Matrix Theory Assignment 14

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***Abstract*—This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_14

1 Problem

Let $\mathbf{V}$ be a real vector space and E an idempotent linear operator on $\mathbf{V}$, i.e., a projection. Prove that $\left(\mathbf{I}+\mathbf{E}\right)$ is invertible. Find $\left(\mathbf{I}+\mathbf{E}\right)^{-1}$.

2 solution

Since $\mathbf{E}$ is an idempotent matrix, that is :

$$\mathbf{E}^2 = \mathbf{E} \tag{2.0.1}$$

Hence it will satisfy the polynomial,

$$x^2 - x = 0 \tag{2.0.2}$$

Thus minimal polynomial will be,

$$m_{\mathbf{E}}(x) = x^2 - x = 0 \tag{2.0.3}$$

$$\implies m_{\mathbf{E}}(x) = x(x-1) = 0 \tag{2.0.4}$$

Hence minimal polynomial $m_{\mathbf{E}}(x)$, of $\mathbf{E}$ divides $x^2 - x$ that is,

$$m_{\mathbf{E}}(x) = x \tag{2.0.5}$$

or,

$$m_{\mathbf{E}}(x) = x - 1 \tag{2.0.6}$$

if

$$m_{\mathbf{E}}(x) = x \tag{2.0.7}$$

$$\implies \mathbf{E} = \mathbf{0} \tag{2.0.8}$$

if

$$m_{\mathbf{E}}(x) = x - 1 \tag{2.0.9}$$

$$\implies \mathbf{E} = \mathbf{I} \tag{2.0.10}$$

Hence, if $\mathbf{E}$ is idempotent then, minimal polynomial of $\mathbf{E}$ is product of distinct polynomial of degree one.Thus matrix $\mathbf{E}$ is similar to diagonal matrix with diagonal entries consisting of characteristic value 0 and 1.
Since $\mathbf{E}$ is diagonalizable, then there exist at least one basis such that :

$$ß = \{\beta_1, \beta_2 \ldots, \beta n\} \tag{2.0.11}$$

Such that

$$\mathbf{E}\beta_i = \beta_i, \forall i = 1, 2 \ldots k \tag{2.0.12}$$

and,

$$\mathbf{E}\beta_i = 0, \forall i = k+1, \ldots n. \tag{2.0.13}$$

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_i = 2\beta_i, \forall i = 1, 2 \ldots k. \tag{2.0.14}$$

And,

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_i = \beta_i, \forall i = k+1 \ldots n. \tag{2.0.15}$$

In matrix form we can write it as:

$$[\mathbf{I}+\mathbf{E}]_ß = \begin{pmatrix} 2\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} \tag{2.0.16}$$

Where, $\mathbf{I_1}$ is $k \times k$ and $\mathbf{I_2}$ is $(n-k) \times (n-k)$ identity matrices, and each $\mathbf{0}$ represents the zero matrix of appropriate dimension.
From 2.0.16 we can calculate the determinant as:

$$\det\left(\mathbf{I}+\mathbf{E}\right) = 2^k \neq 0 \tag{2.0.17}$$

From 2.0.16 we can observe that, eigen values of $[\mathbf{I}+\mathbf{E}]$ are kth number of 2 and $(n-k)^{th}$ number of 1. hence none of the eigen value of the matrix is zero. hence it is invertible.
Since $[\mathbf{I}+\mathbf{E}]_ß$ is combination of identity matrices. Hence, from (2.0.16), inverse of the matrix $[\mathbf{I}+\mathbf{E}]_ß$

is given as :

$$\left([\mathbf{I}+\mathbf{E}]_{ß}\right)^{-1} = \begin{pmatrix} \frac{1}{2}\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} = \mathbf{I} - \frac{1}{2}[\mathbf{E}]_{ß} \quad (2.0.18)$$

Hence,

$$\left([\mathbf{I}+\mathbf{E}]_{ß}\right)^{-1} = \mathbf{I} - \frac{1}{2}[\mathbf{E}]_{ß} \quad (2.0.19)$$

we can also verify our results as:

$$\left(\mathbf{I}+\mathbf{E}\right)\left(\mathbf{I}-\tfrac{1}{2}\mathbf{E}\right) = \mathbf{I}^2 - \frac{1}{2}\mathbf{E} + \mathbf{E} - \frac{1}{2}\mathbf{E}^2 \quad (2.0.20)$$

From (2.0.1), We have $\mathbf{E}^2 = \mathbf{E}$, Hence (2.0.20) becomes,

$$\mathbf{I} - \frac{1}{2}\mathbf{E} + \mathbf{E} + \frac{1}{2}\mathbf{E} = \mathbf{I} \quad (2.0.21)$$

3 Example

Let consider a matrix $\mathbf{E}$ as :

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.1)$$

$$\implies \mathbf{E}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.2)$$

$$\implies \mathbf{E}^2 = E. \quad (3.0.3)$$

Basis for this matix is

$$\beta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{and,} \beta_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.4)$$

We have,

$$\mathbf{E}\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta_1 \quad (3.0.5)$$

And,

$$\mathbf{E}\beta_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \beta_2 \quad (3.0.6)$$

Now,

$$\mathbf{I}+\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (3.0.7)$$

$$\implies \mathbf{I}+\mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.8)$$

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (3.0.9)$$

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_1 = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\beta_1 \quad (3.0.10)$$

Similarly,

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.11)$$

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_1 = 2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \beta_2 \quad (3.0.12)$$

$$\implies [\mathbf{I}+\mathbf{E}]_{ß} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_1} \end{pmatrix} \quad (3.0.13)$$

Now let find the eigen value of matrix $\left(\mathbf{I}+\mathbf{E}\right)$:

$$\implies \begin{pmatrix} 2-ł & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0 \quad (3.0.14)$$

$$\implies (2-\lambda)(1-\lambda) = 0 \quad (3.0.15)$$

$$\implies \lambda_1 = 2, \lambda_2 = 1 \quad (3.0.16)$$

The eigen values of the matrix $(\mathbf{I}+\mathbf{E})$ from (3.0.16) are 2 and 1. Since none of the eigen value is zero, hence matrix is invertible.

Here we have $k = 1$ and $n = 2$ and, $(n-k) = 1$. Hence size of $\mathbf{I_1}$ and $\mathbf{I_2}$ are 1×1. Similarly, size of zero matrix is also 1×1. Now determinant of $\mathbf{I}+\mathbf{E}$ is :

$$\det\left(\mathbf{I}+\mathbf{E}\right) = 2^k = 2^1 = 2 \neq 0. \quad (3.0.17)$$

Inverse of the matrix is :

$$\left([\mathbf{I}+\mathbf{E}]\right)^{-1}_{ß} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.18)$$

$$\implies \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.19)$$

Hence, equation (3.0.19) can be written as:

$$\left([\mathbf{I}+\mathbf{E}]\right)^{-1}_{ß} = \begin{pmatrix} \mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} = \mathbf{I} - \frac{1}{2}[\mathbf{E}]_{ß} \quad (3.0.20)$$