UNIT 5 FIRST ORDER PREDICATE LOGIC

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5.0 INTRODUCTION

In the previous unit, we discussed how propositional logic helps us in solving problems. However, one of the major problems with propositional logic is that, sometimes, it is unable to capture even elementary type of reasoning or argument as represented by the following statements:

Every man is mortal.

Raman is a man.

Hence, he is mortal.

The above reasoning is intuitively correct. However, if we attempt to simulate the reasoning through Propositional Logic and further, for this purpose, we use symbols P, Q and R to denote the statements given above as:

P: Every man is mortal,

Q: Raman is a man,

R: Raman is mortal.

Once, the statements in the argument in English are symbolised to apply tools of propositional logic, we just have three symbols P, Q and R available with us and apparently no link or connection to the original statements or to each other. The connections, which would have helped in solving the problem become invisible. In Propositional Logic, there is no way, to conclude the *symbol* R from the *symbols* P and Q. However, as we mentioned earlier, even in a natural language, the conclusion of the *statement* denoted by R from *the statements* denoted by P and Q is obvious. Therefore, we search for some **symbolic** system of reasoning that helps us in discussing *argument forms* of the above-mentioned type, in addition to those forms which can be discussed within the framework of propositional logic. **First Order Predicate Logic (FOPL)** is the most well-known symbolic system for the pourpose.

The symbolic system of FOPL treats an atomic statement *not as an indivisible unit*. Rather, FOPL not only treats an atomic statement divisible into subject and predicate but even further deeper structures of an atomic statement are considered in order to handle larger class of arguments. How and to what extent

FOPL symbolizes and establishes *validity/invalidity* and *consistency/inconsistency* of *arguments* is the subject matter of this unit.

5.1 OBJECTIVES

After studying this unit, you should be able to:

- explain why FOPL is required over and above PL;
- define, and give appropriate examples for, each of the new concepts required for FOPL including those of quantifier, variable, constant, term, free and bound occurrences of variables, closed and open wff;
- check consistency/validity, if any, of closed formulas;
- reduce a given formula of FOPL to normal forms: Prenex Normal Form (PNF) and (Skolem) Standard Form, and conversion to the clausal form
- use the tools and techniques of FOPL, developed in the unit, to solve problems requiring logical reasoning
- Perform unification and resolution mechanism.

5.2 SYNTAX OF FIRST ORDER PREDICATE LOGIC

We learned about the concept of propositions in Artificial intelligence, in Unit 4 of Block 1. Now it's time to understand the difference between the Proposition and the Predicate (also known as propositional function). In short, a proposition is a specialized statement whereas Predicate is a generalized statement. To be more specific the propositions uses the logical connectives only and the predicates uses logical connectives and quantifiers (universal and existential), both.

Note : ∃ is the symbol used for the Existential quantifier and ∀ is used for the Universal quantifier.

Let's understand the difference through some more detail, as given below.

A propositional function, or a **predicate,** in a variable x is a sentence p(x) involving x that becomes a proposition when we give x a definite value from the set of values it can take. We usually denote such functions by p(x), q(x), etc. The set of values x can take is called the universe of discourse.

So, if p(x) is '$x > 5$', then p(x) is not a proposition. But when we give x particular values, say $x = 6$ or $x = 0$, then we get propositions. Here, p(6) is a true proposition and p(0) is a false proposition.

Similarly, if q(x) is 'x has gone to Patna.', then replacing x by 'Taj Mahal' gives us a false proposition.

Note that a predicate is usually not a proposition. But, of course, every proposition is a prepositional function in the same way that every real number is a real-valued function, namely, the constant function.

Now, can all sentences be written in symbolic from by using only the logical connectives? What about sentences like 'x is prime and x + 1 is prime for some x.'? How would you symbolize the phrase 'for some x', which we can rephrase as 'there exists an x'? You must have come across this term often while studying mathematics. **We use the symbol '∃' to denote this quantifier, 'there exists'.** The way we use it is, for instance, to rewrite 'There is at least one child in the class.' as'(∃ x in U)p(x)',

where p(x) is the sentence 'x is in the class.' and U is the set of all children.

Now suppose we take the negative of the proposition we have just stated. Wouldn't it be 'There is no child in the class.'? We could symbolize this as 'for all x in U, q(x)' where x ranges over all children and q(x) denotes the sentence 'x is not in the class.', i.e., $q(x) \equiv \sim p(x)$.

We have a **mathematical symbol for the quantifier 'for all', which is '$\forall$'.** So the proposition above can be written as

'$(\forall x \in U)q(x)$', or '$q(x), \forall x \in U$'.

An example of the use of the existential quantifier is the true statement.

$(\exists x \in \mathbf{R})(x + 1 > 0)$, which is read as 'There exists an x in **R** for which $x + 1 > 0$.'.

Another example is the false statement

$(\exists x \in \mathbf{N})(x - \frac{1}{2} = 0)$, which is read as 'There exists an x in **N** for which $x - \frac{1}{2} = 0$.'.

An example of the use of the universal quantifier is $(\forall x \notin \mathbf{N})(x^2 > x)$, which is read as 'for every x not in **N**, $x^2 > x$.'. Of course, this is a false statement, because there is at least one $x \notin \mathbf{N}$, $x \in \mathbf{R}$, for which it is false.

As you have already read in the example of a child in the class,

$(\forall x \in U)p(x)$ is logically equivalent to $\sim(\exists x \in U)(\sim p(x))$. Therefore,

$\sim(\forall x \in U)p(x) \equiv \sim\sim(\exists x \in U)(\sim p(x)) \equiv (\exists x \in U)(\sim p(x))$.

This is one of the rules for negation that relate $\forall$ and $\exists$. The two rules are

$\sim(\forall \mathbf{x} \in \mathbf{U})\mathbf{p}(\mathbf{x}) \equiv (\exists x \in \mathbf{U})(\sim \mathbf{p}(\mathbf{x}))$, and

$\sim(\exists \mathbf{x} \in \mathbf{U})\mathbf{p}(\mathbf{x}) \equiv (\forall x \in \mathbf{U})(\sim \mathbf{p}(\mathbf{x}))$

Where U is the set of values that x can take.

5.3 INTERPRETATIONS IN FOPL

In order to have a glimpse at how FOPL extends propositional logic, let us again discuss the earlier argument.

Every man is mortal. Raman is a man.

Hence, he is mortal.

In order to derive the validity of above simple argument, instead of looking at an atomic statement as indivisible, to begin with, we divide each statement into *subject* and *predicate*. The two predicates which occur in the above argument are:

'*is mortal*' and *'is man'*.

Let us use the notation

IL: *is_mortal* and

IN: *is_man*.

In view of the notation, the argument on para-phrasing becomes:
For all x, if IN (x) then IL (x).

IN (Raman).

Hence, IL (RAMAN)

More generally, relations of the form *greater-than (x, y)* denoting the phrase 'x is greater than y', *is_brother_ of* (x, y) denoting '*x is brother of y,*' *Between (x, y, z)* denoting the phrase that '*x lies between y and z*', and *is_tall (x)* denoting '*x is tall*' are some **examples of predicates**. The variables x, y, z etc which appear in a predicate are called **parameters** of the predicate.

The parameters may be given some appropriate values such that after substitution of appropriate value from all possible values of each of the variables, the predicates become *statements*, for each of which we can say whether it is 'True' or it is 'False'.

For example, for the predicate *greater-than (x, y),* if x is given value 3 then we obtain *greater-than (3, y),* for which still it is not possible to tell whether it is True or False. Hence, '*greater-than (3, y)*' is also a predicate. Further, if the variable y is given value 5 then we get *greater (3, 5)* which , as we known, is False. Hence, it is possible to give its Truth-value, which is *False* in this case. Thus, from the *predicate greater-than (x, y)*, we get the *statement greater-than (3, 5)* by assigning values 3 to the variable x and 5 to the variable y. These values 3 and 5 are called parametric values or *arguments* of the predicate *greater-than*.

(Please note 'argument of a function/predicate' is a mathematical concept, different from logical argument)

Similarly, we can represent the phrase *x likes y* by the *predicate LIKE (x, y)*. Then *Ram likes Mohan* can be represented by the statement *LIKE (RAM, MOHAN)*.

Also *function symbols* can be used in the first-order logic. For example, we can use *product (x, y)* to denote $x * y$ and *father (x)* to mean the *'father of x'*. The statement: *Mohan's father loves Mohan* can be symbolised as *LOVE (father (Mohan), Mohan)*. Thus, we need not know name of father of Mohan and still we can talk about him. A function serves such a role.

We may note that LIKE (Ram, Mohan) and LOVE (father (Mohan),Mohan) are atoms or atomic statements of PL, in the sense that, one can associate a truth-value *True* or *False* with each of these, and each of these does not involve a logical operator like $\sim, \wedge, \vee, \rightarrow$ or $\leftrightarrow$.

Summarizing in the above discussion, *LIKE (Ram, Mohan)* and *LOVE (father (Mohan) Mohan)* are **atoms;** where as GREATER, LOVE and LIKE are **predicate symbols;** x and y are **variables and 3**, *Ram* and *Mohan* are **constants;** and *father* and *product* are **function symbols**.

