Matrix Theory Assignment 15

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Abstract—**This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

1 PROBLEM

Let **V** be the space of $n \times n$ matrices over a field $\mathbb{F}$, and let A be a fixed $n \times n$ matrix over field $\mathbb{F}$. Define a linear operator **T** on **V** by the equation

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} - \mathbf{BA} \qquad (1.0.1)$$

Prove that if **A** is a nilpotent matrix, then **T** is a nilpotent operator.

2 SOLUTION

Since **A** is a nilpotent matrix, hence for a positive value K:

$$\mathbf{A}^K = \mathbf{0} \qquad (2.0.1)$$

Now we have

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} - \mathbf{BA} \qquad (2.0.2)$$

$$\implies \mathbf{T}^2(\mathbf{B}) = \mathbf{T}(\mathbf{T}(\mathbf{B})) = \mathbf{T}(\mathbf{AB} - \mathbf{BA}) \qquad (2.0.3)$$

$$\mathbf{T}(\mathbf{AB} - \mathbf{BA}) = \mathbf{A}^2\mathbf{B} - \mathbf{ABA} - \mathbf{ABA} + \mathbf{BA}^2$$
$$= \mathbf{A}^2 - 2\mathbf{ABA} + \mathbf{BA}^2 \qquad (2.0.4)$$

$$\implies \mathbf{T}^3(\mathbf{B}) = \mathbf{T}(\mathbf{T}^2(\mathbf{B}))$$
$$= \mathbf{A}^3 - 3\mathbf{A}^2\mathbf{BA} + 3\mathbf{ABA}^2 - \mathbf{BA}^3 \qquad (2.0.5)$$

Hence, from (2.0.5) we can say that, as we are increasing the power of operator **T**, the power of **A** is also increasing in every terms. Hence there exist a value P such that :

$$\mathbf{T}^P(\mathbf{B}) = \mathbf{0} \qquad (2.0.6)$$

Hence, if **A** is a nilpotent matrix then operator **T** is also a nilpotent operator.

3 EXAMPLE

Let consider a matrices **A** and, B as :

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad (3.0.1)$$

$$\mathbf{A}.\mathbf{B} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} \neq \mathbf{0} \qquad (3.0.2)$$

Now,

$$\implies \mathbf{A}^2 = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad (3.0.3)$$

$$\implies \mathbf{A}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad (3.0.4)$$

Hence $K = 2$.

And we have also,

$$\mathbf{ABA} = \begin{pmatrix} -16 & 16 \\ -16 & 16 \end{pmatrix} \neq \mathbf{0} \qquad (3.0.5)$$

Hence from (2.0.3) we conclude, $P \neq 2$. Now putting the value of $\mathbf{A}^3$ from (3.0.4) and value of $\mathbf{A}^2$ from (3.0.3) in (2.0.5) we get,

$$\mathbf{T}^3(\mathbf{B}) = \mathbf{0} \qquad (3.0.6)$$

Hence, $P = 3$ for this example.