Matrix Theory Assignment 17

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All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_17

1 Problem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. such that

$$\int_{\mathbb{R}^n} |f(\mathbf{x})d\mathbf{x}| < \infty \tag{1.0.1}$$

Let A be a real $n \times n$ invertible matrix and for $x, y \in \mathbb{R}^n$. Let $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes the standard inner product in $\mathbb{R}^n$ then, $\int_{\mathbb{R}^n} f(\mathbf{A}\mathbf{x})e^{i\langle \mathbf{y}, \mathbf{x} \rangle} d\mathbf{x}$ =?

1) $\int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{x} \rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|}$
2) $\int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle \mathbf{A}^T \mathbf{y}, \mathbf{x} \rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|}$.
3) $\int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle (\mathbf{A}^T)^{-1} \mathbf{y}, \mathbf{x} \rangle} d\mathbf{x}$.
4) $\int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle \mathbf{A}^{-1} \mathbf{y}, \mathbf{x} \rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|}$.

2 solution

Let consider,

$$\mathbf{A}\mathbf{x} = \mathbf{t} \tag{2.0.1}$$

$$\implies \mathbf{x} = \mathbf{A}^{-1}\mathbf{t} \tag{2.0.2}$$

$$\implies d\mathbf{x} = \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \tag{2.0.3}$$

Using (2.0.1) to (2.0.3), We can write:

$$\int_{\mathbb{R}^n} f(\mathbf{A}\mathbf{x})e^{i\langle \mathbf{y}, \mathbf{x} \rangle} d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{t})e^{i\langle \mathbf{y}, (\mathbf{A}^{-1}\mathbf{t}) \rangle} \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \tag{2.0.4}$$

We know that,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} \tag{2.0.5}$$

$$\implies \langle \mathbf{y}, (\mathbf{A}^{-1}\mathbf{t}) \rangle = (\mathbf{y}^T \mathbf{A}^{-1}\mathbf{t}) \tag{2.0.6}$$

And

$$\implies \langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle = ((\mathbf{A}^{-1})^T \mathbf{y})^T \mathbf{t} \tag{2.0.7}$$

$$\implies ((\mathbf{A}^{-1})^T \mathbf{y})^T \mathbf{t} = (\mathbf{y}^T ((\mathbf{A}^{-1})^T)^T \mathbf{t}) = (\mathbf{y}^T \mathbf{A}^{-1}\mathbf{t}) \tag{2.0.8}$$

Hence, from (2.0.6) and (2.0.8)

$$\langle \mathbf{y}, (\mathbf{A}^{-1}\mathbf{t}) \rangle = \langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle \tag{2.0.9}$$

Using (2.0.9) in (2.0.4) We can write,

$$\implies \int_{\mathbb{R}^n} f(\mathbf{t})e^{i\langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle} \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \tag{2.0.10}$$

replacing variable $\mathbf{t}$ with $\mathbf{x}$.

$$\implies \int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{x} \rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|} \tag{2.0.11}$$

Hence option 1 is correct.

3 Example

Let consider a matrix $\mathbf{A}$ and $\mathbf{x}$ as :

$$\mathbf{A} = \begin{pmatrix} x_1 & 2x_2 \\ 3x_1 & 4x_2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{A}\mathbf{x} = \mathbf{t} \tag{3.0.2}$$

$$\implies \mathbf{x} = \mathbf{A}^{-1}\mathbf{t} \tag{3.0.3}$$

$$\mathbf{A^{-1}} = \begin{pmatrix} -\frac{2}{x_1} & \frac{1}{x_1} \\ \frac{3}{2x_2} & -\frac{1}{2x_2} \end{pmatrix} \tag{3.0.4}$$

$$\mathbf{t} = \begin{pmatrix} x_1^2 + 2x_2^2 \\ 3x_1^2 + 4x_2^2 \end{pmatrix} \tag{3.0.5}$$

Let consider another matrix $\mathbf{y}$ as :

$$\mathbf{y} = \begin{pmatrix} 5x_1 \\ 7x_2 \end{pmatrix} \tag{3.0.6}$$

Now,

$$\langle \mathbf{y}, (\mathbf{A^{-1}t}) \rangle = \mathbf{y}^T (\mathbf{A}^{-1}\mathbf{t}) = \begin{pmatrix} 5x_1 & 6x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5x_1^2 + 6{x_2}^2 \tag{3.0.7}$$

And,

$$\langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle = ((\mathbf{A}^{-1})^T \mathbf{y})^T \mathbf{t} = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} x_1^2 + 2{x_2}^2 \\ 3x_1^2 + 4{x_2}^2 \end{pmatrix}$$
$$= 5x_1^2 + 6{x_2}^2 \quad (3.0.8)$$

Hence from (3.0.7) and (3.0.8) we can conclude,

$$\langle \mathbf{y}, (\mathbf{A^{-1}t}) \rangle = \langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle \tag{3.0.9}$$

We have also,

$$d\mathbf{x} = \frac{d\mathbf{t}}{|-2x_1 x_2|} \tag{3.0.10}$$

Hence our equation (2.0.11) becomes,

$$\int_{\mathbb{R}^2} f(\mathbf{x}) e^{i(5x_1^2 + 6{x_2}^2)} \frac{d\mathbf{x}}{|-2x_1 x_2|} \tag{3.0.11}$$

$$\implies \int f(x_1, x_2) e^{i(5x_1^2 + 6{x_2}^2)} \frac{1}{|-2x_1 x_2|} d \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.0.12}$$

$$\implies \int f(x_1, x_2) e^{i(5x_1^2 + 6{x_2}^2)} \frac{1}{|-2x_1 x_2|} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \tag{3.0.13}$$