From the above discussion we learned the following concepts of symbols.

i) ***Individual symbols or*** *constant symbols*: These are usually names of objects, such as Ram, Mohan, numbers like 3, 5 etc.
ii) **Variable symbols**: These are usually lowercase unsubscripted or subscripted letters, like x, y, z, x_3.

iii) **Function symbols:** These are usually lowercase letters like f, g, h,….or strings of lowercase letters such as *father* and *product*.
iv) **Predicate symbols**: These are usually uppercase letters like P, Q, R,….or strings of lowercase letters such as *greater-than*, *is_tall* etc.

A function symbol or predicate symbol takes a fixed number of arguments. If a *function symbol f* takes *n* arguments, *f* is called an *n-place function symbol*. Similarly, if a predicate symbol Q takes *m* arguments, P is called an m-*place predicate symbol*. For example, father is a one-place *function* symbol, and GREATER and LIKE are two-place *predicate* symbols. However, *father-of* in father_of (x, y) is a, *two place predicate* symbol.

The symbolic representation of an argument of a function or a predicate is called a *term* where a **term** is defined recursively as follows:

i) A variable is a term.
ii) A constant is a term.
iii) If *f* is an n-place function symbol, and $t_1 \ldots t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.
iv) Any term can be generated only by the application of the rules given above.

For example: Since, *y* and 3 are both terms and *plus* is a two-place function symbol, *plus (y, 3)* is a term according to the above definition.

Furthermore, we can see that *plus (plus (y, 3), y)* and *father (father (Mohan))* are also terms; the former denotes *(y + 3) + y* and the later denotes *grandfather of Mohan*.

A predicate can be thought of as a function that maps a list of constant arguments to T or F. For example, GREATER is a predicate with GREATER (5, 2) as T, but GREATER (1, 3) as F.

We already know that in PL, an atom or atomic statement is an indivisible unit for representing and validating arguments. Atoms in PL are denoted generally by symbols like P, Q, and R etc. But in FOPL,

Definition: An Atom is

(i) either an atom of Propositional Logic, or
(ii) is obtained from an n-place predicate symbol P, and terms $t_1, \ldots t_n$ so that $P(t_1, \ldots, t_n)$ is an atom.

Once, the atoms are defined, by using the logical connectives defined in Propositional Logic, and assuming having similar meaning in FOPL, we can build complex formulas of FOPL. Two special symbol $\forall$ and $\exists$ are used to denote qualifications in FOPL. The symbols $\forall$ and $\exists$ are called, respectively, the *universal* quantifier and *existential* quantifier. For a variable x, $(\forall x)$ is read as *for all x,* and $(\exists x)$ is read as *there exists an x.* Next, we consider some examples to illustrate the concepts discussed above.

In order to symbolize the following statements:

i) There exists a number that is rational.
ii) Every rational number is a real number
iii) For every number *x*, there exists a number *y*, which is greater than *x*.

let us denote *x is a rational number* by Q(x), *x is a real number* by R(x), and *x is less than y* by *LESS(x, y)*. Then the above statements may be symbolized respectively, as

(i) $(\exists x)\ Q(x)$

(ii) $(\forall x)\ (Q(x) \rightarrow R\ (x))$

(iii) $(\forall x)\ (\exists y)$ LESS(x, y).

Each of the expressions (i), (ii), and (iii) is called a **formula** or a well-formed formula or **wff.**

5.4 SEMATICS OF QUANTIFIERS

To understand the semantics of quantifiers we need to first understand the difference between the Proposition and the Predicate(also known as propositional function). In short, a proposition is a specialized statement whereas Predicate is a generalized statement. To be more specific the propositions uses the logical connectives only and the predicates uses logical connectives and quantifiers (universal and existential), both.

Note : $\exists$ is the symbol used for the Existential quantifier and $\forall$ is used for the Universal quantifier.

Let's understand the difference through some more detail, as given below.

A propositional function, or a **predicate,** in a variable x is a sentence p(x) involving x that becomes a proposition when we give x a definite value from the set of values it can take. We usually denote such functions by p(x), q(x), etc. The set of values x can take is called the universe of discourse.

So, if p(x) is '$x > 5$', then p(x) is not a proposition. But when we give x particular values, say $x = 6$ or $x = 0$, then we get propositions. Here, p(6) is a true proposition and p(0) is a false proposition.

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Note that a predicate is usually not a proposition. But, of course, every proposition is a prepositional function in the same way that every real number is a real-valued function, namely, the constant function.

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where p(x) is the sentence 'x is in the class.' and U is the set of all children.

Now suppose we take the negative of the proposition we have just stated. Wouldn't it be 'There is no child in the class.'? We could symbolize this as 'for all x in U, q(x)' where x ranges over all children and q(x) denotes the sentence 'x is not in the class.', i.e., $q(x) \equiv \sim p(x)$.

We have a **mathematical symbol for the quantifier 'for all', which is '$\forall$'.** So the proposition above can be written as

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Another example is the false statement

($\exists$ x $\in$**N**) (x - $\frac{1}{2}$ = 0), which is read as 'There exists an x in **N** for which x - $\frac{1}{2}$ = 0.'.

An example of the use of the universal quantifier is ($\forall$ x $\notin$ **N**) ($x^2 > x$), which is read as 'for every x not in **N**, $x^2 > x$.'. Of course, this is a false statement, because there is at least one x$\notin$ **N**, x $\in$ **R**, for which it is false.

As you have already read in the example of a child in the class,

($\forall$ x $\in$U)p(x) is logically equivalent to ~ ($\exists$ x $\in$ U) (~ p(x)). Therefore,

~($\forall$ x $\in$ U)p(x) $\equiv$ ~~ ($\exists$ x $\in$U) (~ p(x)) $\equiv$ ($\exists$ x $\in$ U) (~ p(x)).

This is one of the rules for negation that relate $\forall$ and $\exists$. The two rules are

~ ($\forall$ **x** $\in$ **U)p(x)** $\equiv$ ($\exists$ x $\in$ **U**) **(~ p(x)),** and

~ ($\exists$ **x** $\in$ **U)p(x)** $\equiv$ ($\forall$ x $\in$ **U**) **(~ p(x))**

Where U is the set of values that x can take.

Next, we discuss three new concepts, viz **Scope** of occurrence of a quantified variable, Bound occurrence of a quantifier variable or quantifier and *Free occurrence* of a variable.

Before discussion of these concepts, we should know the *difference between a **variable** and **occurrence of a variable** in a quantifier expression*.

The variable *x* has THREE occurrences in the formula

($\exists$x) Q(x) $\rightarrow$ P(x, y).

Also, the variable *y* has only one occurrence and the variable *z* has zero occurrence in the above formula. Next, we define the three concepts mentioned above.

Scope of an occurrence of a quantifiers is the smallest but complete formula following the quantifier sometimes delimited by pair *f* parentheses. For example, Q(x) is the scope of ($\exists$x) in the formula

($\exists$x) Q(x) $\rightarrow$ P(x, y).

But the scope of ($\exists$x) in the formula: ($\exists$x) (Q(x) $\rightarrow$ P(x, y)) is (Q(x) $\rightarrow$ P(x, y)).

Further in the formula:

($\exists$x) (P(x) $\rightarrow$ Q(x, y)) $\wedge$ ($\exists$x) (P(x) $\rightarrow$ R(x, 3)),

the scope of **first** occurrence of ($\exists$x) is the formula (P(x) $\rightarrow$ Q (x, y) and the scope of **second** occurrence of ($\exists$x) is the formula

$(P(x) \rightarrow R(x, 3))$.

As another example, the scope of the only occurrence of the quantifier $(\forall y)$ in

$(\exists x)$ (($P(x) \rightarrow Q(x) \leftrightarrow (\forall y) (Q (x) \rightarrow R (y)))$ is ($Q (x) \rightarrow R(y))$. But the scope of the only occurrence of the existential variable $(\exists x)$ in the same formula is the formula:

$(P(x) \rightarrow Q(x))$ P $\leftrightarrow (\forall y) (Q (x) \rightarrow R(y))$

*An **occurrence** of a variable in a formula* is **bound** if and only if the occurrence is within the scope of a quantifier employing the variable, or is the occurrence in that quantifier. An occurrence of a variable in a formula is **free** if and only if this occurrence of the variable is not bound.

Thus, in the formula $(\exists x) P(x, y) \rightarrow Q (x)$, there are three occurrences of *x*, out of which first two occurrences of *x* are *bound*, where, the last occurrence of *x is free*, because scope of $(\exists x)$ in the above formula is P(x, y). The only occurrence of *y* in the formula is free. Thus, *x* is both a bound and a free variable in the above formula and y is only a free variable in the formula so far, we talked of an *occurrence of a variable* as free or bound. Now, we talk of (only) a *variable* as free or bound. *A variable is **free*** in a formula if at least one occurrence of it is free in the formula. A variable is **bound** in a formula if at least one occurrence of it is bound.

