190

Which Drug is Better?

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- $p(C|A) \approx 10\%$ and $p(C|B) \approx 50\%$.
- **Drug II is better than Drug I.**

Which Drug is Better?

	Women		**Men**	
	Drug I	Drug II	Drug I	Drug II
Success	200	10	19	1000
Failure	1800	190	1	1000

- View II: Looking into male and female patients individually. What are $p(C|A)$ and $p(C|B)$ for female and male patients respectively?

Women	Drug I	Drug II
Success	200	10
Failure	1800	190

$p(C|A) = 10\%$, $p(C|B) = 5\%$

Which Drug is Better?

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$p(C|A) \approx 100\%,\ p(C|B) = 50\%$

Drug I is better than Drug II

Conditional Independence

- Event A and B are conditionally independent given C in case

$$p(AB|C) = p(A|C)p(B|C)$$

Conditional Independence

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$$p(AB|C) = p(A|C)p(B|C)$$

- A set of events $\{A_i\}$ is conditionally independent given C in case

$$p(\cup_i A_i|C) = \prod_i p(A_i|C)$$

Conditional Independence (cont'd)

Example: There are three events A, B, C

- $p(A) = p(B) = p(C) = 1/5$
- $p(A, C) = p(B, C) = 1/25, p(A, B) = 1/10$
- $p(A, B, C) = 1/125$

Conditional Independence (cont'd)

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Answer the following question:

- Whether A, B are independent?

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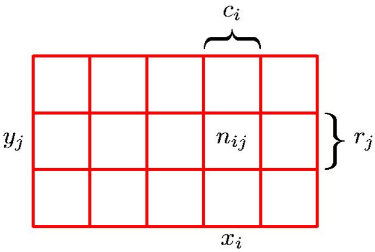
Answer the following question:

- Whether A, B are independent?
- Whether A, B are conditionally independent given C?

A and B are independent $\neq$ A and B are conditionally independent.

Probability Computation



Marginal Probability

$$p(X = x_i) = \frac{c_i}{N}$$

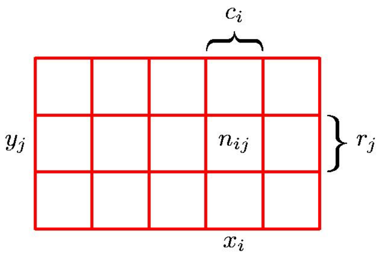
Conditional Probability

$$p(Y = y_i | X = x_i) = \frac{n_{ij}}{c_i}$$

Joint Probability

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

Probability Computation



Sum Rule

$$p(X = x_i) = \frac{c_i}{N} = \frac{1}{N}\sum_{j=1}^{L} n_{ij}$$

$$= \sum_{j=1}^{L} p(X = x_i, Y = y_j)$$

Product Rule

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i}\frac{c_i}{N}$$

$$= p(Y = y_i | X = x_i)p(X = x_i)$$

The Rules of Probability

Sum Rule $$p(X) = \sum_Y p(X, Y)$$

Product Rule $$p(X, Y) = p(Y|X)p(X)$$

These two simple rules form the basis for all of the probabilistic machinery that we need.

Bayes' Theorem

- From the product rule, together with the symmetry property $p(X, Y) = p(Y, X)$, we obtain the following relationship between conditional probabilities:

$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}$$

where $p(X) = \sum_Y p(X|Y)p(Y)$.

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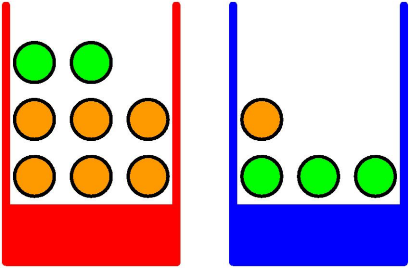
- Bayes' theorem plays a central role in pattern recognition and machine learning.

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

Illustration of Bayes' Theorem



apples and oranges, $p(\text{Box} = \text{red}) = 40\%$, $p(\text{Box} = \text{blue}) = 60\%$.

Suppose we are told that a piece of fruit has been selected and it is an orange, and we would like to know which box it came from.

