

Matrix Theory : Assignment 4

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Abstract—This problem is to demonstrate a method to find the equations of circles who touches both the axes and passes through a common point using matrix algebra.

Download latex and python codes from

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_4

1 PROBLEM

Show that two circles can be drawn to pass through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and touch the coordinate axes, and find their equations.

2 SOLUTION

Let us consider we have a circle which passes through the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and touches x - axis at point $\begin{pmatrix} r \\ 0 \end{pmatrix}$ and y - axis at $\begin{pmatrix} 0 \\ r \end{pmatrix}$. Radius of the circle is r since it touches both axes. Hence we have 3 points which are :

$$\mathbf{P}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2.0.1)$$

$$\mathbf{P}_2 = \begin{pmatrix} r \\ 0 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{P}_3 = \begin{pmatrix} 0 \\ r \end{pmatrix} \quad (2.0.3)$$

The general equation of circle is :

$$\|\mathbf{x} - \mathbf{O}\| = r \quad (2.0.4)$$

Substituting the given coordinates:

$$\|\mathbf{P}_2 - \mathbf{O}\|^2 = r^2 \quad (2.0.5)$$

$$\|\mathbf{P}_3 - \mathbf{O}\|^2 = r^2 \quad (2.0.6)$$

$$\|\mathbf{P}_1 - \mathbf{O}\|^2 = r^2 \quad (2.0.7)$$

From equation 2.0.5, 2.0.6 and 2.0.7 we have

$$\|\mathbf{P}_2 - \mathbf{O}\|^2 - \|\mathbf{P}_1 - \mathbf{O}\|^2 = 0 \quad (2.0.8)$$

And,

$$\|\mathbf{P}_3 - \mathbf{O}\|^2 - \|\mathbf{P}_1 - \mathbf{O}\|^2 = 0 \quad (2.0.9)$$

Simplifying 2.0.8 and 2.0.9,

$$(\mathbf{P}_2 - \mathbf{O})^T (\mathbf{P}_2 - \mathbf{O}) - (\mathbf{P}_1 - \mathbf{O})^T (\mathbf{P}_1 - \mathbf{O}) = 0 \quad (2.0.10)$$

$$\Rightarrow \|\mathbf{P}_2\|^2 - 2\mathbf{P}_2^T \mathbf{O} - \|\mathbf{P}_1\|^2 + 2\mathbf{P}_1^T \mathbf{O} = 0 \quad (2.0.11)$$

Substituting the value of $\|\mathbf{P}_1\|$ and $\|\mathbf{P}_2\|$ and other values then rearranging it, we get :

$$(2 - 2r \ 4)(O) = 5 - r^2 \quad (2.0.12)$$

Similarly,

$$(\mathbf{P}_3 - \mathbf{O})^T (\mathbf{P}_3 - \mathbf{O}) - (\mathbf{P}_1 - \mathbf{O})^T (\mathbf{P}_1 - \mathbf{O}) = 0 \quad (2.0.13)$$

$$\Rightarrow \|\mathbf{P}_3\|^2 - 2\mathbf{P}_3^T \mathbf{O} - \|\mathbf{P}_1\|^2 + 2\mathbf{P}_1^T \mathbf{O} = 0 \quad (2.0.14)$$

Substituting the value of $\|\mathbf{P}_1\|$ and $\|\mathbf{P}_3\|$ and other values then rearranging it, we get :

$$(2 \ 4 - 2r)(O) = 5 - r^2 \quad (2.0.15)$$

Combining 2.0.15 and 2.0.12

$$\begin{pmatrix} 2 - 2r & 4 \\ 2 & 4 - 2r \end{pmatrix} (O) = \begin{pmatrix} 5 - r^2 \\ 5 - r^2 \end{pmatrix} \quad (2.0.16)$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2-2r & 4 & 5-r^2 \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \quad (2.0.17)$$

$$\begin{aligned} & \begin{pmatrix} 2-2r & 4 & 5-r^2 \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \xrightarrow{R1 \leftarrow \frac{R1}{2-2r}} \begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \xrightarrow{R2 \leftarrow R2-2R1} \\ & \begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & \frac{2r(r-3)}{r-1} & \frac{r(r^2-5)}{r-1} \end{pmatrix} \xrightarrow{R2 \leftarrow \left(\frac{1-r}{2r(r-3)}\right)R2} \\ & \begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \xrightarrow{R1 \leftarrow R1 - \left(\frac{-2}{r-1}\right)R2} \\ & \begin{pmatrix} 1 & 0 & \frac{r^2-5}{2(r-3)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \end{aligned} \quad (2.0.18)$$