It may be noted that a variable can *be **both free and bound** in a formula*. In order to further elucidate the concepts of *scope, free and bound occurrences of a variable*, we consider a similar but different formula for the purpose:

$(\exists x) (P(x, y) \rightarrow Q(x))$.

In this formula, *scope* of the only occurrence of the quantifier $(\exists x)$ is the whole of the rest of the formula, viz. scope of $(\exists x)$ in the given formula is $(P(x, y) \rightarrow Q (x))$

Also, all three occurrence of variable *x* are bound. The only occurrence of *y* is free.

Remarks: It may be noted that a bound variable *x* is just a **place holder** or a **dummy variable** in the sense that all occurrences of a bound variable *x* may be replaced by another free variable say *y*, which does not occur in the formula. However, once, *x* is replaced by *y* then *y* becomes bound. For example, $(\forall x) (f (x))$ is the same as $(\forall y) f (y)$. It is something like

$$\int_1^2 x^2 dx = \int_1^2 y^2 dy = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

Replacing a bound variable *x* by another variable *y* under the restrictions mentioned above is called **Renaming of a variable x**

Having defined an atomic formula of FOPL, next, we consider the definition of a general formula formally in terms of atoms, logical connectives, and quantifiers.

Definition A well-formed formula, **wff** a just or formula in FOPL is defined recursively as follows:

i) An atom or atomic formula is a *wff.*

ii) If E and G are wff, then each of $\sim (E)$, $(E \vee G)$, $(E \wedge G)$, $(E \rightarrow G)$, $(E \leftrightarrow G)$ is a **wff.**

iii) If E is a *wff* and x is a free variable in E, then $(\forall x)E$ is a wff.

iv) A *wff* can be obtained only by applications of (i), (ii), and (iii) given above.

We may drop pairs of parentheses by agreeing that quantifiers have the least scope. For example, $(\exists x)\ P(x, y) \rightarrow Q(x)$ stands for

$((\exists x)\ P(x, y)) \rightarrow Q(x)$

We may note the following two cases of translation:

(i) for all x, P(x) is Q(x) is translated as
$(\forall x)\ (P(x) \rightarrow Q(x)\ )$
(the other possibility $(\forall x)\ P(x) \wedge Q(x)$ *is not valid.)*

(ii) for some x, P(x) is Q (x) is translated as $(\exists x)\ P(x) \wedge Q(x)$
(the other possibility $(\exists x)\ P(x) \rightarrow Q(x)$ *is not valid)*

Example
Translate the statement: *Every man is mortal. Raman is a man. Therefore, Raman is mortal.*

As discussed earlier, let us denote "*x is a man*" by MAN (x), and "*x is mortal*" by MORTAL(x). Then "*every man is mortal*" can be represented by

$$(\forall x)\ (\text{MAN}(x) \rightarrow \text{MORTAL}(x)),$$

"Raman is a man" by

$$\text{MORTAL(Raman)}.$$

The whole argument can now be represented by

$(\forall x)\ (\text{MAN}(x) \rightarrow \text{MORTAL}(x)) \wedge \text{MAN (Roman)} \rightarrow \text{MORTAL (Roman)}.$

as a single statement.

In order to further explain symbolisation let us recall the axioms of natural numbers:

(1) For every number, there is one and only one immediate successor,

(2) There is no number for which 0 is the immediate successor.

(3) For every number other than 0, there is one and only one immediate predecessor.

Let the *immediate successor* and *predecessor of x*, respectively be denoted by f(x) and g(x).

Let E *(x, y)* denote *x is equal to y*. Then the axioms of natural numbers are represented respectively by the formulas:

(i) $(\forall x)\ (\exists y)\ (E(y, f(x)) \wedge (\forall z)\ (E(z, f(x)) \rightarrow E(y, z)))$

(ii) $\sim ((\exists x)\ E(0, f(x)))$ and

(iii) $(\forall x)\ (\sim E(x, 0) \rightarrow ((y)\exists, g(x)) \wedge (\forall z)\ (E(z, g(x)) \rightarrow E(y, z)))))$.

From the semantics (for meaning or interpretation) point of view, the **wff of FOPL** may be divided into two categories, each consisting of

(i) wffs, in each of which, **all** occurrences of variables are **bound**.
(ii) wffs, in each of which, at **least one** occurrence of a variable is **free**.

The wffs of FOPL in which there is no occurrence of a free variable, are like *wffs* of PL in the sense that we can call each of the wffs as **True, False, consistent, inconsistent, valid, invalid etc**. Each such a formula is called **closed formula**. However, when a wff involves a free occurrence, then it is not possible to call such a wff as True, False etc. **Each of such a formula is called an open formula**.

For example: Each of the formulas: greater (x, y), greater (x, 3), $(\forall y)$ greater (x, y) has one free occurrence of variable x. Hence, each is an **open** formula.
Each of the formulas: $(\forall x)\ (\exists y)$ greater (x, y), $(\forall y)$ greater (y, 1), greater (9, 2), does not have free occurrence of any variable. Therefore each of these formulas is a closed formula.

Next we discuss some equivalences, and inequalities

The following equivalences hold for any two formulas P(x) and Q(x):

(i) $(\forall x)\ P(x) \wedge (\forall x)\ Q(x) = (\forall x)\ (P(x) \wedge Q(x))$

(ii) $(\exists x)\ P(x) \vee (\exists x)\ Q(x) = (\exists x)\ (P(x) \vee Q(x)$

But the following inequalities hold, in general:

(iii) $(\forall x)\ (P(x) \vee Q(x) \neq (\forall x)\ P(x) \vee (\forall x)\ Q(x)$
(iv) $(\exists x)\ (P(x) \wedge Q(x) \neq (\exists x)\ P(x) \wedge (\exists x)\ Q(x)$

We justify (iii) & (iv) below:
Let P(x): x is odd natural number,

Q(x): x is even natural number.

Then L.H.S of (iii) above states *for every natural number it is either odd or even,* **which is correct**. But R.H.S of (iii) states that *every natural number is odd* or *every natural number is even*, **which is not correct.**

Next, L.H.S. of (iv) states that: there is a natural number which is both even and odd, **which is not correct.** However, R.H.S. of (iv) says *there is an integer which is odd and there is an integer which is even,* correct.

Equivalences involving Negation of Quantifiers

(v) $\sim (\forall x)\ P(x) = (\exists x) \sim P(x)$
(iv) $\sim (\exists x)\ P(x) = (\forall x) \sim P(x)$

Examples: For each of the following closed formula, Prove

(i) $(\forall x)\ P(x) \wedge (\exists y) \sim P(y)$ is inconsistent.

(ii) $(\forall x)\ P(x) \rightarrow (\exists y)\ P(y)$ is valid

Solution: (i) Consider

$(\forall x)\ P(x) \wedge (\exists y) \sim P(y)$

$= (\forall x)\ P(x) \wedge \sim (\forall y)\ P(y)$ (*taking negation out*)

But we know for each bound occurrence, a variable is dummy, and can be replaced in the whole scope of the variable uniformly by another free variable. Hence,

$R = (\forall x)\ P(x) \wedge \sim (\forall x)\ P(x)$

Each conjunct of the formula is either

True of False and, hence, can be thought of as a formula of PL, in stead of formula of FOPL, Let us replace $(\forall x)\ (P(x)$ by Q , a formula of PL.

$R = Q \wedge \sim Q =$ False

Hence, the proof.

(ii) Consider

$(\forall x)\ P(x) \rightarrow (\exists y)\ P(y)$

Replacing '$\rightarrow$' we get

$= \sim (\forall x)\ P(x) \vee (\exists y)\ P(y)$

$= (\exists x) \sim P(x) \vee (\exists y)\ P(y)$

$= (\exists x) \sim P(x) \vee (\exists x)\ P(x)$ *(renaming x as y in the second disjunct)*

In other words,

$= (\exists x)\ (\sim P(x) \vee P(x))$ *(using equivalence)*

The last formula states: *there is at least one element say b, for $\sim P(b) \vee P(b)$ holds i.e., for b, either P(b) is False or P(b) is True.*

But, as P is a predicate symbol and b is a constant $\sim P(b) \vee P(b)$ must be True. Hence, the proof.

Check Your Progress - 1

Ex. 1 Let P(x) and Q(x) represent "x is a rational number" and "x is a real number," respectively. Symbolize the following sentences:

(i) Every rational number is a real number.

(ii) Some real numbers are rational numbers.

(iii) Not every real number is a rational number.

Ex. 2 Let C(x) mean "x is a used-car dealer," and H(x) mean "x is honest." Translate each of the following into English:

(i) $(\exists x)C(x)$

(ii) $(\exists x)\ H(x)$

(iii) $(\forall x)C(x) \rightarrow \sim H(x))$

(iv) $(\exists x)(C(x) \wedge H(x))$

(v) $(\exists x)(H(x) \rightarrow C(x))$.

Ex. 3 Prove the following:

(i) $P(a) \rightarrow \sim ((\exists x) P(x))$ is consistent.

(ii) $(\forall x) P(x) \vee ((\exists y) \sim P(y))$ is valid.

