Matrix Theory Assignment 15

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EE20RESCH11005

***Abstract*—This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_15

1 Problem

Let $\mathbf{V}$ be the space of $n \times n$ matrices over a field $\mathbb{F}$, and let A be a fixed $n \times n$ matrix over field $\mathbb{F}$. Define a linear operator $\mathbf{T}$ on $\mathbf{V}$ by the equation

$$\mathbf{T}(B) = AB - BA \tag{1.0.1}$$

Prove that if A is a nilpotent matrix, then $\mathbf{T}$ is a nilpotent operator.

2 solution

Since A is a nilpotent matrix, hence for a positive value K:

$$A^K = \mathbf{0} \tag{2.0.1}$$

Now we have

$$\mathbf{T}(B) = AB - BA \tag{2.0.2}$$

$$\implies \mathbf{T}^2(B) = \mathbf{T}(\mathbf{T}(B)) = \mathbf{T}(AB - BA) \tag{2.0.3}$$

$$\begin{aligned} \mathbf{T}(AB - BA) &= A^2B - ABA - ABA + BA^2 \\ &= A^2 - 2ABA + BA^2 \end{aligned} \tag{2.0.4}$$

$$\implies \mathbf{T}^3(B) = \mathbf{T}(\mathbf{T}^2(B)) = A^3 - 3A^2BA + 3ABA^2 - BA^3 \tag{2.0.5}$$

Hence, from (2.0.5) we can say that, as we are increasing the power of operator $\mathbf{T}$, the power of A is also increasing in every terms. Hence there exist a value P such that :

$$\mathbf{T}^P(B) = \mathbf{0} \tag{2.0.6}$$

Hence, if A is a nilpotent matrix then operator $\mathbf{T}$ is also a nilpotent operator