So,

$$\mathbf{O} = \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} \quad (2.0.19)$$

Now substituting the 2.0.3 in 2.0.7, we have

$$\|\mathbf{P}_3 - \mathbf{O}\|^2 = r^2 \quad (2.0.20)$$

Substituting the value of \mathbf{O} in 2.0.7 and simplify,

$$(\mathbf{P}_3 - \mathbf{O})^T (\mathbf{P}_3 - \mathbf{O}) = r^2 \quad (2.0.21)$$

$$\Rightarrow \|\mathbf{P}_3\|^2 - \mathbf{P}_3^T \mathbf{O} - \mathbf{P}_3 \mathbf{O}^T + \|\mathbf{O}\|^2 = r^2 \quad (2.0.22)$$

Putting the values of \mathbf{O} from 2.0.19 and $\|\mathbf{P}_3\|^2$

$$\Rightarrow -\mathbf{P}_3^T \mathbf{O} - \mathbf{P}_3 \mathbf{O}^T = -\|\mathbf{O}\|^2 \quad (2.0.23)$$

$$\Rightarrow -\begin{pmatrix} 0 \\ r \end{pmatrix}^T \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} - \begin{pmatrix} 0 \\ r \end{pmatrix} \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix}^T = -\|\mathbf{O}\|^2 \quad (2.0.24)$$

Substituting the value of $\|\mathbf{O}\|^2$ and simplify it,

$$\Rightarrow 2r \left(\frac{r^2-5}{2(r-3)} \right) = 2 \left(\frac{r^2-5}{2(r-3)} \right)^2 \quad (2.0.25)$$

$$\Rightarrow r = \frac{r^2-5}{2(r-3)} \quad (2.0.26)$$

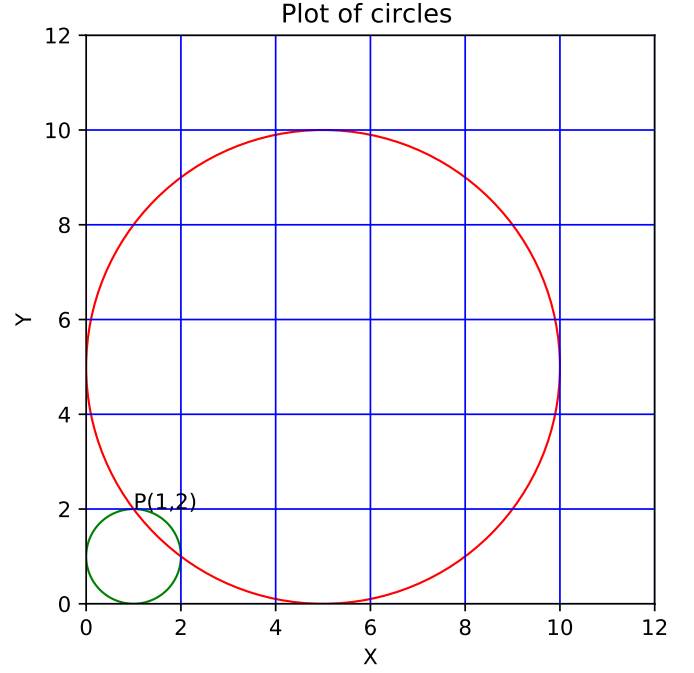


Fig. 1: Two circles passing through a common point

$$\Rightarrow r^2 - 6r + 5 = 0 \quad (2.0.27)$$

$$\Rightarrow (r-1)(r-5) = 0 \quad (2.0.28)$$

$$\Rightarrow r = 1, r = 5. \quad (2.0.29)$$

Hence,

$$\mathbf{O}_1 = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \text{ and } \mathbf{O}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.30)$$

Hence equation of circles are :

$$\left\| \mathbf{x} - \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right\| = 5 \quad (2.0.31)$$

And,

$$\left\| \mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = 1 \quad (2.0.32)$$