5.5 INFERENCING & ENTAILMENT IN FOPL

In the previous unit, we discussed eight inferencing rules of Propositional Logic (PL) and further discussed applications of these rules in exhibiting validity/invalidity of arguments in **PL.** In this section, the earlier eight rules are extended to include four more rules involving quantifiers for inferencing. Each of the new rules, is called a **Quantifier Rule**. The extended set of 12 rules is then used for validating arguments in First Order Predicate Logic (**FOPL)**.

Before introducing and discussing the Quantifier rules, we briefly discuss why, at all, these rules are required. For this purpose, let us recall the argument discussed earlier, which Propositional Logic could not handle:

(i) Every man is mortal.
(ii) Raman is a man.
(iii) Raman is mortal.

The equivalent symbolic form of the argument is given by:

(i') $(\forall x)$ (Man (x) $\rightarrow$ Mortal (x)

(ii') Man (Raman)

(iii') Mortal (Raman)

If, instead of (i') we were given

(iv) Man (Raman) $\rightarrow$ Mortal (Raman) ,

(which is a formula of Propositional Logic also)

then using Modus Ponens on (ii') & (iv) in *Propositional Logic*, we would have obtained (iii') *Mortal (Raman).*

However, from (i') & (ii') we cannot derive in Propositional Logic (iii'). This suggests that there should be mechanisms for dropping and introducing quantifier appropriately, i.e., in such a manner that *validity* of arguments is not violated. Without discussing the validity-preserving characteristics, we introduce the four Quantifier rules.

(i) Universal Instantiation Rule (U.I.):

$$\frac{(\forall x)\, p(x)}{p(a)}$$

Where is an a arbitrary constant.

The rule states if ***(∀x) p(x)*** *is True, then we can assume* ***P(a)*** *as True for any constant* ***a*** *(where a constant* ***a*** *is like Raman). It can be easily seen that the rule associates a formula* ***P(a) of Propositional Logic*** *to a formula* ***(∀x) p(x) of FOPL****. The significance of the rule lies in the fact that once we obtain a formula like* ***P(a)****, then the reasoning process of Propositional Logic may be used. The rule may be used , whenever, its application seems to be appropriate.*

(ii) Universal Generalisation Rule (U.G.)

$$\frac{P(a), \textit{ for all } a}{(\forall x)\, p(x)}$$

The rule says that if it is known that for all constants a, the statement P(a) is True, then we can, instead, use the formula $(\forall x)\, p(x)$.

The rule associates with a set of formulas ***P(a) for all*** **a of Propositional Logic**, a formula $(\forall x)\, p(x)$ *of FOPL.*

Before using the rule, we must ensure that P(a) is True for all a,
Otherwise it may lead to wrong conclusions.

(iii) Existential Instantiation Rule (E. I.)

$$\frac{(\exists x)\, P(x)}{P(a)} \qquad (E.I.)$$

The rule says if the Truth of $(\exists x)\ P(x)$ *is known then we can assume the Truth of P(a) for* ***some fixed*** *a .* *The rule, again, associates a formula P(a) of Propositional Logic to a formula* $(\forall x)\, p(x)$ *of FOPL.*

An inappropriate application of this rule may lead to *wrong* conclusions. The source of possible errors lies in the fact that the choice 'a' in the rule *is not arbitrary* and can not be known at the time of deducing P(a) from $(\exists x)\ P(x)$.

If during the process of deduction some other $(\exists y)\ Q(y)$ or $(\exists x)\,(\,R(x)\,)$ or even another *(∃x)P(x)* is encountered, then each time a new constant say *b, c* etc. should be chosen to infer Q (b) from $(\exists y)\ Q(y)$ *or R(c)* from $(\exists x)\,(\,R(x)\,)$ *or P(d)* from $(\exists x)\ P(x)$.

(iv) Existential Generalization Rule (E.G)

$$\frac{P(a)}{(\exists x) P(x)} \qquad \text{(E.G)}$$

The rule states that if P(a), a formula of Propositional Logic is True, then the Truth of $(\exists x)\ P(x)$, a formula of FOPL , may be assumed to be True.

The Universal Generalisation (U.G) and Existential Instantiation rules should be applied with utmost care, however, other two rules may be applied, whenever, it appears to be appropriate.

Next, The purpose of the two rules, viz.,

(i) Universal Instantiation Rule (U. I.)

(iii) Existantial Rule (E. I.)

is to associate formulas of Propositional Logic (PL) to formulas of FOPL in a manner, the validity of arguments due to these associations, is not disturbed. Once, we get formulas of PL, then any of the eight rules of inference of PL may be used to validate conclusions and solve problems requiring logical reasoning for their solutions.

The purpose of the other Quantification rules viz. for generalisation, i.e.,

(ii) $\dfrac{P(a),\ \text{for all a}}{(\forall x)\ P(x)}$

(iv) $\dfrac{P(a)}{(\exists x)\, P(x)}$

is that the conclusion to be drawn in FOPL is not generally a formula of PL but a formula of FOPL. However, while making inference, we may be first associating formulas of PL with formulas of FOPL and then use inference rules of PL to conclude formulas in PL. But the conclusion to be made in the problem may correspond to a formula of FOPL. These two generalisation rules help us in associating formulas of FOPL with formulas of PL.

Example: Tell, supported with reasons, which one of the following is a correct inference and which one is not a correct inference.

(i) *To conclude* $F(a) \wedge G(a) \rightarrow H(a) \wedge I(a)$
from $(\forall x)\big(F(x) \wedge G(x)\big) \rightarrow H(x) \wedge I(x)$

using Universal Instantiation (U.I.)

The above inference or conclusion is *incorrect* in view of the fact that the scope of universal quantification is only the formula: $F(x) \wedge G(x)$ and not the whole of the formula.

The occurrences of *x* in $H(x) \wedge I(x)$ are free occurrences. Thus, one of the correct inferences would have been:

$$F(a) \wedge G(a) \rightarrow H(x) \wedge I(x)$$

(ii) *To conclude* $F(a) \wedge G(a) \rightarrow H(a) \wedge I(a)$ *from*
$(\forall x)\ (F(x) \wedge G(x) \rightarrow H(x) \vee I(x))$ *using U.I.*

The conclusion is correct in view of the argument given in (i) above.

(iii) *To conclude* $\sim F(a)$ *for an arbitrary a, from* $\sim (\forall x)\, F(x)$ *using U.I.*

The conclusion is incorrect, because actually

$$\sim (\forall x)\ F(x) = (\exists x) \sim F(x)$$

Thus, the inference is not a case of U.I., but of Existential Instantiation (E.I.)

Further, as per restrictions, we can not say for which a, $\sim F(x)$ is True. Of course, $\sim F(x)$ is true for some constant, but not necessarily for a pre-assigned constant *a*.

(iv) *to conclude* $\left((F(b) \wedge G(b) \rightarrow H(c)\right)$

from $(\exists x)\left((F(b) \wedge G(x)\right) \rightarrow H(c))$

Using E.I. is *not* correct

The reason being that the constant to be substituted for *x* cannot be assumed to be the same constant *b,* being given in advance, as an argument of *F.* However,

to conclude $\left((F(b) \wedge G(a) \rightarrow H(c)\right)$

from $(\exists x)\left((F(b) \wedge G(x)\right) \rightarrow H(c))$ is correct.

Step for using Predicate Calculus as a Language for Representing Knowledge & for Reasoning:

Step 1: Conceptualisation: First of all, all the relevant entities and the relations that exist between these entities are explicitly enumerated. Some of the implicit facts like, '*a person dead once is dead for ever*' have to be explicated.

Step 2: *Nomenclature* & Translation: Giving appropriate names to objects and relations. And then translating the given sentences given in English to formulas in FOPL. Appropriate names are essential in order to guide a reasoning system based on FOPL. It is well-established that no reasoning system is complete. In other words, a reasoning system may need help in arriving at desired conclusion.

Step 3: Finding appropriate sequence of reasoning steps, involving selection of appropriate rule and appropriate FOPL formulas to which the selected rule is to be applied, to reach the conclusion.

Applications of the 12 inferrencing rules (8 of Propositional Logic and 4 involving Quantifiers.)

Example: Symbolize the following and then construct a proof for the argument:

(i) Anyone who repairs his own car is highly skilled and saves a lot of money on repairs
(ii) Some people who repair their own cars have menial jobs. Therefore,
(iii) Some people with menial jobs are highly skilled.

Solution: Let us use the notation:

P(x) : x is a person

S(x)	:	x saves money on repairs
M(x)	:	x has a menial job
R(x)	:	x repairs his own car
H(x)	:	x is highly skilled.

