Matrix Theory : Assignment 4

Ritesh Kumar
Roll no. : EE20RESCH11005

Abstract—**This problem is to demonstrate a method to find the equations of circles who touches both the axes and passes through a common point using matrix algebra.**

Download latex and python codes from

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_4

1 Problem

Show that two circles can be drawn to pass through the point $\begin{pmatrix}1\\2\end{pmatrix}$ and touch the coordinate axes, and find their equations.

2 Solution

Let us consider we have a circle which passes through the point $\begin{pmatrix}1\\2\end{pmatrix}$ and touches x - axis at point $\begin{pmatrix}r\\0\end{pmatrix}$ and y - axis at $\begin{pmatrix}0\\r\end{pmatrix}$. Radius of the circle is **r** since it touches both axes. Hence we have 3 points which are :

$$\mathbf{P_1} = \begin{pmatrix}1\\2\end{pmatrix} \tag{2.0.1}$$

$$\mathbf{p_2} = \begin{pmatrix}r\\0\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{P_3} = \begin{pmatrix}0\\r\end{pmatrix} \tag{2.0.3}$$

The general equation of circle is :

$$||\mathbf{x} - \mathbf{O}|| = r \tag{2.0.4}$$

Substituting the given coordinates:

$$||\mathbf{P_2} - \mathbf{O}||^2 = r^2 \tag{2.0.5}$$

$$||\mathbf{P_3} - \mathbf{O}||^2 = r^2 \tag{2.0.6}$$

$$||\mathbf{P_1} - \mathbf{O}||^2 = r^2 \tag{2.0.7}$$

From equation 2.0.5, 2.0.6 and 2.0.7 we have

$$||\mathbf{P_2} - \mathbf{O}||^2 - ||\mathbf{P_1} - \mathbf{O}||^2 = 0 \tag{2.0.8}$$

And,

$$||\mathbf{P_3} - \mathbf{O}||^2 - ||\mathbf{P_1} - \mathbf{O}||^2 = 0 \tag{2.0.9}$$

Simplifying 2.0.8 and 2.0.9,

$$\left(\mathbf{P_2} - \mathbf{O}\right)^T \left(\mathbf{P_2} - \mathbf{O}\right) - \left(\mathbf{P_1} - \mathbf{O}\right)^T \left(\mathbf{P_1} - \mathbf{O}\right) = \mathbf{0} \tag{2.0.10}$$

$$\implies ||\mathbf{P_2}||^2 - 2\mathbf{P_2}^T\mathbf{O} - ||\mathbf{P_1}||^2 + 2\mathbf{P_1}^T\mathbf{O} = \mathbf{0} \tag{2.0.11}$$

Substituting the value of $||\mathbf{P_1}||$ and $||\mathbf{P_2}||$ and other values then rearranging it, we get :

$$\begin{pmatrix}2 - 2r & 4\end{pmatrix}\begin{pmatrix}O\end{pmatrix} = 5 - r^2 \tag{2.0.12}$$

Similarly,

$$\left(\mathbf{P_3} - \mathbf{O}\right)^T \left(\mathbf{P_3} - \mathbf{O}\right) - \left(\mathbf{P_1} - \mathbf{O}\right)^T \left(\mathbf{P_1} - \mathbf{O}\right) = \mathbf{0} \tag{2.0.13}$$

$$\implies ||\mathbf{P_3}||^2 - 2\mathbf{P_3}^T\mathbf{O} - ||\mathbf{P_1}||^2 + 2\mathbf{P_1}^T\mathbf{O} = \mathbf{0} \tag{2.0.14}$$

Substituting the value of $||\mathbf{P_1}||$ and $||\mathbf{P_3}||$ and other values then rearranging it, we get :

$$\begin{pmatrix}2 & 4 - 2r\end{pmatrix}\begin{pmatrix}O\end{pmatrix} = 5 - r^2 \tag{2.0.15}$$

Combining 2.0.15 and 2.0.12

$$\begin{pmatrix}2 - 2r & 4\\ 2 & 4 - 2r\end{pmatrix}\begin{pmatrix}O\end{pmatrix} = \begin{pmatrix}5 - r^2\\ 5 - r^2\end{pmatrix} \tag{2.0.16}$$

