Matrix Theory Assignment 16

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EE20RESCH11005

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_16

1 Problem

Consider a matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix} \tag{1.0.1}$$

and,

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{1.0.2}$$

Then which of following is true,

1) **A** and **B** is similar over the field of rational numbers.
2) **A** is diagonalizable over the field of rational numbers $\mathbb{Q}$.
3) **B** is the Jordan canonical form of **A**.
4) The minimal polynomial and the characteristic polynomial of **A** are the same.

2 solution

2.1 Part 1

Two matrix are said to be similar if their eigen values are same.
Eigen value of **A** is given as:

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0 \tag{2.1.1}$$

$$\implies -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0 \tag{2.1.2}$$

$$\implies \lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 3. \tag{2.1.3}$$

Similarally, eigen values of **B** is givem as:

$$\begin{pmatrix} 2-\lambda & 10 & \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} \tag{2.1.4}$$

$$\implies -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0 \tag{2.1.5}$$

$$\implies \lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 3. \tag{2.1.6}$$

Hence, matrices **A** and **B** are similar.

2.2 Part B

Matrix **A** is diagonalizable if and only if there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of **A**.

From (2.1.3) , our eigenvalues for **A** are,

$$\lambda_1 = \lambda_2 = 2 \tag{2.2.1}$$

and,

$$\lambda_3 = 3. \tag{2.2.2}$$

Hence $\lambda_1 = \lambda_2$ is a repeated root with multiplicity two. Hence, We can get only two linearly independent eigenvectors for **A**, are given as :

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} and, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \tag{2.2.3}$$

But any basis for $\mathbb{R}^3$ consists of three vectors. Therefore there is no third eigenbasis for **A**, hence **A** is not diagonalizable.

2.3 part 3

From (2.1.3) we have eigenvalue $\lambda_1 = 2$ with geometic multiplicity 2. Hence the Jordon canonical form of **A** can be written as :

$$\mathbf{J_A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{2.3.1}$$

Hence **B** is the Jordan canonical form of **A**.

2.4 Part 4

From (2.1.3), the characteristic polynomial of this matrix is:

$$f(\lambda) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = (\lambda - 2)^2(\lambda - 3) \tag{2.4.1}$$

Minimal polynomial for a matrix is a smallest polynomial for which

$$M_{\mathbf{A}}(x) = 0 \tag{2.4.2}$$

Using (2.4.2), we found minimal polynomial of **A** is :

$$M_{\mathbf{A}}(x) = (x - 2)^2(x - 3) \tag{2.4.3}$$

We can relate the minimal polynomial with the size of Jordan block.

Size of Jordan block = degree of minimal polynomial with geometic multiplicity of the eigen values.

From (2.4.3) we can observe that, geometric multiplicity of eigen value 2 is 2. Hence size of Jordan block is 2. which is given as:

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \tag{2.4.4}$$

if geometric multiplicity of $\lambda = 2$ would be 3, then Jordan block would be:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \tag{2.4.5}$$

In (2.4.3) geometric multiplicity of eigen value 2 is 2, and geometric multiplicity of eigen value 3 is one hence jardon block is:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{2.4.6}$$