Therefore, (i), (ii) and (iii) can be symbolized as:

(i) $(\forall x)\ (R(x) \rightarrow (H(x) \wedge S(x)))$
(ii) $\exists(x)\ (R(x) \wedge M(x))$
(iii) $(\exists x)\ (M(x) \wedge H(x))$ *(to be concluded)*

From (ii) using Existential Instantiation (E.I), we get, for some fixed a

(iv) $R(a) \wedge M(a)$

Then by simplification rule of Propositional Logic, we get

(v) $R(a)$

From (i), using Universal Instantiation (U.I.), we get

(vi) $R(a) \rightarrow H(a) \wedge S(a)$

Using modus ponens w.r.t. (v) and (vi) we get

(vii) $H(a) \wedge S(a)$

By specialisation of (vii) we get

(viii) $H(a)$

By specialisation of (iv) we get

(ix) $M(a)$

By conjunctions of (viii) & (ix) we get

$M(a) \wedge H(a)$

By Existential Generalisation, we get

$(\exists x)\ (M(x) \wedge H(x))$

Hence, (iii) is concluded.

Example:

(i) Some juveniles who commit minor offences are thrown into prison, and any juvenile thrown into prison is exposed to all sorts of hardened criminals.
(ii) A juvenile who is exposed to all sorts of hardened criminals will become bitter and learn more techniques for committing crimes.
(iii) Any individual who learns more techniques for committing crimes is a menace to society, if he is bitter.
(iv) Therefore, some juveniles who commit minor offences will be menaces to the society.

Example: Let us symbolize the statement in the given argument as follows:

(i)	J(x)	:	x is juvenile.
(ii)	C(x)	:	x commits minor offences.
(iii)	P(x)	:	x is thrown into prison.
(iv)	E(x)	:	x is exposed to hardened criminals.
(v)	B(x)	:	x becomes bitter.
(vi)	T(x)	:	x learns more techniques for committing crimes.
(vii)	M(x)	:	x is a menace to society.

The statements of the argument may be translated as:

(i) $(\exists x)\ (J(x) \wedge C(x) \wedge P(x)) \wedge ((\forall y)\ (J(y) \rightarrow E(y))$
(ii) $(\forall x)\ (J(x) \wedge E(x) \rightarrow B(x) \wedge T(x))$
(iii) $(\forall x)\ (T(x) \wedge B(x) \rightarrow M(x))$
Therefore,

(iv) $(\exists x)\ (J(x) \wedge C(x) \wedge M(x))$

By simplification (i) becomes

(v) $(\exists x)\ (J(x) \wedge C(x) \wedge P(x))$ and
(vi) $(\forall y)\ (J(y) \rightarrow E(y))$

From (v) through Existential Instantiation, for some fixed *b*, we get

(vii) $J(b) \wedge C(b) \wedge P(b)$
Through simplification (vii) becomes

(viii) $J(b)$
(ix) $C(b)$ and
(x) $P(b)$
Using Universal Instantiation, on (vi), we get

(xi) $J(b) \rightarrow E\ (b)$
Using Modus Ponens in (vii) and (xi) we get

(xii) $E(b)$
Using conjunction for (viii) & (xii) we get

(xiii) $J(b) \wedge E(b)$
Using Universal Instantiation on (ii) we get

(xiv) $J(b) \wedge E(b) \rightarrow B(b) \wedge T(b)$
Using Modus Ponens for (xiii) & (xiv), we get

(xv) $T(b) \wedge B(b)$

Using Universal Instantiation for (iii) we get

(xvi) $T(b) \wedge B(b) \rightarrow M(b)$

Using Modus Ponens with (xv) and (xvi) we get

(xvii) $M(b)$

Using conjunction for (viii), (ix) and (xvii) we get

(xviii) $J(b) \wedge C(b) \wedge M(b)$

From (xviii), through Existential Generalization we get the required (iv), i.e.

$(\exists x)\ (J(x) \wedge C(x) \wedge M(x))$

Remark: It may be noted the occurrence of quantifiers is not, in general, commutative i.e.,

$(Q_1x)\ (Q_2x) \neq (Q_2x)\ (Q_1x)$

For example

$(\forall x)\ (\exists y)\ F(x,y) \neq (\exists y)\ (\forall x)\ F(x,y)$ (A)

The occurrence of $(\exists y)$ on L.H.S depends on *x* i.e., occurrence of *y* on L.H.S is a function of *x*. However, the occurrence of $(\exists y)$ on R.H.S is independent of *x*, hence, occurrence of *y* on R.H.S is not a function of *x*.

For example, if we take F(x,y) to mean:

y and x are integers such that y>x,

then*, L.H.S of (A)* above states: *For each x there is a y such that y>x.*

The statement is true in the domain of real numbers.

On the other hand, *R.H.S of (A)* above states that: There is an integer *y* which is greater than *x*, for all *x*.

This statement is not true in the domain of real numbers.

When the logical statements are interconnected in a manner that one is consequence of other then such Logical consequences (also called entailment) are the fundamental concept in logical reasoning, which describes the relationship between statements that hold true when one statement logically follows from one or more statements.

A valid logical argument is one in which the conclusion is entailed by the premises, because the conclusion is the consequence of the premises. The philosophical analysis of logical consequence involves the questions: In what sense does a conclusion follow from its premises? and What does it mean for a conclusion to be a consequence of premises? All of philosophical logic is meant to provide accounts of the nature of logical consequence and the nature of logical truth.

Logical consequence is necessary and formal, by way of examples that explain with formal proof and models of interpretation. A sentence is said to be a logical consequence of a set of sentences, for a given language, if and only if, using only logic (i.e., without regard to any personal interpretations of the sentences) the sentence must be true if every sentence in the set is true.

5.6 CONVERSION TO CLAUSAL FORM

In order to facilitate problem solving through Propositional Logic, we discussed two normal forms, viz, the conjunctive normal form **CNF** and the disjunctive normal form **DNF**. In **FOPL**, there is a normal form called the **prenex normal form**. Further the statement in Prenex Normal Form is required to be skolomized to get the clausal form, which can be used for the purpose of Resolution.

So, first step towards the Clausal form is to begin with Prenex Normal Form (PNF), and the second step is skolomization, which will be discussed after PNF.

Prenex Normal Form (PNF): In broad sense it relates to re-alignment of the quantifiers, i.e. to bring all the quantifiers in the beginning of the expression and then replacement the existential and universal quantifiers with constants and the functions is performed for skolomization i.e. to bring the statement in the clausal form.

The use of a prenex normal form of a formula simplifies the proof procedures, to be discussed.

Definition A formula G in FOPL is said to be in a **prenex normal form** if and only if the formula G is in the form

$$(Q_1x_1)\ldots.(Q_n\ x_n)\ P$$

where each (Q_ix_i), for i = 1, ….,n, is either $(\forall x_i)$ or $(\exists x_i)$, and P is a quantifier free formula. The expression $(Q_1x_1)\ldots.(Q_n\ x_n)$ is called the **prefix** and P is called the **matrix of the formula G**.

Examples of some formulas in prenex normal form:

(i) $(\exists x)\ (\forall y)\ (R(x, y) \vee Q(y))$, $(\forall x)\ (\forall y)\ (\sim P(x, y) \rightarrow S(y))$,

(ii) $(\forall x)\ (\forall y)\ (\exists z)\ (P(x, y) \rightarrow R\ (z))$.

Next, we consider a method of transforming a given formula into a prenex normal form. For this, first we discuss equivalence of formulas in FOPL. Let us recall that two formulas E and G are **equivalent**, denoted by **E = G**, *if and only if the truth values of F and G are identical under every interpretation.* The pairs of equivalent formulas given in Table of equivalent Formulas of previous unit are still valid as these are quantifier–free formulas of FOPL. However, there are pairs of equivalent formulas of FOPL that contain quantifiers. Next, we discuss these additional pairs of equivalent formulas. We introduce some notation specific to FOPL: *the symbol G denote a formula that does not contain any free variable x*. Then we have the following pairs of equivalent formulas, where Q denotes a quantifier which is either ∀ or ∃. Next, we introduce four laws for **pairs of equivalent formulas**.

In the rest of the discussion of FOPL, P[x] is used to denote the fact that *x* is a free variable in the formula P, for example, P[x] = (∀y) P (x, y). Similarly, R [x, y] denotes that variables *x* and *y* occur as free variables **in the formula** R Some of these equivalences, we have discussed earlier.

Then, the following laws involving quantifiers hold good in FOPL

(i) $(\ Qx\ )\ P\ [\ x]\ \vee G = (\ Qx\ )\ (\ P\ [x\ ] \vee G)$.

(ii) $(\ Qx\ )\ P\ [x\ ]\ \wedge G = (\ Qx\ )\ (\ P\ [x] \wedge G)$.

In the above two formulas, Q may be either ∀ or ∃.

(iii) $\sim((\forall x)\, P[x]) = (\exists x)(\sim P[x])$.

(iv) $\sim((\exists x)\, P[x]) = (\forall x)(\sim P[x])$.

(v) $(\forall x)\, P[x] \wedge (\forall x)\, H[x] = (\forall x)(P[x] \wedge H[x])$.

(vi) $(\exists x)\, P[x] \vee (\exists x)\, H[x] = (\exists x)(P[x] \vee H[x])$.

That is, the universal quantifier $\forall$ and the existential quantifier $\exists$ can be distributed respectively over $\wedge$ and $\vee$.

