

Matrix Theory Assignment 14

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Abstract—This problem is all about to introducing the concept of characteristic polynomial over a field.

All the codes for this document can be found at

[https://github.com/Ritesh622/
Assignment_EE5609/tree/master/
Assignment_14](https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_14)

1 PROBLEM

Let \mathbf{V} be a real vector space and \mathbf{E} an idempotent linear operator on \mathbf{V} , i.e., a projection. Prove that $(\mathbf{I} + \mathbf{E})$ is invertible. Find $(\mathbf{I} + \mathbf{E})^{-1}$.

2 SOLUTION

Since \mathbf{E} is an idempotent matrix, that is :

$$\mathbf{E}^2 = \mathbf{E} \quad (2.0.1)$$

Hence it will satisfy the polynomial,

$$x^2 - x = 0 \quad (2.0.2)$$

Thus minimal polynomial will be,

$$m_{\mathbf{E}}(x) = x^2 - x = 0 \quad (2.0.3)$$

$$\Rightarrow m_{\mathbf{E}}(x) = x(x - 1) = 0 \quad (2.0.4)$$

Hence minimal polynomial $m_{\mathbf{E}}(x)$, of \mathbf{E} divides $x^2 - x$ that is,

$$m_{\mathbf{E}}(x) = x \quad (2.0.5)$$

or,

$$m_{\mathbf{E}}(x) = x - 1 \quad (2.0.6)$$

if

$$m_{\mathbf{E}}(x) = x \quad (2.0.7)$$

$$\Rightarrow \mathbf{E} = \mathbf{0} \quad (2.0.8)$$

if

$$m_{\mathbf{E}}(x) = x - 1 \quad (2.0.9)$$

$$\Rightarrow \mathbf{E} = \mathbf{I} \quad (2.0.10)$$

Hence, if \mathbf{E} is idempotent then, minimal polynomial of \mathbf{E} is product of distinct polynomial of degree one. Thus matrix \mathbf{E} is similar to diagonal matrix with diagonal entries consisting of characteristic value 0 and 1.

Since \mathbf{E} is diagonalizable, then there exist at least one basis such that :

$$\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_n\} \quad (2.0.11)$$

Such that

$$\mathbf{E}\beta_i = \beta_i, \forall i = 1, 2, \dots, k \quad (2.0.12)$$

and,

$$\mathbf{E}\beta_i = 0, \forall i = k + 1, \dots, n. \quad (2.0.13)$$

$$\Rightarrow (\mathbf{I} + \mathbf{E})\beta_i = 2\beta_i, \forall i = 1, 2, \dots, k. \quad (2.0.14)$$

And,

$$\Rightarrow (\mathbf{I} + \mathbf{E})\beta_i = \beta_i, \forall i = k + 1, \dots, n. \quad (2.0.15)$$

In matrix form we can write it as:

$$[\mathbf{I} + \mathbf{E}]_{\mathcal{B}} = \begin{pmatrix} 2\mathbf{I}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix} \quad (2.0.16)$$

Where, \mathbf{I}_1 is $k \times k$ and \mathbf{I}_2 is $(n - k) \times (n - k)$ identity matrices, and each $\mathbf{0}$ represents the zero matrix of appropriate dimension.

From 2.0.16 we can calculate the determinant as:

$$\det(\mathbf{I} + \mathbf{E}) = 2^k \neq 0 \quad (2.0.17)$$

Hence Matrix $(\mathbf{I} + \mathbf{E})$ is invertible.

Since $[\mathbf{I} + \mathbf{E}]_{\mathcal{B}}$ is combination of identity matrices. Hence, from (2.0.16), inverse of the matrix $[\mathbf{I} + \mathbf{E}]_{\mathcal{B}}$ is given as :

$$\begin{aligned} ([\mathbf{I} + \mathbf{E}]_{\mathcal{B}})^{-1} &= \begin{pmatrix} \frac{1}{2}\mathbf{I}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{I}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\mathbf{I}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix} = \mathbf{I} - \frac{1}{2}[\mathbf{E}]_{\mathcal{B}} \end{aligned} \quad (2.0.18)$$

Hence,

$$([\mathbf{I} + \mathbf{E}]_{\mathcal{B}})^{-1} = \mathbf{I} - \frac{1}{2} [\mathbf{E}]_{\mathcal{B}} \quad (2.0.19)$$

we can also verify our results as:

$$(\mathbf{I} + \mathbf{E})(\mathbf{I} - \frac{1}{2}\mathbf{E}) = \mathbf{I}^2 - \frac{1}{2}\mathbf{E} + \mathbf{E} - \frac{1}{2}\mathbf{E}^2 \quad (2.0.20)$$

From (2.0.1), We have $\mathbf{E}^2 = \mathbf{E}$, Hence (2.0.20) becomes,

$$\mathbf{I} - \frac{1}{2}\mathbf{E} + \mathbf{E} + \frac{1}{2}\mathbf{E} = \mathbf{I} \quad (2.0.21)$$

Similarly,

$$\Rightarrow (\mathbf{I} + \mathbf{E})\beta_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.11)$$

$$\Rightarrow (\mathbf{I} + \mathbf{E})\beta_1 = 2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \beta_2 \quad (3.0.12)$$

$$\Rightarrow [\mathbf{I} + \mathbf{E}]_{\mathcal{B}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\mathbf{I}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_1 \end{pmatrix} \quad (3.0.13)$$

Here we have $k = 1$ and $n = 2$ and, $(n - k) = 1$. Hence size of \mathbf{I}_1 and \mathbf{I}_2 are 1×1 . Similarly, size of zero matrix is also 1×1 . Now determinant of $\mathbf{I} + \mathbf{E}$ is :

$$\det(\mathbf{I} + \mathbf{E}) = 2^k = 2^1 = 2 \neq 0. \quad (3.0.14)$$

Inverse of the matrix is :

$$([\mathbf{I} + \mathbf{E}]_{\mathcal{B}})^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.15)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.16)$$

Let consider a matrix \mathbf{E} as :

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.1)$$

$$\Rightarrow \mathbf{E}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.2)$$

$$\Rightarrow \mathbf{E}^2 = \mathbf{E}. \quad (3.0.3)$$

Basis for this matix is

$$\beta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \beta_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.4)$$

We have,

$$\mathbf{E}\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta_1 \quad (3.0.5)$$

And,

$$\mathbf{E}\beta_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \beta_2 \quad (3.0.6)$$

Now,

$$\mathbf{I} + \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.7)$$

$$\Rightarrow \mathbf{I} + \mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.8)$$

$$\Rightarrow (\mathbf{I} + \mathbf{E})\beta_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.0.9)$$

$$\Rightarrow (\mathbf{I} + \mathbf{E})\beta_1 = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\beta_1 \quad (3.0.10)$$

Hence, equation (3.0.16) can be written as:

$$([\mathbf{I} + \mathbf{E}]_{\mathcal{B}})^{-1} = \begin{pmatrix} \mathbf{I}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\mathbf{I}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 \end{pmatrix} = \mathbf{I} - \frac{1}{2} [\mathbf{E}]_{\mathcal{B}} \quad (3.0.17)$$