Matrix Theory : Assignment 3

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***Abstract*—This problem is to demonstrate the way to prove the triangles are congruent and to prove a triangle as isosceles using matrix algebra.**

1 Problem

ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that

1) $\triangle ABC \cong \triangle ACF$

2 Solution

2.1 part 1

Let consider we have a triangle $\triangle ABC$. There are two altitudes BE and CF being drawn from the vertices B and C respectively.
In $\triangle ABE$, taking inner product of sides AE and EB we can write :

$$(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B}) = \|\mathbf{A}-\mathbf{E}\|\,\|\mathbf{E}-\mathbf{B}\|\cos AEB \tag{2.1.1}$$

$$\implies \cos AEB = \frac{(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B})}{\|\mathbf{A}-\mathbf{E}\|\,\|\mathbf{E}-\mathbf{B}\|} \tag{2.1.2}$$

In $\triangle ACF$, taking inner product of sides AF and FC :

$$(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C}) = \|\mathbf{A}-\mathbf{F}\|\,\|\mathbf{F}-\mathbf{C}\|\cos AFC \tag{2.1.3}$$

$$\implies \cos AFC = \frac{(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C})}{\|\mathbf{A}-\mathbf{F}\|\,\|\mathbf{F}-\mathbf{C}\|} \tag{2.1.4}$$

In triangle $\triangle ABC$,

$$\cos AFC = \cos AEB \tag{2.1.5}$$

$$\frac{(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B})}{\|\mathbf{A}-\mathbf{E}\|\,\|\mathbf{E}-\mathbf{B}\|} = \frac{(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C})}{\|\mathbf{A}-\mathbf{F}\|\,\|\mathbf{F}-\mathbf{C}\|} \tag{2.1.6}$$

Given,

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{2.1.7}$$

$$\frac{(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B})}{(\mathbf{A}-\mathbf{E})^T(\mathbf{A}-\mathbf{E})} = \frac{(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C})}{(\mathbf{A}-\mathbf{F})^T(\mathbf{A}-\mathbf{F})} \tag{2.1.8}$$

$$\frac{(\mathbf{E}-\mathbf{B})}{(\mathbf{A}-\mathbf{E})} = \frac{(\mathbf{F}-\mathbf{C})}{(\mathbf{A}-\mathbf{F})} \tag{2.1.9}$$

Taking norms both side,

$$\frac{\|\mathbf{E}-\mathbf{B}\|}{\|\mathbf{A}-\mathbf{E}\|} = \frac{\|\mathbf{F}-\mathbf{C}\|}{\|\mathbf{A}-\mathbf{F}\|} \tag{2.1.10}$$

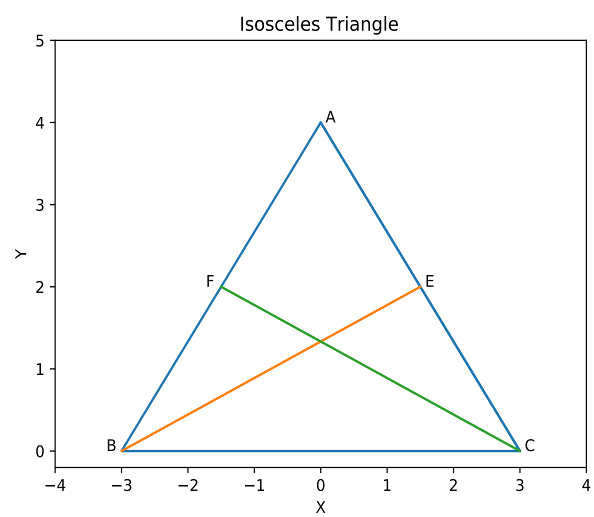

Fig. 1: Isosceles triangle

Using 2.1.7 in 2.1.10 we get :

$$\|\mathbf{A}-\mathbf{E}\| = \|\mathbf{A}-\mathbf{F}\| \tag{2.1.11}$$

Now, we have

$$\|\mathbf{A}-\mathbf{E}\| = \|\mathbf{A}-\mathbf{F}\| (\text{ we got the result}) \tag{2.1.12}$$

$$\|\mathbf{C}-\mathbf{F}\| = \|\mathbf{B}-\mathbf{E}\| (\text{given}) \tag{2.1.13}$$

since BE $\perp$ AC and CF $\perp$ AB , hence we have,

$$\cos BEA = \cos CFA\,(\ 90^{\circ}) \tag{2.1.14}$$

Hence by SAS (Side - Angle - Side) We can say that , $\triangle$ABC $\cong$ $\triangle$ACF