But we must be careful about *(we have already mentioned these inequalities)*
(vii) $(\forall x)\, E[x] \vee (\forall x)\, H[x] \neq (\forall x)(P[x] \vee H[x])$ and
(viii) $(\exists x)\, P[x] \wedge (\exists x)\, H[x] \neq (\exists x)(P[x] \wedge H[x])$

Steps for Transforming an FOPL Formula into Prenex Normal Form

Step 1 Remove the connectives '$\leftrightarrow$' and '$\rightarrow$' using the equivalences

$$P \leftrightarrow G = (P \rightarrow G) \wedge (G \rightarrow P)$$

$$P \rightarrow G = \sim P \vee G$$

Step 2 Use the equivalence to remove even number of ~'s

$$\sim(\sim P) = P$$

Step 3 Apply De Morgan's laws in order to bring the negation signs immediately before atoms.

$$\sim(P \vee G) = \sim P \wedge \sim G \qquad \sim(P \wedge G) = \sim P \vee \sim G$$

and the quantification laws

$$\sim((\forall x)\, P[x]) = (\exists x)(\sim P[x]) \qquad \sim((\exists x)\, P[x]) = (\forall x)(\sim F[x])$$

Step 4 rename bound variables **if necessary**

Step 5 Bring quantifiers to the left before any predicate symbol appears in the formula. This is achieved by using (i) to (vi) discussed above.

We have already discussed that, if all occurrences of a bound variable are replaced uniformly throughout by another variable not occurring in the formula, then the equivalence is preserved. Also, we mentioned under (vii) that $\forall$ does not distribute over $\wedge$ and under (viii) that $\exists$ does not distribute over $\vee$. In such cases, in order to bring quantifiers to the left of the rest of the formula, we may have to first rename one of bound variables, say x, may be renamed as z, which does not occur either as free or bound in the other component formulas. And then we may use the following equivalences.

$$(Q1\ x)\, P[x] \vee (Q2\ x)\, H[x] = (Q1\ x)(Q2\ z)(P[x] \vee H[z])$$

$$(Q3\ x)\, P[x] \wedge (Q4\ x)\, H[x] = (Q3\ x)(Q4\ z)(P[x] \wedge H[z])$$

Example: Transform the following formulas into prenex normal forms:

(i) $(\forall x)\ (Q(x) \to (\exists x)\ R\ (x, y))$

(ii) $(\exists x)\ (\sim (\exists y)\ Q(x, y) \to ((\exists z)\ R(z) \to S\ (x)))$

(iii) $(\forall x)\ (\forall y)\ ((\exists z)\ Q(z, y, z) \wedge ((\exists u)\ R\ (x, u) \to (\exists v)\ R\ (y, v)))$.

Part (i)

Step 1: By removing '→', we get

$$(\forall x)\ (\sim Q\ (x) \vee (\exists x)\ R\ (x, y))$$

Step 2: By renaming x as z in (∃x) R (x, y) the formula becomes

$$(\forall x)\ (\sim Q\ (x) \vee (\exists z)\ R\ (z, y))$$

Step 3: As ~ Q(x) does not involve z, we get

$$(\forall x)\ (\exists z)\ (\sim Q\ (x) \vee R\ (z, y))$$

Part (ii)

$(\exists x)\ (\sim (\exists y)\ Q\ (x, y) \to ((\exists z)\ R\ (z) \to S\ (x)))$

Step 1: Removing outer '→' we get

$$(\exists x)\ (\sim (\sim ((\exists y)\ Q\ (x, y))) \vee ((\exists z)\ R\ (z) \to S\ (x)))$$

Step 2: Removing inner '→' , and simplifying ~ (~ ( )) we get

$$(\exists x)\ ((\exists y)\ Q\ (x, y) \vee (\sim ((\exists z)\ R(z)) \vee S\ (x)))$$

Step 3: Taking '~' inner most, we get

$$(\exists x)\ (\exists y)\ Q\ (x, y) \vee ((\forall z) \sim R(z) \vee S(x)))$$

As first component formula Q (x, y) does not involve z and S(x) does not involve both y and z and ~ R(z) does not involve y. Therefore, we may take out (∃ y) and (∀z) so that, we get

$(\exists x)\ (\exists y)\ (\forall z)\ (Q\ (x, y) \vee (\sim R(z) \vee S\ (x)\ )$, which is the required formula in prenex normal form.

Part (iii)

$(\forall x)\ (\forall y)\ ((\exists z)\ Q\ (x, y, z) \wedge ((\exists u)\ R\ (x, u) \to (\exists v)\ R\ (y\ v)))$

Step 1: Removing '→', we get

$$(\forall x)\ (\forall y)\ ((\exists z)\ Q\ (x, y, z) \wedge (\sim ((\exists u)\ R\ (x, u)) \vee (\exists v)\ R\ (y, v)))$$

Step 2: Taking '~' inner most, we get

$$(\forall x)\ (\forall y)\ ((\exists z)\ Q\ (x, y, z) \wedge ((\forall u) \sim R\ (x, u) \vee (\exists v)\ R\ (y, v)))$$

Step 3: As variables z, u & v do not occur in the rest of the formula except the formula which is in its scope, therefore, we can take all quantifiers outside, preserving the order of their occurrences, Thus we get

$(\forall x)\ (\forall y)\ (\exists z)\ (\forall u)\ (\exists v)\ (Q\ (x, y, z) \wedge (\sim R\ (x, u) \vee R\ (y, v)))$

Skolomization : A further refinement of Prenex Normal Form (PNF) called (Skolem) Standard Form, is the basis of problem solving through Resolution Method. The Resolution Method will be discussed next.

The **Standard Form of a formula of FOPL is** obtained through the following three steps:

(1) The given formula should be converted to Prenex Normal Form (PNF), and then

(2) Convert the Matrix of the PNF, i.e, quantifier-free part of the PNF into conjunctive normal form
(3) Skolomization: Eliminate the existential quantifiers using skolem constants and functions

Before illustrating the process of conversion of a formula of FOPL to Standard Normal Form, through examples, we discuss briefly skolem functions.

Skolem Function
We in general, mentioned earlier that $(\exists x)\ (\forall y)\ P(x,y) \neq\ (\forall y)\ (\exists x)\ P(x,y)$…….(1)

For example, if P(x,y) stands for the relation 'x>y' in the set of integers, then the L.H.S. of the inequality (i) above states: *some (fixed) integer (x) is greater than all integers (y)*. This statement is False.

On the other hand, R.H.S. of the inequality (1) states: *for each integer y, there is an integer x so that x>y*. This statement is True.

The difference in meaning of the two sides of the inequality arises because of the fact that on L.H.S. *x* in $(\exists x)$ is independent of *y* in $(\forall y)$ **whereas** on R.H.S *x* of dependent on *y*. In other words, *x* on L.H.S. of the inequality can be replaced by some constant say 'c' whereas on the right hand side *x* is some function, say, f(y) of *y*.

Therefore, the two parts of the inequality (i) above may be written as

LH.S. of (1) $= (\exists x)\ (\forall y)\ P\ (x,y) = (\forall y)\ P(c,y)$,

Dropping x because there is no x appearing in $(\forall y)\ P(c,y)$

R.H.S. of (1) $= (\forall y)\ (\exists x)\ P(f(y),y) = (\forall y)\ P(f(y), y)$

The above argument, in essence, explains what is meant by each of the terms viz. *skolem constant, skolem function and skolomisation.*

The constants and functions which replace existential quantifiers are respectively called **skolem constants and skolem functions.** The process of replacing all existential variables by skolem constants and variables is called **skolemisation**.

A form of a formula which is obtained after applying the steps for

(i) reduction to PNF and then to

(ii) CNF and then

(iii) applying skolomization is called **Skolem Standard Form** or just **Standard Form.**

We explain through examples, the skolomisation process after PNF and CNF have already been obtained.

Example: Skolomize the following:

(i) $(\exists x_1)\ (\exists x_2)\ (\forall y_1)\ (\forall y_2)(\exists x_3)(\forall y_3)\ P(x_1, x_2, x_3, y_1, y_2, y_3)$

(ii) $(\exists x_1)(\forall y_1)(\exists x_2)(\forall y_2)\ (\exists x_3)P(x_1, x_2, x_3, y_1, y_2) \wedge (\exists x_1)(\forall y_3)(\exists x_2)\ (\forall y_4)Q(x_1, x_2,$

$y_3, y_4)$

Solution (i) As existential quantifiers x_1 and x_2 precede all universal quantifiers, therefore, x_1 and x_2 are to be replaced by *constants*, but by distinct constants, say by 'c' and 'd' respectively. As existential variable x_3 is preceded by universal quantifiers y_1 and y_2, therefore, x_3 is replaced by some function $f(y_1, y_2)$ of the variables y_1 and y_2. After making these substitutions and dropping universal and existential variables, we get the skolemized form of the given formula as

$(\forall y_1)\ (\forall y_2)\ (\forall y_3)\ (c, d, f\ (y_1, y_2), y_1, y_2, y_3)$.