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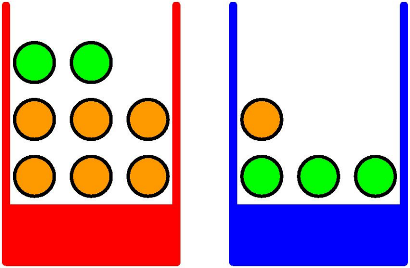

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$$p(B = r | F = o) = ?, p(B = b | F = o) = ?$$

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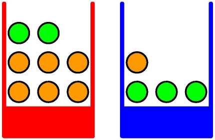

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$$p(B = r | F = o) = \frac{p(F = o | B = r) p(B = r)}{p(F = o)}$$

$$p(B = b | F = o) = \frac{p(F = o | B = b) p(B = b)}{p(F = o)}$$

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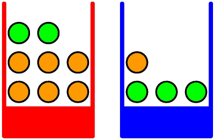

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$$p(B = b|F = o) = \frac{p(F = o|B = b)p(B = b)}{p(F = o)}$$

We have

$$p(B = r|F = o)/p(B = b|F = o) = 2/1$$
$$p(B = r|F = o) + p(B = b|F = o) = 1$$

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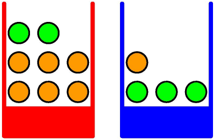

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$$p(B = r|F = o) + p(B = b|F = o) = 1$$

Therefore

$$p(B = r|F = o) = 2/3$$
$$p(B = b|F = o) = 1/3$$

Interpretation of Bayes' Theorem

$$p(B|F) = \frac{p(F|B)p(B)}{p(F)}$$

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

- $p(B)$: prior probability because it is the probability available before we observe the identity of the fruit.
- $p(B|F)$: posterior probability because it is the probability obtained after we have observed F.

Random Variable and Distribution

- A **random variable** X is a numerical outcome of a random experiment
- The **distribution** of a random variable is the collection of possible outcomes along with their probabilities:
  - Discrete case: $p(X = x) = p(x)$
  - Continuous case: $p(a \leq X \leq b) = \int_a^b p(x)dx$

Expectation

- For a random variable $X \sim p(X = x)$, its **expectation** is

$$\mathbb{E}(X) = \sum_x xp(X = x)$$

  - In an empirical sample, $x_1, x_2, \ldots, x_N$, $\mathbb{E}[X] = \frac{1}{N}\sum_{i=1}^N x_i$
- Continuous case: $E[X] = \int_{-\infty}^{\infty} xp(x)dx$
- Expectation of sum of random variables

$$\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2]$$

Expectation of a function

The average value of some function $f(x)$ under a probability distribution $p(x)$ is called the **expectation** of $f(x)$:

- Discrete $\mathbb{E} = \sum_x p(x)f(x)$
  Approximate Expectation $\mathbb{E} \approx \frac{1}{N}\sum_{n=1}^{N} f(x_n)$
- Continuous $\mathbb{E}[f] = \int p(x)f(x)dx$

Variances

The **variance** of $f(x)$ denoted as var$[f]$ provides a measure of how much variability there is in $f(x)$ around its mean value $\mathbb{E}[f(x)]$.

$$var[f] = \mathbb{E}\left[(f(x) - \mathbb{E}[f(x)])^2\right] = \mathbb{E}[f(x)^2] - \mathbb{E}[f(x)]^2$$

Covariances

- For two random variables x and y, the **covariance** is defined by

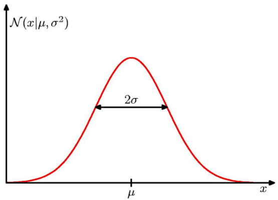
$$
\begin{aligned}
\mathrm{cov}[x, y] &= \mathbb{E}_{x,y}[\{x - \mathbb{E}[x]\}\{y - \mathbb{E}[y]\}] \\
&= \mathbb{E}_{x,y}[xy] - \mathbb{E}[x]\mathbb{E}[y]
\end{aligned}
$$

- It expresses the extent to which x and y vary together. If x and y are independent, then their covariance vanishes.
- The covariance between two vectors of random variables x and y is a matrix

$$
\begin{aligned}
\mathrm{cov}[\mathrm{x}, \mathrm{y}] &= \mathbb{E}_{\mathrm{x},\mathrm{y}}[\{\mathrm{x} - \mathbb{E}[\mathrm{x}]\}\{\mathrm{y} - \mathbb{E}[\mathrm{y}]\}] \\
&= \mathbb{E}_{\mathrm{x},\mathrm{y}}[\mathrm{x}\mathrm{y}^T] - \mathbb{E}[\mathrm{x}]\mathbb{E}[\mathrm{y}^T]
\end{aligned}
$$

The Gaussian Distribution

The normal or Gaussian distribution is one of the most important probability distributions for continuous variables.



