

# Matrix Theory Assignment 17

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All the codes for this document can be found at And

[https://github.com/Ritesh622/Assignment\\_EE5609/tree/master/Assignment\\_17](https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_17)

## 1 PROBLEM

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous function. such that

$$\int_{\mathbb{R}^n} |f(\mathbf{x}) d\mathbf{x}| < \infty \quad (1.0.1)$$

Let  $A$  be a real  $n \times n$  invertible matrix and for  $x, y \in \mathbb{R}^n$ . Let  $\langle \mathbf{x}, \mathbf{y} \rangle$  denotes the standard inner product in  $\mathbb{R}^n$  then,  $\int_{\mathbb{R}^n} f(A\mathbf{x}) e^{i\langle \mathbf{y}, \mathbf{x} \rangle} d\mathbf{x} = ?$

$$1) \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\langle (A^{-1})^T \mathbf{y}, \mathbf{x} \rangle} \frac{d\mathbf{x}}{|\det(A)|}$$

$$2) \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\langle A^T \mathbf{y}, \mathbf{x} \rangle} \frac{d\mathbf{x}}{|\det(A)|}.$$

$$3) \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\langle (A^T)^{-1} \mathbf{y}, \mathbf{x} \rangle} d\mathbf{x}.$$

$$4) \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\langle A^{-1} \mathbf{y}, \mathbf{x} \rangle} \frac{d\mathbf{x}}{|\det(A)|}.$$

## 2 SOLUTION

Let consider,

$$\mathbf{Ax} = \mathbf{t} \quad (2.0.1)$$

$$\Rightarrow \mathbf{x} = \mathbf{A}^{-1} \mathbf{t} \quad (2.0.2)$$

$$\Rightarrow d\mathbf{x} = \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \quad (2.0.3)$$

Using (2.0.1) to (2.0.3), We can write:

$$\int_{\mathbb{R}^n} f(A\mathbf{x}) e^{i\langle \mathbf{y}, \mathbf{x} \rangle} d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{t}) e^{i\langle \mathbf{y}, (\mathbf{A}^{-1} \mathbf{t}) \rangle} \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \quad (2.0.4)$$

We know that,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} \quad (2.0.5)$$

$$\Rightarrow \langle \mathbf{y}, (\mathbf{A}^{-1} \mathbf{t}) \rangle = (\mathbf{y}^T \mathbf{A}^{-1} \mathbf{t}) \quad (2.0.6)$$

$$\Rightarrow \langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle = ((\mathbf{A}^{-1})^T \mathbf{y})^T \mathbf{t} \quad (2.0.7)$$

$$\Rightarrow ((\mathbf{A}^{-1})^T \mathbf{y})^T \mathbf{t} = (\mathbf{y}^T ((\mathbf{A}^{-1})^T)^T \mathbf{t}) = (\mathbf{y}^T \mathbf{A}^{-1} \mathbf{t}) \quad (2.0.8)$$

Hence, from (2.0.6) and (2.0.8)

$$\langle \mathbf{y}, (\mathbf{A}^{-1} \mathbf{t}) \rangle = \langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle \quad (2.0.9)$$

Using (2.0.9) in (2.0.4) We can write,

$$\Rightarrow \int_{\mathbb{R}^n} f(\mathbf{t}) e^{i\langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle} \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \quad (2.0.10)$$

replacing variable  $\mathbf{t}$  with  $\mathbf{x}$ .

$$\Rightarrow \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{x} \rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|} \quad (2.0.11)$$

Hence option 1 is correct.

## 3 EXAMPLE

Let consider a matrix  $\mathbf{A}$  and  $\mathbf{x}$  as :

$$\mathbf{A} = \begin{pmatrix} x_1 & 2x_2 \\ 3x_1 & 4x_2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{Ax} = \mathbf{t} \quad (3.0.2)$$

$$\Rightarrow \mathbf{x} = \mathbf{A}^{-1} \mathbf{t} \quad (3.0.3)$$

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{2}{3x_1} & \frac{1}{-2x_2} \\ \frac{3}{2x_2} & -\frac{1}{2x_2} \end{pmatrix} \quad (3.0.4)$$

$$\mathbf{t} = \begin{pmatrix} x_1^2 + 2x_2^2 \\ 3x_1^2 + 4x_2^2 \end{pmatrix} \quad (3.0.5)$$

Let consider another matrix  $\mathbf{y}$  as :

$$\mathbf{y} = \begin{pmatrix} 5x_1 \\ 7x_2 \end{pmatrix} \quad (3.0.6)$$

Now,

$$\langle \mathbf{y}, (\mathbf{A}^{-1} \mathbf{t}) \rangle = \mathbf{y}^T (\mathbf{A}^{-1} \mathbf{t}) = \begin{pmatrix} 5x_1 & 6x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5x_1^2 + 6x_2^2 \quad (3.0.7)$$

And,

$$\begin{aligned}\langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle &= ((\mathbf{A}^{-1})^T \mathbf{y})^T \mathbf{t} = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} x_1^2 + 2x_2^2 \\ 3x_1^2 + 4x_2^2 \end{pmatrix} \\ &= 5x_1^2 + 6x_2^2 \quad (3.0.8)\end{aligned}$$

Hence from (3.0.7) and (3.0.8) we can conclude,

$$\langle \mathbf{y}, (\mathbf{A}^{-1} \mathbf{t}) \rangle = \langle (\mathbf{A}^{-1})^T \mathbf{y}, \mathbf{t} \rangle \quad (3.0.9)$$

We have also,

$$d\mathbf{x} = \frac{d\mathbf{t}}{|-2x_1x_2|} \quad (3.0.10)$$

Hence our equation (2.0.11) becomes,

$$\int_{\mathbb{R}^2} f(\mathbf{x}) e^{i(5x_1^2 + 6x_2^2)} \frac{d\mathbf{x}}{|-2x_1x_2|} \quad (3.0.11)$$