Matrix Theory Assignment 14

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***Abstract*—This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_14

1 Problem

Let $\mathbf{V}$ be a real vector space and E an idempotent linear operator on $\mathbf{V}$, i.e., a projection. Prove that $\left(\mathbf{I}+\mathbf{E}\right)$ is invertible. Find $\left(\mathbf{I}+\mathbf{E}\right)^{-1}$.

2 solution

we have $\mathbf{E}$ and it is idempotent. And we know that the eigen value of idempotent matrix is either 0 or 1. When we add the identity matrix in this:

$$\mathbf{I}+\mathbf{E} \tag{2.0.1}$$

Then eigen value will be either 1 or 2. Hence $\left(\mathbf{I}+\mathbf{E}\right)$ is invertible. Since $\mathbf{E}$ is an idempotent matrix, that is :

$$\mathbf{E}^2 = \mathbf{E} \tag{2.0.2}$$

Let,

$$\mathbf{A} = \mathbf{I}+\mathbf{E} \tag{2.0.3}$$

$$\implies \mathbf{E} = \mathbf{A}-\mathbf{I} \tag{2.0.4}$$

$$\implies \mathbf{E}^2 = (\mathbf{A}-\mathbf{I})(\mathbf{A}-\mathbf{I}) = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}^2 \tag{2.0.5}$$

From 2.0.2,

$$\implies \mathbf{E} = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} \tag{2.0.6}$$

Using (2.0.4) we have,

$$\implies \mathbf{A}-\mathbf{I} = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = \mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = 0 \tag{2.0.7}$$

$$\implies \mathbf{I} = \frac{3\mathbf{A}-\mathbf{A}^2}{2} \tag{2.0.8}$$

multiplying $\mathbf{A}^{-1}$ both side,

$$\mathbf{A}^{-1} = \frac{3\mathbf{I}-\mathbf{A}}{2} = \frac{3\mathbf{I}-(\mathbf{I}+\mathbf{E})}{2} \tag{2.0.9}$$

$$\implies \mathbf{A}^{-1} = \frac{2\mathbf{I}-\mathbf{E}}{2} \tag{2.0.10}$$

Using (2.0.3), we have,

$$(\mathbf{I}+\mathbf{E})^{-1} = \mathbf{I} - \frac{1}{2}\mathbf{E} \tag{2.0.11}$$

3 example

Let consider a matrix $\mathbf{E}$ as :

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.1}$$

$$\implies \mathbf{E}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.2}$$

$$\implies \mathbf{E}^2 = E. \tag{3.0.3}$$

Now,

$$\mathbf{I}+\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.4}$$

$$\implies \mathbf{I}+\mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.5}$$

$$\tag{3.0.6}$$

Now let find the eigen value of matrix $\left(\mathbf{I}+\mathbf{E}\right)$:

$$\implies \begin{pmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0 \tag{3.0.7}$$

$$\implies (2-\lambda)(1-\lambda) = 0 \tag{3.0.8}$$

$$\implies \lambda_1 = 2, \lambda_2 = 1 \tag{3.0.9}$$

The eigen values of the matrix ($\mathbf{I}$+$\mathbf{E}$) from (3.0.9) are 2 and 1. Since none of the eigen value is zero, hence matrix is invertible.
Inverse of the matrix from (2.0.11) is :

$$\left(\mathbf{I}+\mathbf{E}\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.10}$$