Matrix Theory Assignment 15

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***Abstract*—This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_15

1 Problem

Let $\mathbf{V}$ be the space of $n \times n$ matrices over a field $\mathbb{F}$, and let A be a fixed $n \times n$ matrix over field $\mathbb{F}$. Define a linear operator $\mathbf{T}$ on $\mathbf{V}$ by the equation

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} - \mathbf{BA} \tag{1.0.1}$$

Prove that if $\mathbf{A}$ is a nilpotent matrix, then $\mathbf{T}$ is a nilpotent operator.

2 solution

Since $\mathbf{A}$ is a nilpotent matrix, hence for a positive value K:

$$\mathbf{A}^K = \mathbf{0} \tag{2.0.1}$$

Now we have

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} - \mathbf{BA} \tag{2.0.2}$$

$$\implies \mathbf{T}^2(\mathbf{B}) = \mathbf{T}(\mathbf{T}(\mathbf{B})) = \mathbf{T}(\mathbf{AB} - \mathbf{BA}) \tag{2.0.3}$$

$$\begin{aligned}\mathbf{T}(\mathbf{AB} - \mathbf{BA}) &= \mathbf{A}^2\mathbf{B} - \mathbf{ABA} - \mathbf{ABA} + \mathbf{BA}^2 \\ &= \mathbf{A}^2 - 2\mathbf{ABA} + \mathbf{BA}^2\end{aligned} \tag{2.0.4}$$

$$\begin{aligned}\implies \mathbf{T}^3(\mathbf{B}) &= \mathbf{T}(\mathbf{T}^2(\mathbf{B})) \\ &= \mathbf{A}^3 - 3\mathbf{A}^2\mathbf{BA} + 3\mathbf{ABA}^2 - \mathbf{BA}^3\end{aligned} \tag{2.0.5}$$

Hence, from (2.0.5) we can say that, as we are increasing the power of operator $\mathbf{T}$, the power of $\mathbf{A}$ is also increasing in every terms. Hence there exist a value P such that :

$$\mathbf{T}^P(\mathbf{B}) = \mathbf{0} \tag{2.0.6}$$

Hence, if $\mathbf{A}$ is a nilpotent matrix then operator $\mathbf{T}$ is also a nilpotent operator.
Let consider $K = 2$ for which $\mathbf{A}^K = \mathbf{0}$ that is :

$$\mathbf{A}^2 = \mathbf{0} \tag{2.0.7}$$

$$\implies \mathbf{A}^3 = \mathbf{0} \tag{2.0.8}$$

Now using (2.0.7) and (2.0.8) in (2.0.5), we get:

$$\mathbf{T}^3(\mathbf{B}) = \mathbf{0} \tag{2.0.9}$$

Hence $P = 3$.

3 example

Let consider a matrices $\mathbf{A}$ and, B as :

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{A}.\mathbf{B} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} \neq \mathbf{0} \tag{3.0.2}$$

Now,

$$\implies \mathbf{A}^2 = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.3}$$

$$\implies \mathbf{A}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.4}$$

Hence $K = 2$.

And we have also,

$$\mathbf{ABA} = \begin{pmatrix} -16 & 16 \\ -16 & 16 \end{pmatrix} \neq \mathbf{0} \tag{3.0.5}$$

Hence from (2.0.3) we conclude, $P \neq 2$. Now putting the value of $\mathbf{A}^3$ from (3.0.4) and value of $\mathbf{A}^2$ from (3.0.3) in (2.0.5) we get,

$$\mathbf{T}^3(\mathbf{B}) = \mathbf{0} \tag{3.0.6}$$

Hence, $P = 3$ for this example.