Matrix Theory : Assignment 4

Ritesh Kumar
Roll no. : EE20RESCH11005

Abstract—**This problem is to demonstrate a method to find the equations of circles who touches both the axes and passes through a common point using matrix algebra.**

Download latex and python codes from

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_4

1 Problem

Show that two circles can be drawn to pass through the point $\begin{pmatrix}1\\2\end{pmatrix}$ and touch the coordinate axes, and find their equations.

2 Solution

Let us consider we have a circle which passes through the point $\begin{pmatrix}1\\2\end{pmatrix}$ and touches x - axis at point $\begin{pmatrix}r\\0\end{pmatrix}$ and y - axis at $\begin{pmatrix}0\\r\end{pmatrix}$. Radius of the circle is **r** since it touches both axes. Hence we have 3 points which are :

$$\mathbf{P_1} = \begin{pmatrix}1\\2\end{pmatrix} \tag{2.0.1}$$

$$\mathbf{p_2} = \begin{pmatrix}r\\0\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{P_3} = \begin{pmatrix}0\\r\end{pmatrix} \tag{2.0.3}$$

The general equation of circle is :

$$\|\mathbf{x} - \mathbf{O}\| = r \tag{2.0.4}$$

Substituting the given coordinates:

$$\left\|\begin{pmatrix}r\\0\end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{2.0.5}$$

$$\left\|\begin{pmatrix}0\\r\end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{2.0.6}$$

$$\left\|\begin{pmatrix}1\\2\end{pmatrix} - \mathbf{O}\right\|^2 = r^2 \tag{2.0.7}$$

From equation 2.0.5, 2.0.6 and 2.0.7 we have

$$\left\|\begin{pmatrix}r\\0\end{pmatrix} - \mathbf{O}\right\|^2 - \left\|\begin{pmatrix}1\\2\end{pmatrix} - \mathbf{O}\right\|^2 = 0 \tag{2.0.8}$$

$$\left\|\begin{pmatrix}0\\r\end{pmatrix} - \mathbf{O}\right\|^2 - \left\|\begin{pmatrix}1\\2\end{pmatrix} - \mathbf{O}\right\|^2 = 0 \tag{2.0.9}$$

Let consider simplifying 2.0.8 and 2.0.8,

$$\left(\begin{pmatrix}r\\0\end{pmatrix} - \left(O\right)\right)^T \left(\begin{pmatrix}r\\0\end{pmatrix} - \left(O\right)\right) - \left(\begin{pmatrix}1\\2\end{pmatrix} - \left(O\right)\right)^T \left(\begin{pmatrix}1\\2\end{pmatrix} - \left(O\right)\right) = \mathbf{0} \tag{2.0.10}$$

$$r^2 - 2\begin{pmatrix}r\\0\end{pmatrix}^T \left(O\right) - 5 + 2\begin{pmatrix}1\\2\end{pmatrix}^T \left(O\right) = \mathbf{0} \tag{2.0.11}$$

$$\begin{pmatrix}2-2r & 4\end{pmatrix}\left(O\right) = 5 - r^2 \tag{2.0.12}$$

Similarly,

$$\left(\begin{pmatrix}0\\r\end{pmatrix} - \left(O\right)\right)^T \left(\begin{pmatrix}0\\r\end{pmatrix} - \left(O\right)\right) - \left(\begin{pmatrix}1\\2\end{pmatrix} - \left(O\right)\right)^T \left(\begin{pmatrix}1\\2\end{pmatrix} - \left(O\right)\right) = \mathbf{0} \tag{2.0.13}$$

$$r^2 - 2\begin{pmatrix}0\\r\end{pmatrix}^T \left(O\right) - 5 + 2\begin{pmatrix}1\\2\end{pmatrix}^T \left(O\right) = \mathbf{0} \tag{2.0.14}$$

$$\begin{pmatrix}2 & 4-2r\end{pmatrix}\left(O\right) = 5 - r^2 \tag{2.0.15}$$

combining 2.0.15 and 2.0.12

$$\begin{pmatrix}2-2r & 4\\ 2 & 4-2r\end{pmatrix}\left(O\right) = \begin{pmatrix}5-r^2\\5-r^2\end{pmatrix} \tag{2.0.16}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2-2r & 4 & 5-r^2 \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \tag{2.0.17}$$