Solution (ii) As a first step we must bring all the quantifications in the beginning of the formula through Prenex Normal Form reduction. Also,

$$(\exists x)\ldots P(x,\ldots) \wedge (\exists x)\ldots\ Q\ (x,\ldots) \neq (\exists x)\ (\ldots P(x) \wedge \ldots Q(x,\ldots),$$

therefore, we rename the second occurrences of quantifiers $(\forall x_1)$ and $(\forall x_2)$ by renaming these as x_5 and x_6. Hence, after renaming and pulling out all the quantifications to the left, we get

$(\exists x_1)\ (\forall y_1)\ (\exists x_2)\ (\forall y_2)\ (\exists x_3)\ (\exists x_5)\ (\forall y_3)\ (\exists x_6)\ (\forall y_4)$

$$(P(x_1, x_2, x_3, y_1, y_2) \wedge Q\ (x_5, x_6, y_3, y_4)$$

Then the existential variable x_1 is independent of all the universal quantifiers. Hence, x_1 may be replaced by a constant say, 'c'. Next x_2 is preceded by the universal quantifier y_1 hence, x_2 may be replaced by $f\ (y_1)$. The existential quantifier x_3 is preceded by the universal quantifiers y_1 and y_2. Hence x_3 may be replaced by g

(y_1, y_2). The existential quantifier x_5 is preceded by again universal quantifier y_1 and y_2. In other words, x_5 is also a function of y_1 and y_2. But, we have to use a different function symbol say h and replace x_5 by $h\ (y_1, y_2)$. Similarly x_6 may be replaced by

$j\ (y_1, y_2, y_3)$.

Thus, (Skolem) Standard Form becomes
$(\forall y_1)\ (\forall y_2)\ (\forall y_3)\ (P\ (c, f(y_1), g(y_1, y_2), y_1, y_2) \wedge Q\ (h\ (y_1, y_2), j\ (y_1, y_2, y_3)))$.

Check Your Progress -2

Ex: 4 (i) Transform the formula $(\forall x)\ P(x) \rightarrow (\exists x)\ Q(x)$ into prenex normal form.

(ii) Obtain a prenex normal form for the formula

$(\forall x)\ (\forall y)\ ((\exists z)\ (P(x, y) \wedge P(y, z)) \rightarrow (\exists u)\ Q\ (x, y, u))$

Ex 5. Obtain a (skolem) standard form for each of the following formula:

(i) $(\exists x)\ (\forall y)\ (\forall v)\ (\exists z)\ (\forall w)\ (\exists u)\ P\ (x, y, z, u, v, w)$
(ii) $(\forall x)\ (\exists y)\ (\exists z)\ ((P\ (x, y) \vee \sim Q\ (x, z)) \rightarrow R\ (x, y, z))$

5.7 RESOLUTION & UNIFICATION

In the beginning of the previous section, we mentioned that resolution method for FOPL requires discussion of a number of complex new concepts. Also, , we discussed (Skolem) Standard Form and also discussed how to obtain Standard Form for a given formula of FOPL. In this section, along with Resolution we will introduce two new, and again complex, concepts, viz., *substitution and unification.*

The complexity of the resolution method for FOPL mainly results from the fact that a clause in FOPL is generally of the form : $P(x) \vee Q(f(x), x, y) \vee \ldots$, in which the variables x, y, z, may assume any one of the values of their domain.

Thus, the atomic formula $(\forall x)\ P(x)$, *which after dropping of universal quantifier, is written as just P(x)* stands for $P(a_1) \wedge P(a_2) \ldots \wedge P(a_n)$ where the set $\{a_1\ a_2 \ldots, a_n\}$ is assumed here to be domain (x).

Similarly, $(\exists x)\ P(x)$ *stands for* $(P(a_1) \vee P(a_2) \vee \ldots \vee P(a_n))$

However, in order to resolve two clauses – one containing say P(x) and the other containing ~ P(y) where *x* and *y* are universal quantifiers, possibly having some restrictions, we have to know which values of x and y satisfy both the clauses. For this purpose we need the concepts of **substitution** and **unification** as defined and discussed in the rest of the section.

Instead of giving formal definitions of substitution, unification, unifier, most general unifier and resolvent, resolution of clauses in FOPL, we illustrate the concepts through examples and minimal definitions, if required

Example: Let us consider our old problem:

To conclude

(i) Raman is mortal

From the following two statements:

(ii) Every man is mortal and

(iii) Raman is a man

Using the notations

MAN (x) : x is a man

MORTAL (x) : x is mortal,

the problem can be formulated in symbolic logic as: Conclude

MORTAL (Raman)

from

(ii) $((\forall x)\ (MAN(x) \rightarrow MORTAL\ (x))$

(iii) MAN (Raman).

As resolution is a refutation method, assume

(i) ~ MORTAL (Raman)

After Skelomization and dropping ($\forall$x), (ii) in standard form becomes

(i) ~ MAN (x) $\vee$ MORTAL (x)
(ii) MAN (Raman)

In the above *x* varies over the set of human beings including Raman. Hence, one special instance of (iv) becomes

(vi) ~ MAN (Raman) $\vee$ MORTAL (Raman)

At the stage, we may observe that

(a) *MAN(Raman)* and *MORTAL(Raman)* do not contain any variables, and, hence, their truth or falsity can be determined directly. Hence, each of like a formula of PL. In term of formula which does not contain any variable is called **ground term** or **ground formula**.

(b) Treating *MAN (Raman)* as formula of PL and using resolution method on (v) and (vi), we conclude

(vii) MORTAL (Raman),

Resolving (i) and (vii), we get **False**. Hence, the solution.

Unification: In the process of solution of the problem discussed above, we tried to make the two expression *MAN(x)* and *MAN(Raman)* identical. Attempt to make identical two or more expressions is called *unification.*

In order to unify *MAN (x)* and *MAN (Raman)* identical, we found that because one of the possible values of *x* is *Raman* also. And, hence, we replaced *x* by one of its possible values : *Raman*.

This replacement of a variable like *x*, by a term (*which may be another variable also*)

which is one of the possible values of *x*, is called **substitution**. The substitution, in this case is denoted formally as {Raman/x}

Substitution, in general, **notationally** is of the form $\{t_1 / x_1, t_2 / x_2 \ldots t_m / x_m\}$ where $x_1, x_2 \ldots, x_m$ are variables and $t_2, t_2 \ldots t_m$ are terms and t_i replaces the variable x_i in some expression.

Example: (i) Assume Lord Krishna is loved by everyone who loves someone (ii) Also assume that no one loves nobody. Deduce Lord Krishna is loved by everyone.

Solution: Let us use the symbols

Love (x, y): *x loves y (or y is loved by x)*

LK : *Lord Krishna*

Then the given problem is formalized as :

(i) ($\forall$x) (($\exists$y) Love (x, y)$\rightarrow$Love (x, LK))

(ii) $\sim (\exists x) ((\forall y) \sim \text{Love}(x, y))$

To show : $(\forall x) (\text{Love}(x, LK))$

As resolution is a refutation method, assume negation of the last statement as an axiom.

(iii) $\sim (\forall x) \text{ Love}(x, LK)$

The formula in (i) above is reduced in standard form as follows:

$(\forall x) (\sim (\exists y) \text{ Love}(x, y) \vee \text{Love}(x, LK))$

$= (\forall x) ((\forall y) \sim \text{Love}(x, y) \vee \text{Love}(x, LK))$

$= (\forall x) (\forall y) (\sim \text{Love}(x, y) \vee \text{Love L}(x, LK))$

($\because$ ($\forall$y) does not occurs in Love (x, LK))

After dropping universal quantifications, we get

(iv) $\sim \text{Love}(x, y) \vee \text{Love}(x, LK)$

Formula (ii) can be reduced to standard form as follows:

(ii) $= (\forall x) (\exists y) \text{ Love}(x, y)$

y is replaced through skolomization by f(x)

so that we get

$(\forall x) \text{ Love}(x, f(x))$

Dropping the universal quantification

(v) Love (x, f(x))

The formula in (iii) can be brought in standard form as follows:

(iii) $= (\exists x) (\sim \text{Love}(x, LK))$

As existential quantifier x is not preceded by any universal quantification, therefore, x may be substituted by a constant a , i.e., we use the substitution $\{a/x\}$ in (iii) to get the standard form:

(vi) $\sim \text{Love}(a, LK)$.

Thus, to solve the problem, we have the following standard form formulas for resolution:

(iv) $\sim \text{Love}(x, y) \vee \text{Love}(x, LK)$

(v) Love (x, f(x))

(vi) $\sim \text{Love}(a, LK)$.

Two possibilities of resolution exist for two pairs of formulas viz.

one possibility: resolving (v) and (vi).

second possibility : resolving (iv) and (vi).

The possibilities exist because for each possibility pair, the predicate *Love* occurs in complemented form in the respective pair.

Next we attempt to resolve (v) and (vi)

For this purpose we attempt to make the two formulas *Love(x, f(x))* and *Love (a, LK)* identical, through unification involving substitutions. We start from the left, matching the two formulas, term by term. First place where matching may fail is when 'x' occurs in one formula and 'a' occurs in the other formula. **As, one of these happens to be a variable**, hence, the substitution {a/x} can be used to unify the portions so far.