$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

- The square root of the variance, given by σ, is called the standard deviation, and the reciprocal of the variance is called the precision.
- $\mathcal{N}(x|\mu,\sigma^2) > 0$ and $\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\sigma^2)dx = 1$

Gaussian Mean and Variance

- The average value of x is given by

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) x dx = \mu$$

$$\mathbb{E}[x^2] = \int_{-\infty}^{\infty} \mathcal{N}(x|\mu, \sigma^2) x^2 dx = \mu^2 + \sigma^2$$

- The variance of x is given by

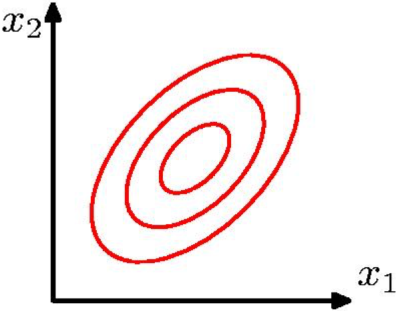
$$\text{var}[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 = \sigma^2$$

The Multivariate Gaussian

The Gaussian distribution defined over a d-dimensional vector x of continuous variables is given by

$$\mathcal{N}(\mathrm{x}|\boldsymbol{\mu}, \boldsymbol{\sigma}^2) = \frac{1}{(2\pi)^{d/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\mathrm{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\mathrm{x}-\boldsymbol{\mu})\right\}$$

where $\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}$ is the covariance



Gaussian Parameter Estimation

- We are given a data set of N observations $\mathrm{x} = \{x_1, x_2, \ldots, x_n\}$ of the scalar variable x.
- We shall suppose that the observations are drawn independently from a Gaussian distribution whose mean and variance are unknown, and need to be estimated.

Gaussian Parameter Estimation

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- We shall suppose that the observations are drawn independently from a Gaussian distribution whose mean and variance are unknown, and need to be estimated.
- Likelihood function $p(\mathrm{x}|\mu, \sigma^2) = \prod_{n=1}^{N} \mathcal{N}(x_n|\mu, \sigma^2)$
- **Maximum Likelihood**: Determine values for the unknown parameters in the Gaussian by maximizing the likelihood function.

Maximum (Log) Likelihood

- It is more convenient to maximize the **log of the likelihood** function:

$$\ln p(\mathrm{x}|\mu, \sigma^2) = -\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n - \mu)^2 - \frac{N}{2}\ln \sigma^2 - \frac{N}{2}\ln(2\pi)$$

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- Therefore we have

$$\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n \qquad \text{Sample mean}$$

$$\sigma^2_{ML} = \frac{1}{N}\sum_{n=1}^{N}(x_n - \mu_{ML})^2 \qquad \text{Sample variance}$$

Maximum a Posterior

- We take a step towards a more Bayesian approach and introduce a prior distribution over the parameters.
- **MAP (maximum posterior)**: Determine the parameters by finding the most probable values given the data, in other words by maximizing the posterior distribution.
-

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

Full Bayesian Approach

- In MAP, we are still making a **point estimate** and so this does not yet amount to a Bayesian treatment.
- In a fully Bayesian approach, we should integrate over all values of the parameter (marginalization).

Decision Theory

- Suppose we have an input vector x together with a corresponding vector t of target variables, and our goal is to predict t given a new value for x.
  - Regression: t comprises continuous variables
  - Classification: t represents class labels
- **Inference Step**: Determine either $p(\mathrm{x}, t)$ or $p(t|\mathrm{x})$. It gives us the most complete probabilistic description of the situation.
- **Decision Step**: How to make optimal decision.
- Three approaches.

Inference and Decision: (1)

- First solve the inference problem of determining the class-conditional densities $p(\mathrm{x}|\mathcal{C}_k)$ for each class $\mathcal{C}_k$ individually. Also separately infer the prior class probabilities $p(\mathcal{C}_k)$. Then use Bayes' theorem to find the posterior probabilities in the form:

$$p(\mathcal{C}_k|\mathrm{x}) = \frac{p(\mathrm{x}|\mathcal{C}_k)p(\mathcal{C}_k)}{p(\mathrm{x})}$$

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- Generative model: Equivalently, we can model the joint distribution $p(\mathrm{x}, \mathcal{C}_k)$ directly and then normalize to obtain the posterior probabilities.

Inference and Decision: (2)

- First solve the inference problem of determining the posterior class probabilities $p(\mathcal{C}_k|\mathrm{x})$, and then subsequently use decision theory to assign each new x to one of the classes. Approaches that model the posterior probabilities directly are called discriminative models.

Inference and Decision: (2)

- First solve the inference problem of determining the posterior class probabilities $p(\mathcal{C}_k|\mathrm{x})$, and then subsequently use decision theory to assign each new x to one of the classes. Approaches that model the posterior probabilities directly are called discriminative models.
- Find a function $f(\mathrm{x})$, called a discriminant function, which maps each input x directly onto a class label. For instance, in the case of two-class problems, $f(\cdot)$ might be binary valued and such that $f = 0$ represents class $\mathcal{C}_1$ and $f = 1$ represents class $\mathcal{C}_2$. In this case, probabilities play no role.

Next Class

- Topic
  - Linear Algebra Basics
- Reading
  - Book Ch. 1,2