# Matrix Theory Assignment 14

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*Abstract*—**This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

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| https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_14 |

## 1 Problem

Let $\mathbf{V}$ be a real vector space and E an idempotent linear operator on $\mathbf{V}$, i.e., a projection. Prove that $\left(\mathbf{I}+\mathbf{E}\right)$ is invertible. Find $\left(\mathbf{I}+\mathbf{E}\right)^{-1}$.

## 2 solution

Since $\mathbf{E}$ is an idempotent matrix, that is :

$$\mathbf{E}^2 = \mathbf{E} \tag{2.0.1}$$

Hence it will satisfy the polynomial,

$$x^2 - x = 0 \tag{2.0.2}$$

Thus minimal polynomial will be,

$$m_{\mathbf{E}}(x) = x^2 - x = 0 \tag{2.0.3}$$

$$\implies m_{\mathbf{E}}(x) = x(x-1) = 0 \tag{2.0.4}$$

Hence minimal polynomial $m_{\mathbf{E}}(x)$, of $\mathbf{E}$ divides $x^2 - x$ that is,

$$m_{\mathbf{E}}(x) = x \tag{2.0.5}$$

or,

$$m_{\mathbf{E}}(x) = x - 1 \tag{2.0.6}$$

if

$$m_{\mathbf{E}}(x) = x \tag{2.0.7}$$

$$\implies \mathbf{E} = \mathbf{0} \tag{2.0.8}$$

if

$$m_{\mathbf{E}}(x) = x - 1 \tag{2.0.9}$$

$$\implies \mathbf{E} = \mathbf{I} \tag{2.0.10}$$

Hence, if $\mathbf{E}$ is idempotent then, minimal polynomial of $\mathbf{E}$ is product of distinct polynomial of degree one.Thus matrix $\mathbf{E}$ is similar to diagonal matrix with diagonal entries consisting of characteristic value 0 and 1.
Since $\mathbf{E}$ is diagonalizable, then there exist at least one basis such that :

$$ß = \{\beta_1, \beta_2 \ldots, \beta n\} \tag{2.0.11}$$

Such that

$$\mathbf{E}\beta_i = \beta_i, \forall i = 1, 2 \ldots k \tag{2.0.12}$$

and,

$$\mathbf{E}\beta_i = 0, \forall i = k+1, \ldots n. \tag{2.0.13}$$

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_i = 2\beta_i, \forall i = 1, 2 \ldots k. \tag{2.0.14}$$

And,

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_i = \beta_i, \forall i = k+1 \ldots n. \tag{2.0.15}$$

In matrix form we can write it as:

$$[\mathbf{I}+\mathbf{E}]_ß = \begin{pmatrix} \mathbf{2I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} \tag{2.0.16}$$

Where, $\mathbf{I_1}$ is $k \times k$ and $\mathbf{I_2}$ is $(n-k) \times (n-k)$ identity matrices, and each $\mathbf{0}$ represents the zero matrix of appropriate dimension.
From 2.0.16 we can calculate the determinant as:

$$det\left(\mathbf{I}+\mathbf{E}\right) = 2^k \neq 0 \tag{2.0.17}$$

Since $[\mathbf{I}+\mathbf{E}]_ß$ is combination of identity matrices. Hence, from (2.0.16), inverse of the matrix $[\mathbf{I}+\mathbf{E}]_ß$ is given as :

$$\begin{aligned}\left([\mathbf{I}+\mathbf{E}]_ß\right)^{-1} &= \begin{pmatrix} \frac{1}{2}\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} = \mathbf{I} - \frac{1}{2}[E]_ß \end{aligned} \tag{2.0.18}$$

Hence,

$$\left([\mathbf{I} + \mathbf{E}]_{ß}\right)^{-1} = \mathbf{I} - \frac{1}{2}\,[E]_{ß} \qquad (2.0.19)$$

we can also verify our results as:

$$\left(\mathbf{I} + \mathbf{E}\right)\left(\mathbf{I} - \tfrac{1}{2}\mathbf{E}\right) = \mathbf{I}^2 - \frac{1}{2}\mathbf{E} + \mathbf{E} - \frac{1}{2}\mathbf{E}^2 \qquad (2.0.20)$$

From (2.0.1), We have $\mathbf{E}^2 = \mathbf{E}$, Hence (2.0.20) becomes,

$$\mathbf{I} - \frac{1}{2}\mathbf{E} + \mathbf{E} + \frac{1}{2}\mathbf{E} = \mathbf{I} \qquad (2.0.21)$$