

Matrix Theory Assignment 5

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Abstract—This document demonstrates a method to trace a curve with given equation using matrix algebra.

Download latex and python codes from

[https://github.com/Ritesh622/
Assignment_EE5609/tree/master/
Assignment_5](https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_5)

1 PROBLEM STATEMENT

Trace the curve

$$(x - y)^2 = x + y + 1 \quad (1.0.1)$$

2 SOLUTION

We have given equation as :

$$(x - y)^2 = x + y + 1 \quad (2.0.1)$$

$$\Rightarrow x^2 - 2xy + y^2 - x - y - 1 = 0 \quad (2.0.2)$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.3)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.4)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.5)$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.6)$$

Comparing (2.0.2) with (2.0.3), we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (2.0.7)$$

$$\mathbf{u}^T = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (2.0.8)$$

$$f = -1 \quad (2.0.9)$$

Expanding the determinant of V we observe,

$$|\mathbf{V}| = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0 \quad (2.0.10)$$

Also

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = \begin{vmatrix} 1 & -1 & -\frac{1}{2} \\ -1 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 \end{vmatrix} \neq 0 \quad (2.0.11)$$

Hence from (2.0.10) and (2.0.11) we conclude that given equation is an parabola. The characteristic equation of V is given as follows,

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = 0 \quad (2.0.12)$$

$$\Rightarrow (\lambda - 1)^2 - 1 = 0 \quad (2.0.13)$$

The eigenvalues are the roots of (2.0.13) given by

$$\lambda_1 = 0, \lambda_2 = 2 \quad (2.0.14)$$

The eigenvector \mathbf{p} is defined as:

$$\mathbf{V} \mathbf{p} = \lambda \mathbf{p} \quad (2.0.15)$$

$$\Rightarrow (\lambda \mathbf{I} - \mathbf{V}) \mathbf{p} = 0 \quad (2.0.16)$$

where λ is the eigenvalue. For $\lambda_1 = 0$,

$$\mathbf{V} \mathbf{p} = 0 \quad (2.0.17)$$

Row reducing V yields,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (2.0.18)$$

Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$(\lambda_2 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.19)$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^{-1} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (2.0.20)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T \mathbf{V} \mathbf{P} \quad (2.0.21)$$

From equation (2.0.18) and (2.0.19), we have

$$\mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (2.0.22)$$

Thus, the eigenvector rotation matrix and the eigenvalue matrix are

$$\mathbf{P} = \frac{1}{\sqrt{2}} (\mathbf{p}_1 \quad \mathbf{p}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.0.23)$$

$$\mathbf{D} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad (2.0.24)$$

$$\begin{aligned} \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} &\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow -1R_1]{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &\xleftrightarrow{R_3 \leftarrow R_3 + R_2} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_1 \leftarrow \frac{R_1}{-2}} \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (2.0.30)$$

So,

$$\mathbf{c} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad (2.0.31)$$

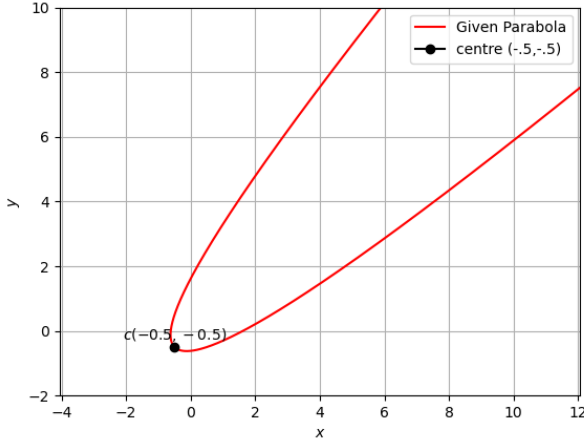


Fig. 1: Parabola with the center \mathbf{c}

The focal length of the parabola is given by

$$\frac{|2\mathbf{u}^T \mathbf{p}_1|}{\lambda_2} = \frac{\sqrt{2}}{2} = \sqrt{2} \quad (2.0.25)$$

and its equation is

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -2\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad (2.0.26)$$

where,

$$\eta = \mathbf{u}^T \mathbf{p}_1 = -\frac{1}{\sqrt{2}} \quad (2.0.27)$$

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p}_1^T \\ \mathbf{v} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p}_1 - \mathbf{u} \end{pmatrix} \quad (2.0.28)$$

$$\Rightarrow \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.29)$$

Forming the augmented matrix and row reducing it: