Matrix Theory Assignment 13

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Abstract—**This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_13

1 Problem

Let A be the an $n \times n$ diagonal matrix with characteristic polynomial

$$(x-c_1)^{d_1}(x-c_2)^{d_2}\dots(x-c_k)^{d_k} \tag{1.0.1}$$

Where $c_1, c_2, \dots c_k$ are distinct. Let $\mathbf{V}$ the space of $n \times n$ matrices B such that

$$AB = BA \tag{1.0.2}$$

Prove that the dimension of $\mathbf{V}$ is,

$${d_1}^2 + {d_2}^2 \dots + {d_k}^2 \tag{1.0.3}$$

2 solution

Let consider we have a matrix A which is a diagonal matrix, which is given as

$$A = \begin{pmatrix} c_1 I & 0 & 0 & \dots & 0 & 0 \\ 0 & c_2 I & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & . & \dots & . & c_k I \end{pmatrix} \tag{2.0.1}$$

Consider B as :

$$B = \begin{pmatrix} B_{11} & B_{12} & . & \dots & . & B_{1k} \\ B_{21} & B_{22} & . & \dots & . & B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ B_{k1} & B_{k2} & . & \dots & . & B_{kk} \end{pmatrix} \tag{2.0.2}$$

Where B_{ij} has dimension $d_i \times d_j$. Since we have given ,

$$AB = BA \tag{2.0.3}$$

$$\implies \begin{pmatrix} c_1 B_{11} & c_1 B_{12} & . & \dots & . & c_1 B_{1k} \\ c_2 B_{21} & c_2 B_{22} & . & \dots & . & c_2 B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ c_k B_{k1} & c_k B_{k2} & . & \dots & . & c_k B_{kk} \end{pmatrix} =$$

$$\begin{pmatrix} c_1 B_{11} & B_{12} & . & \dots & . & c_1 B_{1k} \\ c_2 B_{21} & B_{22} & . & \dots & . & c_2 B_{2k} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ c_k B_{k1} & c_k B_{k2} & . & \dots & . & c_k B_{kk} \end{pmatrix} \tag{2.0.4}$$

Hence, from above equation 2.0.4 we can conclude,

$$c_i \neq c_j, \forall i \neq j \tag{2.0.5}$$

$$\implies B_{ij} = 0, \forall i \neq j \tag{2.0.6}$$

We can have $B_{11}, B_{22} \dots$ any arbitrary matrices. From (2.0.4) we can have

$$D(B_{ij}) = d_i^2 \tag{2.0.7}$$

Where D represents dimension of matrix. Therefore the dimension of the space of all such B_{ij}'s matrices is given as :

$${d_1}^2 + {d_2}^2 \dots + {d_k}^2 \tag{2.0.8}$$

3 Example

Let suppose we have matrix A as :

$$A = \begin{pmatrix} c_1 I & \mathbf{0} \\ \mathbf{0} & c_2 I \end{pmatrix} \tag{3.0.1}$$

Where,

$$c_1 I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c_2 I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{3.0.2}$$

$$\implies A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{3.0.3}$$

and B as :

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{3.0.4}$$

Where,

$$B_{11} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, B_{12} = \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} \tag{3.0.5}$$

$$\tag{3.0.6}$$

$$B_{21} = \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}, B_{22} = \begin{pmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{pmatrix} \tag{3.0.7}$$

$$\tag{3.0.8}$$

$$\implies B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \tag{3.0.9}$$

Consider,

$$C = AB \tag{3.0.10}$$

$$\implies C = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ 2b_{31} & 2b_{32} & 2b_{33} & 2b_{34} \\ 2b_{41} & 2b_{42} & 2b_{43} & 2b_{44} \end{pmatrix} \tag{3.0.11}$$

Let another matrix D as :

$$D = BA \tag{3.0.12}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & 2b_{13} & 2b_{14} \\ b_{21} & b_{22} & 2b_{23} & 2b_{24} \\ b_{31} & b_{32} & 2b_{33} & 2b_{34} \\ b_{41} & b_{42} & 2b_{43} & 2b_{44} \end{pmatrix} \tag{3.0.13}$$

We have given as,

$$BA = AB \tag{3.0.14}$$

$$\implies C = D \tag{3.0.15}$$

it is possible only when,

$$b_{13} = b_{14} = b_{23} = b_{24} = 0 \tag{3.0.16}$$

And,

$$b_{31} = b_{32} = b_{41} = b_{42} = 0 \tag{3.0.17}$$

$$B_{12} = \begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.18}$$

And,

$$B_{21} = \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.19}$$

Hence, therefore matrix B becomes,

$$B = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ 0 & 0 & 2b_{33} & 2b_{34} \\ 0 & 0 & 2b_{43} & 2b_{44} \end{pmatrix} \tag{3.0.20}$$

$$\implies B = \begin{pmatrix} c_1 B_{11} & \mathbf{0} \\ \mathbf{0} & c_2 B_{22} \end{pmatrix} \tag{3.0.21}$$

$$\implies B_{ij} = 0, \forall i \neq j \tag{3.0.22}$$

Now the basis of the $n \times n$ matrices for vector space of all $n \times n$ matrix B are,

$$\beta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.23}$$

$$\beta_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.24}$$

$$\beta_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.0.25}$$

$$\beta_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \beta_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.0.26}$$

Thus, Dimension of $\mathbf{V}$ (vector space of all $n \times n$ matrices B) = 8,

Also

$${d_1}^2 + {d_2}^2 = 2^2 + 2^2 = 8. \tag{3.0.27}$$

Therefore, Dimension of $\mathbf{V}$ (vector space of all $n \times n$ matrix B is :

$$d_1^2 + d_2^2 \tag{3.0.28}$$