Next, possible disagreement through term-by-term matching is obtained when we get the two disagreeing terms from two formulas as f(x) and LK. **As none of f(x) and LK is a variable** *(note f(x) involves a variable but is itself not a variable),* hence, no unification and, hence, no resolution of (v) and (vi) is possible.

Next, we attempt unification of **(vi) Love (a, LK)** *with* **Love (x, LK)** of (iv).

Then first term-by-term possible disagreement occurs when the corresponding terms are 'a' and 'x' respectively. As one of these is a variable, hence, the substitution{a/x} unifies the parts of the formulas so far. Next, the two occurrences of LK, one each in the two formulas, match. Hence, the whole of each of the two formulas can be unified through the substitution {a/x}. Though the unification has been *attempted* in corresponding smaller parts, substitution has to be carried **in the whole of the formula**, in this case in whole of (iv). Thus, after substitution, (iv) becomes
(viii) $\sim$ Love (a, y) $\vee$ Love (a, L K)

resolving (viii) with (vi) we get

(ix) $\sim$ Love (a, y)

In order to resolve (v) and (ix), we attempt to unify **Love (*x*, f(x))** of (v) with

Love (a, y) of (ix).

The term-by-term matching leads to possible disagreement of *a* of (ix) with *x* of (v).

As, one of these is a variable, hence, the substitution {a/x} will unify the portions considered so far.

Next, possible disagreement may occur with f (x) of (v) and y of (ix). As one of these are a variable viz. y, therefore, we can unify the two terms through the substitution {f(x)/y}. Thus, the complete substitution *{a/x, f (x)/y}* is required to match the formulas. Making the substitutions, we get (v) becomes Love (a, f(x)) and (ix) becomes $\sim$ Love (a, f (x))

Resolving these formulas we get **False.** Hence, the proof.

Check you Progress - 3

Ex. 6: Unify, if possible, the following three formulas:

(i) Q (u, f (y, z)),
(ii) Q (u, a)
(iii) Q (u, g (h (k (u))))

Ex. 7: Determine whether the following formulas are unifiable or not:

(i) Q (f (a), g(x))
(ii) Q (x, y)

Example: Find resolvents, if possible for the following pairs of clauses:

(i) ~ Q (x, z, x) ∨ Q (w, z, w) and
(ii) Q (w, h (v, v), w)

Solution: As two literals with predicate Q occur and are mutually negated in (i) and (ii),therefore, there is possibility of resolution of ~ *Q (x, z, x)* from (i) with *Q (w, h (v, v), w)* of (ii). We attempt to unify Q (x, z, x) and Q (w, h (v, v), w), if possible, by finding an appropriate substitution. First terms x and w of the two are variables, hence, unifiable with either of the substitutions {x/w} or {w/x}. Let us take {w/x}.

Next pair of terms from the two formulas, viz, z and h(v, v) are also unifiable, because, one of the terms is a variable, and the required substitution for unification is { h (v, v)/z}.

Next pair of terms at corresponding positions is again {w, x} for which, we have determined the substitution {w/x}. Thus, the substitution {w/x, h(v, v)/z} unfies the two formulas. Using the substitutions, (i) and (ii) become resp. as

(iii) ~ Q (w, h (v, v), w) ∨ Q (w, h (v, v), w)
(iv) Q (w, h (v, v), w)

Resolving, we get

Q (w, h (v, v), w),

which is the required resolvent.

5.8 SUMMARY

In this unit, initially, we discuss how PL is inadequate to solve even simple problems, requires some extension of PL or some other formal inferencing system so as to compensate for the inadequacy. First Order Predicate Logic (FOPL), is such an extension of PL that is discussed in the unit.
Next, syntax of proper structure of a formula of FOPL is discussed. In this respect, a number of new concepts including those of quantifier, variable, constant, term, free and bound occurrences of variables; closed and open wff, consistency/validity of wffs etc. are introduced.

Next, two normal forms viz. Prenex Normal Form (PNF) and Skolem Standard Normal Form are introduced. Finally, tools and techniques developed in the unit, are used to solve problems involving logical reasoning.

5.9 SOLUTIONS/ANSWERS

<u>Check Your Progress - 1</u>

Ex. 1 *(i)* $(\forall x)\ (P(x) \rightarrow Q(x))$

(ii) $(\exists x)\ (P(x) \wedge Q(x))$

(iii) $\sim (\forall x)\ (Q(x) \rightarrow P(x))$

Ex. 2

(i) There is (at least) one (person) who is a used-car dealer.

(ii) There is (at least) one (person) who is honest.

(iii) All used-car dealers are dishonest.

(iv) (At least) one used-car dealer is honest.

(v) There is at least one thing in the universe, (for which it can be said that) if that something is Honest then that something is a used-car dealer

Note: *the above translation is not the same as: Some no gap one honest, is a used-car dealer.*

Ex 3: *(i) After removal of '$\rightarrow$' we get the given formula*

$$= \sim P(a) \vee \sim ((\exists x)\ P(x))$$

$$= \sim P(a) \vee (\forall x)\ (\sim P(x))$$

Now P(a) is an atom in PL which may assume any value T or F. On taking P(a) as F the given formula becomes T, hence, consistent.

(ii) The formula can be written

$(\forall x)\ P(x) \vee \sim (\forall x)\ (P(x))$*, by taking negation outside the second disjunct and then renaming.*

The $(\forall x)\ P(x)$ *being closed is either T or F and hence can be treated as formula of PL.*
Let $\forall x\ P(x)$ *be denoted by Q. Then the given formula may be denoted by* $Q \vee \sim Q$ *= True (always)*
Therefore, formula is valid.

Check Your Progress - 2

Ex: 4 (i) $(\forall x)\ P(x) \rightarrow (\exists x)\ Q(x) = \sim ((\forall x)\ P(x)) \vee (\exists x)\ Q(x)$ *(by removing the connective $\rightarrow$)*

$= (\exists x)\ (\sim P(x)) \vee (\exists x)\ Q(x)$ *(by taking '~' inside)*
$= (\exists x)\ (\sim P(x) \vee Q(x))$ *(By taking distributivity of $\exists x$ over $\vee$)*

Therefore, a prenex normal form of $(\forall x)\ P(x) \rightarrow (\exists x)\ Q(x)$ *is* $(\exists x)\ (\sim P(x) \vee Q(x))$.

(ii) $(\forall x)\ (\forall y)\ ((\exists z)\ (P(x, y) \wedge P(y, z)) \rightarrow (\exists u)\ Q(x, y, u))$ *(removing the connective $\rightarrow$)*

$= (\forall x)\ (\forall y)\ (\sim ((\exists z)\ (P(x, z) \wedge P(y, z)))$
$\vee (\exists u)\ Q(x, y, u))$ *(using De Morgan's Laws)*

$= (\forall x)\ (\forall y)\ ((\forall z)\ (\sim P(x, z) \vee \sim P(y, z))$
$\vee (\exists u)\ Q(x, y, u))$

$= (\forall x)\ (\forall y)\ (\forall z)\ (\sim P(x, z)$

$\vee \sim P(y, z) \vee Q(x, y, u)$ *(as z and u do not occur in the rest of the formula except their respective scopes)*

Therefore, we obtain the last formula as a prenex normal form of the first formula.

Ex 5 (i) *In the given formula $(\exists x)$ is not preceded by any universal quantification. Therefore, we replace the variable x by a (skolem) constant c in the formula and drop $(\exists x)$.*
Next, the existential quantifier $(\exists z)$ is preceded by two universal quantifiers viz., v and y. we replace the variable z in the formula, by some function, say, f (v, y) and drop $(\exists z)$. Finally, existential variable $(\exists u)$ is preceded by three universal quantifiers, viz., $(\forall y)$, $(\forall y)$ and $(\forall w)$. Thus, we replace in the formula the variable u by, some function g(y, v, w) and drop the quantifier $(\exists u)$. Finally, we obtain the standard form for the given formula as
$(\forall y)\ (\forall v)\ (\forall w)\ P(x, y, z, u, v, w)$

(ii) *First of all, we reduce the matrix to CNF.*

$= (P(x, y) \vee \sim Q(x, z)) \rightarrow R(x, y, z)$
$= (\sim P(x, y) \wedge Q(x, z)) \vee R(x, y, z)$
$= (\sim P(x, y) \vee R(x, y, z)) \wedge (Q(x, z) \vee R(x, y, z))$

Next, in the formula, there are two existential quantifiers, viz., $(\exists y)$ and $(\exists z)$. Each of these is preceded by the only universal quantifier, viz. $(\forall x)$.

Thus, each variable y and z is replaced by a function of x. But the two functions of x for y and z must be different functions. Let us assume, variable, y is replaced in the formula by f(x) and the variable z is replaced by g(x). Thus the initially given formula, after dropping of existential quantifiers is in the standard form:
$(\forall x)\ ((\sim P(x, y) \vee R(x, y, z)) \wedge (Q(x, z) \vee R(x, y, z)))$

Check Your Progress - 3

Ex 6 : Refer to section 5.7

Ex 7 : Refer to section 5.7

5.10 FURTHER READINGS

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