Transforming the matrix into row-echelon form

$$\begin{pmatrix} 2-2r & 4 & 5-r^2 \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \tag{2.0.17}$$

$$\begin{pmatrix} 2-2r & 4 & 5-r^2 \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \xleftrightarrow{R1 \leftarrow \frac{R1}{2-2r}}$$
$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-}{2(r-1)} \\ 2 & 4-2r & 5-r^2 \end{pmatrix} \xleftrightarrow{R2 \leftarrow R2-2R1}$$
$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & \frac{2r(r-3)}{r-1} & \frac{r\left(r^2-5\right)}{r-1} \end{pmatrix} \xleftrightarrow{R2 \leftarrow \left(\frac{1-r}{2r(r-3)}\right)R2}$$
$$\begin{pmatrix} 1 & \frac{-2}{r-1} & \frac{r^2-5}{2(r-1)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \xleftrightarrow{R1 \leftarrow R1-\left(\frac{-2}{r-1}\right)R2}$$
$$\begin{pmatrix} 1 & 0 & \frac{r^2-5}{2(r-3)} \\ 0 & 1 & \frac{r^2-5}{2(r-3)} \end{pmatrix} \tag{2.0.18}$$

So,

$$\mathbf{O} = \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} \tag{2.0.19}$$

Now subsituting the 2.0.3 in 2.0.7, we have

$$\|\mathbf{P_3} - \mathbf{O}\|^2 = r^2 \tag{2.0.20}$$

Subsituting the value of $\mathbf{O}$ in 2.0.7 and simplify,

$$\left(\mathbf{P_3} - \mathbf{O}\right)^T \left(\mathbf{P_3} - \mathbf{O}\right) = r^2 \tag{2.0.21}$$

$$\implies \|\mathbf{P_3}\|^2 - \mathbf{P_3}^T\mathbf{O} - \mathbf{P_3}\mathbf{O}^T + \|\mathbf{O}\|^2 = r^2 \tag{2.0.22}$$

Putting the values of $\mathbf{O}$ from 2.0.19 and $\|\mathbf{P_3}\|^2$

$$\implies -\mathbf{P_3}^T\mathbf{O} - \mathbf{P_3}\mathbf{O}^T = -\|\mathbf{O}\|^2 \tag{2.0.23}$$

$$\implies -\begin{pmatrix} 0 \\ r \end{pmatrix}^T \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix} - \begin{pmatrix} 0 \\ r \end{pmatrix} \begin{pmatrix} \frac{r^2-5}{2(r-3)} \\ \frac{r^2-5}{2(r-3)} \end{pmatrix}^T = -\|\mathbf{O}\|^2 \tag{2.0.24}$$

Subsituting the value of $\|\mathbf{O}\|^2$ and simplify it,

$$\implies 2r\left(\frac{r^2-5}{2(r-3)}\right) = 2\left(\frac{r^2-5}{2(r-3)}\right)^2 \tag{2.0.25}$$

$$\implies r = \frac{r^2-5}{2(r-3)} \tag{2.0.26}$$

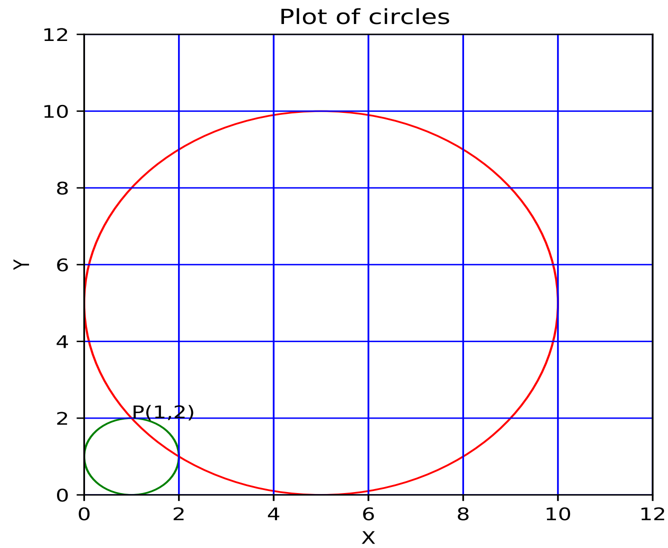

Fig. 1: Two circles passing through a common point

$$\implies r^2 - 6r + 5 = 0 \tag{2.0.27}$$

$$\implies (r-1)(r-5) = 0 \tag{2.0.28}$$

$$\implies r = 1, r = 5. \tag{2.0.29}$$

Hence,

$$\mathbf{O_1} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \text{and,} \mathbf{O_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.30}$$

Hence equation of circles are :

$$\left\|\mathbf{x} - \begin{pmatrix} 5 \\ 5 \end{pmatrix}\right\| = 5 \tag{2.0.31}$$

And,

$$\left\|\mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| = 1 \tag{2.0.32}$$