Matrix Theory Assignment 17

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EE20RESCH11005

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_17

1 Problem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. such that

$$\int_{\mathbb{R}^n} |f(\mathbf{x})d\mathbf{x}| < \infty \tag{1.0.1}$$

Let A be a real $n \times n$ invertible matrix and for $x, y \in \mathbb{R}^n$. Let $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes the standard inner product in $\mathbb{R}^n$ then, $\int_{\mathbb{R}^n} f(\mathbf{Ax})e^{i\langle \mathbf{y},\mathbf{x}\rangle} d\mathbf{x}$ =?

1) $\int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle (\mathbf{A}^{-1})^T\mathbf{y},\mathbf{x}\rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|}$
2) $\int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle \mathbf{A}^T\mathbf{y},\mathbf{x}\rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|}$.
3) $\int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle (\mathbf{A}^T)^{-1}\mathbf{y},\mathbf{x}\rangle} d\mathbf{x}$.
4) $\int_{\mathbb{R}^n} f(\mathbf{x})e^{i\langle \mathbf{A}^{-1}\mathbf{y},\mathbf{x}\rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|}$.

2 solution

Let consider,

$$\mathbf{Ax} = \mathbf{t} \tag{2.0.1}$$

$$\implies \mathbf{x} = \mathbf{A}^{-1}\mathbf{t} \tag{2.0.2}$$

$$\implies d\mathbf{x} = \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \tag{2.0.3}$$

Using (2.0.1) to (2.0.3), We can write:

$$\int_{\mathbb{R}^n} f(\mathbf{Ax})e^{i\langle \mathbf{y},\mathbf{x}\rangle} d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{t})e^{i\langle \mathbf{y},(\mathbf{A}^{-1}\mathbf{t})\rangle} \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \tag{2.0.4}$$

We know that,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x} \tag{2.0.5}$$

$$\implies \langle \mathbf{y}, (\mathbf{A}^{-1}\mathbf{t}) \rangle = (\mathbf{y}^T\mathbf{A}^{-1}\mathbf{t}) \tag{2.0.6}$$

And

$$\implies \langle (\mathbf{A}^{-1})^T\mathbf{y}, \mathbf{t} \rangle = ((\mathbf{A}^{-1})^T\mathbf{y})^T\mathbf{t} \tag{2.0.7}$$

$$\implies ((\mathbf{A}^{-1})^T\mathbf{y})^T\mathbf{t} = (\mathbf{y}^T((\mathbf{A}^{-1})^T)^T\mathbf{t}) = (\mathbf{y}^T\mathbf{A}^{-1}\mathbf{t}) \tag{2.0.8}$$

Hence, from (2.0.6) and (2.0.8)

$$\langle \mathbf{y}, (\mathbf{A}^{-1}\mathbf{t}) \rangle = \langle (\mathbf{A}^{-1})^T\mathbf{y}, \mathbf{t} \rangle \tag{2.0.9}$$

Using (2.0.9) in (2.0.4) We can write,

$$\implies \int_{\mathbb{R}^n} f(\mathbf{t})e^{i\langle (\mathbf{A}^{-1})^T\mathbf{y},\mathbf{t}\rangle} \frac{d\mathbf{t}}{|\det(\mathbf{A})|} \tag{2.0.10}$$

replacing variable $\mathbf{t}$ with $\mathbf{x}$.

$$\implies \int_{\mathbb{R}^n} f(\mathbf{t})e^{i\langle (\mathbf{A}^{-1})^T\mathbf{y},\mathbf{x}\rangle} \frac{d\mathbf{x}}{|\det(\mathbf{A})|} \tag{2.0.11}$$

Hence option 1 is correct.