Matrix Theory : Assignment 3

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***Abstract*—This problem is to demonstrate the way to prove the triangles are congruent and to prove a triangle as isosceles using matrix algebra.**

1 Problem

ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that

1) $\triangle$ABC $\cong$ $\triangle$ACF
2) AB = AC i.e $\triangle$ABC is an isosceles triangle.

2 Solution

2.1 part 1

Let consider we have a triangle $\triangle$ABC. There are two altitudes BE and CF being drawn from the vertices B and C respectively.
In $\triangle$ABE, taking inner product of sides AE and EB we can write :

$$(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B}) = \|\mathbf{A}-\mathbf{E}\|\,\|\mathbf{E}-\mathbf{B}\|\cos\text{AEB} \tag{2.1.1}$$

$$\implies \cos\text{AEB} = \frac{(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B})}{\|\mathbf{A}-\mathbf{E}\|\,\|\mathbf{E}-\mathbf{B}\|} \tag{2.1.2}$$

In $\triangle$ACF, taking inner product of sides AF and FC :

$$(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C}) = \|\mathbf{A}-\mathbf{F}\|\,\|\mathbf{F}-\mathbf{C}\|\cos\text{AFC} \tag{2.1.3}$$

$$\implies \cos\text{AFC} = \frac{(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C})}{\|\mathbf{A}-\mathbf{F}\|\,\|\mathbf{F}-\mathbf{C}\|} \tag{2.1.4}$$

In triangle $\triangle$ABC,

$$\cos\text{AFC} = \cos\text{AEB} \tag{2.1.5}$$

$$\frac{(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B})}{\|\mathbf{A}-\mathbf{E}\|\,\|\mathbf{E}-\mathbf{B}\|} = \frac{(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C})}{\|\mathbf{A}-\mathbf{F}\|\,\|\mathbf{F}-\mathbf{C}\|} \tag{2.1.6}$$

Given,

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{2.1.7}$$

$$\frac{(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B})}{(\mathbf{A}-\mathbf{E})^T(\mathbf{A}-\mathbf{E})} = \frac{(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C})}{(\mathbf{A}-\mathbf{F})^T(\mathbf{A}-\mathbf{F})} \tag{2.1.8}$$

$$\frac{(\mathbf{E}-\mathbf{B})}{(\mathbf{A}-\mathbf{E})} = \frac{(\mathbf{F}-\mathbf{C})}{(\mathbf{A}-\mathbf{F})} \tag{2.1.9}$$

Taking norms both side,

$$\frac{\|\mathbf{E}-\mathbf{B}\|}{\|\mathbf{A}-\mathbf{E}\|} = \frac{\|\mathbf{F}-\mathbf{C}\|}{\|\mathbf{A}-\mathbf{F}\|} \tag{2.1.10}$$

Using 2.1.7 in 2.1.10 we get :

$$\|\mathbf{A}-\mathbf{E}\| = \|\mathbf{A}-\mathbf{F}\| \tag{2.1.11}$$

Now, we have

$$\|\mathbf{A}-\mathbf{E}\| = \|\mathbf{A}-\mathbf{F}\|\,(\text{ we got the result}) \tag{2.1.12}$$

$$\|\mathbf{C}-\mathbf{F}\| = \|\mathbf{B}-\mathbf{E}\|\,(\text{given}) \tag{2.1.13}$$

since BE $\perp$ AC and CF $\perp$ AB , hence we have,

$$\cos BEA = \cos CFA\,(\ 90^\circ) \tag{2.1.14}$$

Hence by SAS (Side - Angle - Side) We can say that , $\triangle$ABC $\cong$ $\triangle$ACF

2.2 part 2

In $\triangle$ABC we can have :

$$\cos\text{ABC} = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.2.1}$$

And,

$$\cos\text{ACB} = \frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{A}-\mathbf{C}\|\,\|\mathbf{C}-\mathbf{B}\|} \tag{2.2.2}$$

Since,

$$\|\mathbf{A}-\mathbf{C}\| = \|\mathbf{A}-\mathbf{B}\|\,(\triangle\text{AEB} \cong \triangle\ \text{AFC}) \tag{2.2.3}$$

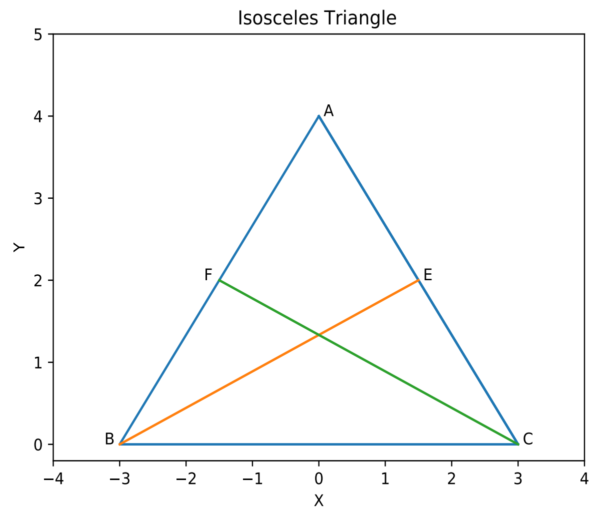

Fig. 2: Isosceles triangle

$$\|\mathbf{B}-\mathbf{C}\| = \|\mathbf{C}-\mathbf{B}\| \tag{2.2.4}$$

Divindg 2.2.1 to 2.2.2, and we get :

$$\frac{\cos ABC}{\cos ACB} = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C})}{(\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{B})} \tag{2.2.5}$$

By multiplying right hand side of equation 2.2.5 by $\frac{(\mathbf{A}-\mathbf{C})(\mathbf{A}-\mathbf{B})}{(\mathbf{A}-\mathbf{C})(\mathbf{A}-\mathbf{B})}$ we get,

$$\frac{\cos ABC}{\cos ACB} = \frac{\|\mathbf{A}-\mathbf{B}\|(\mathbf{B}-\mathbf{C})(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{C}\|(\mathbf{C}-\mathbf{B})(\mathbf{A}-\mathbf{B})} \tag{2.2.6}$$

Substituting equation 2.2.3 in equation 2.2.6 we have :

$$\frac{\cos ABC}{\cos ACB} = \frac{(\mathbf{B}-\mathbf{C})(\mathbf{A}-\mathbf{C})}{(\mathbf{C}-\mathbf{B})(\mathbf{A}-\mathbf{B})} \tag{2.2.7}$$

Taking norms both side

$$\frac{\cos ABC}{\cos ACB} = \frac{\|\mathbf{B}-\mathbf{C}\|\,\|\mathbf{A}-\mathbf{C}\|}{\|\mathbf{C}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{B}\|} \tag{2.2.8}$$

using equation 2.2.3 and 2.2.4

$$\frac{\cos ABC}{\cos ACB} = 1 \tag{2.2.9}$$

$$\implies \cos ABC = \cos ACB \tag{2.2.10}$$

$$\implies \angle B = \angle C \tag{2.2.11}$$

Hence we can say that the $\triangle$ABC is an isosceles triangle since its two sides and respective angles are equal.