Matrix Theory Assignment 14

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***Abstract*—This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_14

1 Problem

Let $\mathbf{V}$ be a real vector space and E an idempotent linear operator on $\mathbf{V}$, i.e., a projection. Prove that $\left(\mathbf{I}+\mathbf{E}\right)$ is invertible. Find $\left(\mathbf{I}+\mathbf{E}\right)^{-1}$.

2 solution

we have $\mathbf{E}$ and it is idempotent. And we know that the eigen value of idempotent matrix is either 0 or 1. When we add the identity matrix in this:

$$\mathbf{I}+\mathbf{E} \tag{2.0.1}$$

Then eigen value will be either 1 or 2. Hence $\left(\mathbf{I}+\mathbf{E}\right)$ is invertible. Since $\mathbf{E}$ is an idempotent matrix, that is :

$$\mathbf{E}^2 = \mathbf{E} \tag{2.0.2}$$

Let,

$$A = I + E \tag{2.0.3}$$
$$\implies E = A - I \tag{2.0.4}$$

$$\implies E^2 = (A-I)(A-I) = A^2 - 2A + I^2 \tag{2.0.5}$$

(from 2.0.2 $E^2 = E$),

$$\implies E = A^2 - 2A + I \tag{2.0.6}$$

Using (2.0.4) we have,

$$\implies A - I = A^2 - 2A + I = A^2 - 3A + 2I = 0 \tag{2.0.7}$$

$$\implies I = \frac{3A - A^2}{2} \tag{2.0.8}$$

multiplying A^{-1} both side,

$$A^{-1} = \frac{3I - A}{2} = \frac{3I - (I+E)}{2} \tag{2.0.9}$$
$$A^{-1} = \frac{2I - E}{2} \tag{2.0.10}$$

Using (2.0.4), we have,

$$(I+E)^{-1} = I - \frac{E}{2} \tag{2.0.11}$$

Hence,

$$\left(\mathbf{I}+\mathbf{E}\right)^{-1} = \mathbf{I} - \frac{1}{2}\mathbf{E} \tag{2.0.12}$$

3 example

Let consider a matrix $\mathbf{E}$ as :

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.1}$$
$$\implies \mathbf{E}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.2}$$
$$\implies \mathbf{E}^2 = E. \tag{3.0.3}$$
$$\tag{3.0.4}$$

Now,

$$\mathbf{I}+\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.5}$$
$$\implies \mathbf{I}+\mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.6}$$
$$\tag{3.0.7}$$

Now let find the eigen value of matrix $\left(\mathbf{I}+\mathbf{E}\right)$:

$$\implies \begin{pmatrix} 2-ł & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0 \tag{3.0.8}$$
$$\implies (2-\lambda)(1-\lambda) = 0 \tag{3.0.9}$$
$$\implies \lambda_1 = 2, \lambda_2 = 1 \tag{3.0.10}$$

The eigen values of the matrix ($\mathbf{I}$ + $\mathbf{E}$) from (3.0.10) are 2 and 1. Since none of the eigen value is zero, hence matrix is invertible.
Inverse of the matrix from (2.0.12) is :

$$\left(\mathbf{I}+\mathbf{E}\right)^{-1}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}-\frac{1}{2} & 0\\ 0 & 1\end{pmatrix} \tag{3.0.11}$$