Matrix Theory Assignment 6

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EE20RESCH11005

Abstract—**This problem demonstrate a method to find the foot perpendicular from a given point to a given plane using Singular Value Decomposition.**

All the codes for the figure in this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_6

1 Problem

Write the equation of a plane through the point A $(-3, 4, -1)$ and perpendicular to the line

$$\frac{x+2}{-3} = \frac{y-2}{1} = \frac{z-4}{2} \tag{1.0.1}$$

2 Solution

Let the equation of plane is

$$ax + by + cz + d = 0 \tag{2.0.1}$$

Direction ratio of the line (1.0.1) is given as

$$\mathbf{D} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \tag{2.0.2}$$

Now let consider

$$\mathbf{A} = \begin{pmatrix} -3 & 4 & -1 \end{pmatrix} \tag{2.0.3}$$

Since plane is passing through the point A (-3, 4, -1) and perpendicular to the line (1.0.1), hence

$$\mathbf{AD} + d = 0 \tag{2.0.4}$$

$$\implies d = -11 \tag{2.0.5}$$

Hence equation of the plane is

$$-3x + y + 2z - 11 = 0 \tag{2.0.6}$$

$$\implies -3x + y + 2z = 11 \tag{2.0.7}$$

equation (2.0.7) can written as :

$$\begin{pmatrix} -3 & 1 & 2 \end{pmatrix} \mathbf{x} = 11 \tag{2.0.8}$$

For foot perpendicular we need to find the distance between the plane and point P $(-2, 2, 4)$.
First we find orthogonal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the given normal vector $\mathbf{n}$. Let, $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then

$$\mathbf{m^T n} = 0 \tag{2.0.9}$$

$$\implies \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 0 \tag{2.0.10}$$

$$\implies -3a + b + 2c = 0 \tag{2.0.11}$$

Putting a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} \tag{2.0.12}$$

Putting a=0 and b=1 we get,

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \tag{2.0.13}$$

Now we solve the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.14}$$

Putting values in (2.0.14),

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} \tag{2.0.15}$$

Now, to solve (2.0.15), we perform Singular Value Decomposition on $\mathbf{M}$ as follows,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.16}$$

Where the columns of $\mathbf{V}$ are the eigen vectors of $\mathbf{M}^T\mathbf{M}$,the columns of $\mathbf{U}$ are the eigen vectors of $\mathbf{MM}^T$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} \frac{13}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix} \tag{2.0.17}$$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \end{pmatrix} \tag{2.0.18}$$

From (2.0.14) putting (2.0.16) we get,

$$\mathbf{U}\mathbf{S}\mathbf{V}^T\mathbf{x} = \mathbf{b} \tag{2.0.19}$$

$$\implies \mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^\mathbf{T}\mathbf{b} \tag{2.0.20}$$

Where $\mathbf{S}_+$ is Moore-Penrose Pseudo-Inverse of $\mathbf{S}$.Now, calculating eigen value of $\mathbf{M}\mathbf{M}^T$,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.21}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & \frac{3}{2} \\ 0 & 1-\lambda & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{5}{2}-\lambda \end{pmatrix} = 0 \tag{2.0.22}$$

$$\implies \lambda(\lambda-1)(\lambda-\frac{7}{2}) = 0 \tag{2.0.23}$$

Hence eigen values of $\mathbf{M}\mathbf{M}^T$ are,

$$\lambda_1 = \frac{7}{2} \tag{2.0.24}$$

$$\lambda_2 = 1 \tag{2.0.25}$$

$$\lambda_3 = 0 \tag{2.0.26}$$

Hence the eigen vectors of $\mathbf{M}\mathbf{M}^T$ are,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}, \tag{2.0.27}$$

Normalizing the eigen vectors we get,

$$\mathbf{u}_1 = \begin{pmatrix} \frac{3}{\sqrt{35}} \\ -\frac{1}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -\frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix} \tag{2.0.28}$$

