Matrix Theory Assignment 5

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***Abstract*—This document demonstrates a method to trace a curve with given equation using matrix algebra.**

Download latex and python codes from

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_5

1 Problem Statement

Trace the curve

$$(x-y)^2 = x+y+1 \tag{1.0.1}$$

2 Solution

We have given equation as :

$$(x-y)^2 = x+y+1 \tag{2.0.1}$$

$$\implies x^2 - 2xy + y^2 - x - y - 1 = 0 \tag{2.0.2}$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.3}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.4}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.5}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.6}$$

Comparing (2.0.2) with (2.0.3), we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{2.0.7}$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{2.0.8}$$

$$f = -1 \tag{2.0.9}$$

Expanding the determinant of V we observe,

$$|\mathbf{V}| = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0 \tag{2.0.10}$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 1 & -1 & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{vmatrix} \neq 0 \tag{2.0.11}$$

Hence from (2.0.10) and (2.0.11) we conclude that given equation is an parabola.The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = 0 \tag{2.0.12}$$

$$\implies (\lambda-1)^2 - 1 = 0 \tag{2.0.13}$$

The eigenvalues are the roots of (2.0.13) given by

$$\lambda_1 = 0, \lambda_2 = 2 \tag{2.0.14}$$

The eigenvector $\mathbf{p}$ is defined as:

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.15}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{2.0.16}$$

where λ is the eigenvalue. For $\lambda_1 = 0$,

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.17}$$

Row reducing $\mathbf{V}$ yields,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.18}$$

Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$(\lambda_2\mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.19}$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \tag{2.0.20}$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \tag{2.0.21}$$

From equation (2.0.18) and (2.0.19), we have

$$\mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{and, } \mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{2.0.22}$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{p_1} & \mathbf{p_2}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{2.0.23}$$

$$\mathbf{D} = \begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix} \tag{2.0.24}$$

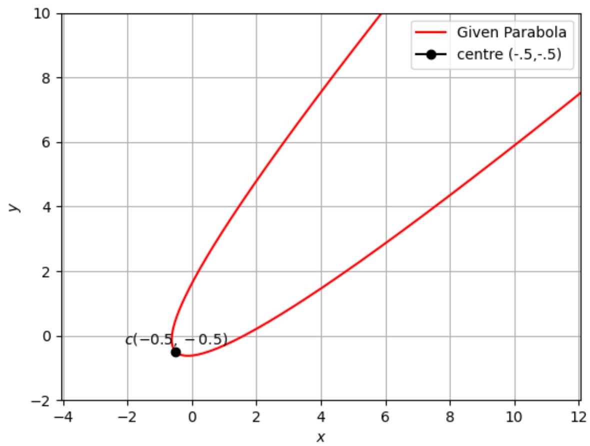

Fig. 1: Parabola with the center c

The focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{\sqrt{2}}{2} = \sqrt{2} \tag{2.0.25}$$

and its equation is

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -2\eta\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{y} \tag{2.0.26}$$

where,

$$\eta = \mathbf{u}^T\mathbf{p_1} = -\frac{1}{\sqrt{2}} \tag{2.0.27}$$

$$\begin{pmatrix}\mathbf{u}^T + \eta\mathbf{p_1}^T\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f\\ \eta\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.0.28}$$

$$\implies \begin{pmatrix}-1 & -1\\ 1 & -1\\ -1 & 1\end{pmatrix}\mathbf{c} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix} \tag{2.0.29}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix}-1 & -1 & 1\\ 1 & -1 & 1\\ -1 & 1 & 0\end{pmatrix} \xleftrightarrow{R_2\leftarrow R_2+R_1} \begin{pmatrix}-1 & -1 & 1\\ 0 & -2 & 1\\ -1 & 1 & 0\end{pmatrix} \xleftrightarrow[R_1\leftarrow -1R_1]{R_3\leftarrow R_3-R_1} \begin{pmatrix}1 & 1 & -1\\ 0 & -2 & 1\\ 0 & 2 & -1\end{pmatrix} \xleftrightarrow{R_3\leftarrow R_3+R_2} \begin{pmatrix}1 & 1 & -1\\ 0 & -2 & 1\\ 0 & 0 & 0\end{pmatrix} \xleftrightarrow[R_1\leftarrow R_1-R_2]{R_1\leftarrow \frac{R_1}{-2}} \begin{pmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.30}$$

So,

$$\mathbf{c} = \begin{pmatrix}-\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix} \tag{2.0.31}$$

2.1 QR decomposition of $\mathbf{V}$

We have ,

$$\mathbf{V} = \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \tag{2.1.1}$$

Let $\mathbf{x}$ and $\mathbf{y}$ be the column vectors of the given matrix.

$$\mathbf{x} = \begin{pmatrix}1\\ -1\end{pmatrix} \tag{2.1.2}$$

$$\mathbf{y} = \begin{pmatrix}-1\\ 1\end{pmatrix} \tag{2.1.3}$$

The column vectors can be expressed as follows,

$$\mathbf{x} = k_1\mathbf{u}_1 \tag{2.1.4}$$

$$\mathbf{y} = r_1\mathbf{u}_1 + k_2\mathbf{u}_2 \tag{2.1.5}$$

$$k_1 = \|\mathbf{x}\| \tag{2.1.6}$$

$$\mathbf{u}_1 = \frac{\mathbf{x}}{k_1} \tag{2.1.7}$$

$$r_1 = \frac{\mathbf{u}_1^T\mathbf{y}}{\|\mathbf{u}_1\|^2} \tag{2.1.8}$$

$$\mathbf{u}_2 = \frac{\mathbf{y} - r_1\mathbf{u}_1}{\|\mathbf{y} - r_1\mathbf{u}_1\|} \tag{2.1.9}$$

$$k_2 = \mathbf{u}_2^T\mathbf{y} \tag{2.1.10}$$

The (2.1.4) and (2.1.5) can be written as,

$$\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{pmatrix}\begin{pmatrix}k_1 & r_1\\ 0 & k_2\end{pmatrix} \tag{2.1.11}$$

$$\begin{pmatrix}\mathbf{x} & \mathbf{y}\end{pmatrix} = \mathbf{QR} \tag{2.1.12}$$

Now using equations (2.1.6) to (2.1.10) we get,

$$k_1 = \sqrt{1^2 + (-1)^2} = \sqrt{2} \tag{2.1.13}$$

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{2.1.14}$$

$$r_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\sqrt{2} \tag{2.1.15}$$

$$\tag{2.1.16}$$

Similarly using equation (2.1.9) and (2.1.10) we can have

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.1.17}$$

$$k_2 = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \tag{2.1.18}$$

Thus putting the values from (2.1.13) to (2.1.18) in (2.1.11) we obtain QR decomposition,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 0 & 1 \end{pmatrix} \tag{2.1.19}$$