$$\begin{pmatrix} 2-2r & 4 & 5-r^2 \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \xleftrightarrow{R1\leftarrow \frac{R1}{2-2r}}$$
$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-}{2(r-1)} \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \xleftrightarrow{R2\leftarrow R2-2R1}$$
$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & \frac{2r(r-3)}{r-1} & \frac{r\left(r^2-5\right)}{r-1} \end{pmatrix} \xleftrightarrow{R2\leftarrow\left(\frac{1-r}{2r(r-3)}\right)R2}$$
$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \xleftrightarrow{R1\leftarrow R1-\left(\frac{-2}{r-1}\right)R2}$$
$$\begin{pmatrix} 1 & 0 & \frac{r^2-5}{2(r-3)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \tag{2.0.18}$$

So,

$$\mathbf{O} = \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} \tag{2.0.19}$$

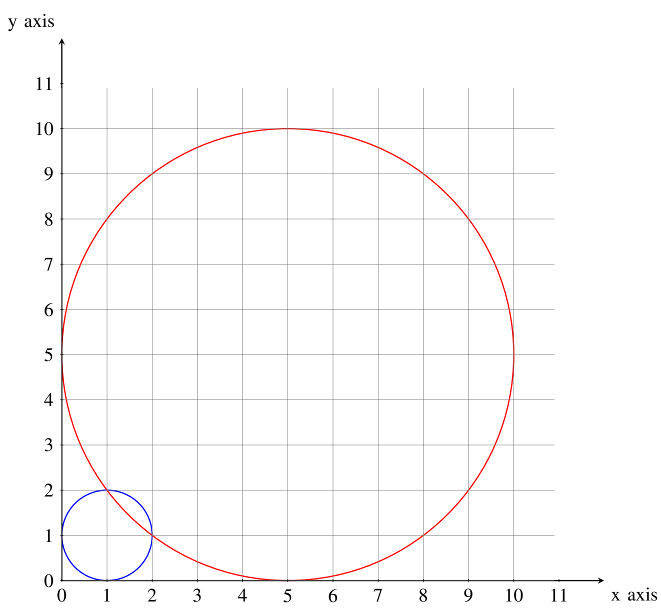

Fig. 0: Two circles passes through the point

Now subsituting the 2.0.3 in 2.0.7, we have

$$\left\| \begin{pmatrix} 0 \\ r \end{pmatrix} - \mathbf{O} \right\|^2 = r^2 \tag{2.0.20}$$

subsituting the value of $\mathbf{O}$ in 2.0.7 and simplify,

$$\left(\begin{pmatrix} 0 \\ r \end{pmatrix} - \left(O\right)\right)^T \left(\begin{pmatrix} 0 \\ r \end{pmatrix} - \left(O\right)\right) = r^2 \tag{2.0.21}$$

$$\implies r^2 - \begin{pmatrix} 0 \\ r \end{pmatrix}^T \left(O\right) - \begin{pmatrix} 0 \\ r \end{pmatrix} \left(O\right)^T + \|O\|^2 = r^2 \tag{2.0.22}$$

Putting the value of $\left(O\right)$ from 2.0.19

$$\implies -\begin{pmatrix} 0 \\ r \end{pmatrix}^T \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} - \begin{pmatrix} 0 \\ r \end{pmatrix} \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix}^T = -\|O\|^2 \tag{2.0.23}$$

$$\implies 2r\left(\frac{r^2-5}{2\left(r-3\right)}\right) = 2\left(\frac{r^2-5}{2\left(r-3\right)}\right)^2 \tag{2.0.24}$$

$$r = \frac{r^2-5}{2\left(r-3\right)} \tag{2.0.25}$$

$$\implies r^2 - 6r + 5 = 0 \tag{2.0.26}$$

$$\implies \left(r-1\right)\left(r-5\right) = 0 \tag{2.0.27}$$

$$\implies r = 1, r = 5. \tag{2.0.28}$$

Hence,

$$\mathbf{O_1} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \text{and,} \mathbf{O_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.29}$$

Hence equation of circles are :

$$\left\| x - \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right\| = 5 \tag{2.0.30}$$

And,

$$\left\| x - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = 1 \tag{2.0.31}$$