Hence we obtain $\mathbf{U}$ of (2.0.16) as follows,

$$\mathbf{U} = \begin{pmatrix} \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}} \end{pmatrix} \tag{2.0.29}$$

After computing the singular values from eigen values $\lambda_1, \lambda_2, \lambda_3$ we get $\mathbf{S}$ of (2.0.16) as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\frac{7}{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.30}$$

Now, calculating eigen value of $\mathbf{M}^T\mathbf{M}$,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.31}$$

$$\implies \begin{pmatrix} \frac{13}{4}-\lambda & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4}-\lambda \end{pmatrix} = 0 \tag{2.0.32}$$

$$\implies \lambda^2 - \frac{9}{2}\lambda + \frac{7}{2} = 0 \tag{2.0.33}$$

Hence eigen values of $\mathbf{M}^T\mathbf{M}$ are,

$$\lambda_4 = \frac{7}{2} \tag{2.0.34}$$

$$\lambda_5 = 1 \tag{2.0.35}$$

Hence the eigen vectors of $\mathbf{M}^T\mathbf{M}$ are,

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{2.0.36}$$

Normalizing the eigen vectors we get,

$$\mathbf{v}_1 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \tag{2.0.37}$$

Hence we obtain $\mathbf{V}$ of (2.0.16) as follows,

$$\mathbf{V} = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \tag{2.0.38}$$

Finally from (2.0.16) we get the Singualr Value Decomposition of $\mathbf{M}$ as follows,

$$\mathbf{M} = \begin{pmatrix} \frac{3}{\sqrt{35}} & \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{7}{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}^T \tag{2.0.39}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \sqrt{\frac{2}{7}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.40}$$

From (2.0.20) we get,

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12}{\sqrt{35}} \\ \frac{2\sqrt{2}}{\sqrt{5}} \\ \frac{8\sqrt{2}}{\sqrt{7}} \end{pmatrix} \tag{2.0.41}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{12\sqrt{10}}{35} \\ \frac{2\sqrt{10}}{5} \end{pmatrix} \tag{2.0.42}$$

$$\mathbf{x} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{10}{7} \\ \frac{6}{7} \end{pmatrix} \tag{2.0.43}$$

Verifying the solution of (2.0.43) using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.44}$$

Evaluating the R.H.S in (2.0.44) we get,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \begin{pmatrix} -7 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.45}$$

$$\implies \begin{pmatrix} \frac{13}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{5}{4} \end{pmatrix}\mathbf{x} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{2.0.46}$$

Solving the augmented matrix of (2.0.46) we get,

$$\begin{pmatrix} \frac{13}{4} & -\frac{3}{4} & 4 \\ -\frac{3}{4} & \frac{5}{4} & 0 \end{pmatrix} \xleftrightarrow{R_1=\frac{4}{13}R_1} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ -\frac{3}{4} & \frac{5}{4} & 0 \end{pmatrix} \tag{2.0.47}$$

$$\xleftrightarrow{R_2=R_2+\frac{3}{4}R_1} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ 0 & \frac{14}{13} & \frac{12}{13} \end{pmatrix} \tag{2.0.48}$$

$$\xleftrightarrow{R_2=\frac{13}{14}R_2} \begin{pmatrix} 1 & -\frac{3}{13} & \frac{16}{13} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \tag{2.0.49}$$

$$\xleftrightarrow{R_1=R_1+\frac{3}{13}R_1} \begin{pmatrix} 1 & 0 & \frac{10}{7} \\ 0 & 1 & \frac{6}{7} \end{pmatrix} \tag{2.0.50}$$

Hence, Solution of (2.0.44) is given by,

$$\mathbf{x} = \begin{pmatrix} \frac{10}{7} \\ \frac{6}{7} \end{pmatrix} \tag{2.0.51}$$

Comparing results of $\mathbf{x}$ from (2.0.43) and (2.0.51) we conclude that the solution is verified.