# Matrix Theory Assignment 13

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*Abstract*—This problem is all about to introducing the concept of characteristic polynomial over a filed.

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/ tree/master/Assignment_13

## 1 PROBLEM

Let $A$ be the an $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \ldots (x - c_k)^{d_k} \quad (1.0.1)$$

Where $c_1, c_2, \ldots c_k$ are distinct. Let $\mathbf{V}$ the space of $n \times n$ matrices $B$ such that

$$AB = BA \quad (1.0.2)$$

Prove that the dimension of $\mathbf{V}$ is,

$$d_1{}^2 + d_2{}^2 \cdots + d_k{}^2 \quad (1.0.3)$$

## 2 SOLUTION

Let consider we have a matrix $A$ which is a diagonal matrix, which is given as

$$A = \begin{bmatrix} c_1 I & 0 & 0 & \ldots & 0 & 0 \\ 0 & c_2 I & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & . & \ldots & . & c_k I \end{bmatrix} \quad (2.0.1)$$

Consider $B$ as :

$$B = \begin{bmatrix} B_{11} & B_{12} & . & \ldots & . & B_{1k} \\ B_{21} & B_{22} & . & \ldots & . & B_{2k} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ B_{k1} & B_{k2} & . & \ldots & . & B_{kk} \end{bmatrix} \quad (2.0.2)$$

Where $B_{ij}$ has dimension $d_i \times d_j$. Since we have given ,

$$AB = BA \quad (2.0.3)$$

$$\implies \begin{bmatrix} c_1 B_{11} & c_1 B_{12} & . & \ldots & . & c_1 B_{1k} \\ c_2 B_{21} & c_2 B_{22} & . & \ldots & . & c_2 B_{2k} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ c_k B_{k1} & c_k B_{k2} & . & \ldots & . & c_k B_{kk} \end{bmatrix} =$$

$$\begin{bmatrix} c_1 B_{11} & B_{12} & . & \ldots & . & c_1 B_{1k} \\ c_2 B_{21} & B_{22} & . & \ldots & . & c_2 B_{2k} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ c_k B_{k1} & c_k B_{k2} & . & \ldots & . & c_k B_{kk} \end{bmatrix} \quad (2.0.4)$$

Hence, from above equation 2.0.4 we can conclude,

$$c_i \neq c_j, \forall i \neq j \quad (2.0.5)$$

$$\implies B_{ij} = 0, \forall i \neq j \quad (2.0.6)$$

We can have $B_{11}, B_{22} \ldots$ any arbitrary matrices. From (2.0.4) we can have

$$D(B_{ij}) = d_i^2 \quad (2.0.7)$$

Where D represents dimension of matrix. Therefore the dimension of the space of all such $B_{ij}'$s matrices is given as :

$$d_1{}^2 + d_2{}^2 \cdots + d_k{}^2 \quad (2.0.8)$$

## 3 EXAMPLE

Let suppose we have matrix $A$ as :

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (3.0.1)$$

and $B$ as :

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad (3.0.2)$$

Where,

$$B_{11} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, B_{12} = \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \quad (3.0.3)$$

$$(3.0.4)$$

$$B_{21} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}, B_{22} = \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \quad (3.0.5)$$

$$(3.0.6)$$

$$\implies B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \quad (3.0.7)$$

Consider,

$$C = AB \quad (3.0.8)$$

$$\implies C = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ 2b_{31} & 2b_{32} & 2b_{33} & 2b_{34} \\ 2b_{41} & 2b_{42} & 2b_{43} & 2b_{44} \end{bmatrix} \quad (3.0.9)$$

Let another matrix D as :

$$D = BA \quad (3.0.10)$$

$$B = \begin{bmatrix} b_{11} & b_{12} & 2b_{13} & 2b_{14} \\ b_{21} & b_{22} & 2b_{23} & 2b_{24} \\ b_{31} & b_{32} & 2b_{33} & 2b_{34} \\ b_{41} & b_{42} & 2b_{43} & 2b_{44} \end{bmatrix} \quad (3.0.11)$$

We have given as,

$$BA = AB \quad (3.0.12)$$

$$\implies C = D \quad (3.0.13)$$

it is possible only when,

$$b_{13} = b_{14} = b_{23} = b_{24} = 0 \quad (3.0.14)$$

And,

$$b_{31} = b_{32} = b_{41} = b_{42} = 0 \quad (3.0.15)$$

$$B_{12} = \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (3.0.16)$$

And,

$$B_{21} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (3.0.17)$$

Hence, therefore matrix $B$ becomes,

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 2b_{33} & 2b_{34} \\ 0 & 0 & 2b_{43} & 2b_{44} \end{bmatrix} \quad (3.0.18)$$

$$\implies B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} \quad (3.0.19)$$

$$\implies B_{ij} = 0, \forall i \neq j \quad (3.0.20)$$

Now the basis of the $n \times n$ matrices for vector space of all $n \times n$ matrix B are,

$$\beta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.0.21)$$

$$\beta_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.0.22)$$

$$\beta_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \beta_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.0.23)$$

$$\beta_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \beta_8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (3.0.24)$$

Thus, Dimension of **V** (vector space of all $n \times n$ matrices $B$) = 8,

Also

$$d_1{}^2 + d_2{}^2 = 2^2 + 2^2 = 8. \quad (3.0.25)$$

Therefore, Dimension of **V** (vector space of all $n \times n$ matrix $B$ is :

$$d_1^2 + d_2^2 \quad (3.0.26)$$