Matrix Theory Assignment 15

Ritesh Kumar
EE20RESCH11005

***Abstract*—This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_15

1 Problem

Let $\mathbf{V}$ be the space of $n \times n$ matrices over a field $\mathbb{F}$, and let A be a fixed $n \times n$ matrix over field $\mathbb{F}$. Define a linear operator $\mathbf{T}$ on $\mathbf{V}$ by the equation

$$\mathbf{T}(B) = AB - BA \tag{1.0.1}$$

Prove that if A is a nilpotent matrix, then $\mathbf{T}$ is a nilpotent operator.

2 solution

Since A is a nilpotent matrix, hence for a positive value K:

$$A^K = \mathbf{0} \tag{2.0.1}$$

Now we have

$$\mathbf{T}(B) = AB - BA \tag{2.0.2}$$

$$\implies \mathbf{T}^2(B) = \mathbf{T}(\mathbf{T}(B)) = \mathbf{T}(AB - BA) \tag{2.0.3}$$

$$\begin{aligned}\mathbf{T}(AB - BA) &= A^2B - ABA - ABA + BA^2 \\ &= A^2 - 2ABA + BA^2\end{aligned} \tag{2.0.4}$$

$$\begin{aligned}\implies \mathbf{T}^3(B) &= \mathbf{T}(\mathbf{T}^2(B)) \\ &= A^3 - 3A^2BA + 3ABA^2 - BA^3\end{aligned} \tag{2.0.5}$$

Hence, from (2.0.5) we can say that, as we are increasing the power of operator $\mathbf{T}$, the power of A is also increasing in every terms. Hence there exist a value P such that :

$$\mathbf{T}^P(B) = \mathbf{0} \tag{2.0.6}$$

Hence, if A is a nilpotent matrix then operator $\mathbf{T}$ is also a nilpotent operator.

3 example

Let consider a matrices A and, B as :

$$A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \tag{3.0.1}$$

$$A.B = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} \neq \mathbf{0} \tag{3.0.2}$$

Now,

$$\implies A^2 = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.3}$$

Hence $K = 2$.

And we have also,

$$ABA = \begin{pmatrix} -16 & 16 \\ -16 & 16 \end{pmatrix} \neq \mathbf{0} \tag{3.0.4}$$

Hence from (2.0.3) we conclude, $P \neq 2$. Now putting the value of A^2 from (3.0.3) in (2.0.5) we get,

$$\mathbf{T}^3(B) = \mathbf{0} \tag{3.0.5}$$

Hence, $P = 3$ for this example.