Matrix Theory : Assignment 3

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***Abstract*—This problem is to demonstrate the way to prove a triangle as isosceles using matrix algebra.**

1 Problem

ABC is a triangle in which altitudes BE and CF to sides AC and AB, are equal. Show that

1) AB = AC i.e, $\triangle$ABC is an isosceles triangle.

2 Solution

Let consider we have a triangle $\triangle$ABC. There are two altitudes BE and CF being drawn from the vertices B and C respectively. And we have ,

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{2.0.1}$$

Let $\mathbf{m}_{AB}$ and $\mathbf{m}_{CF}$ are the direction vectors of AB and CF respectively. Since AB $\perp$ CF hence,

$$\mathbf{m}_{AB}\mathbf{m}_{CF} = 0 \tag{2.0.2}$$

$$(\mathbf{B}-\mathbf{E})^{\mathbf{T}}(\mathbf{A}-\mathbf{C}) = \mathbf{0} \tag{2.0.3}$$

$$(\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{E})^{\mathbf{T}}(\mathbf{A}-\mathbf{C}) = \mathbf{0} \tag{2.0.4}$$

$$(\mathbf{B}-\mathbf{A})^{\mathbf{T}}(\mathbf{A}-\mathbf{C}) + \|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{C}) = \mathbf{0} \tag{2.0.5}$$

Similarly, AC $\perp$ BE hence,

$$\mathbf{m}_{AC}\mathbf{m}_{BE} = 0 \tag{2.0.6}$$

$$(\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) = \mathbf{0} \tag{2.0.7}$$

$$(\mathbf{C}-\mathbf{A}+\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) = \mathbf{0} \tag{2.0.8}$$

$$(\mathbf{C}-\mathbf{A})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) + \|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) = \mathbf{0} \tag{2.0.9}$$

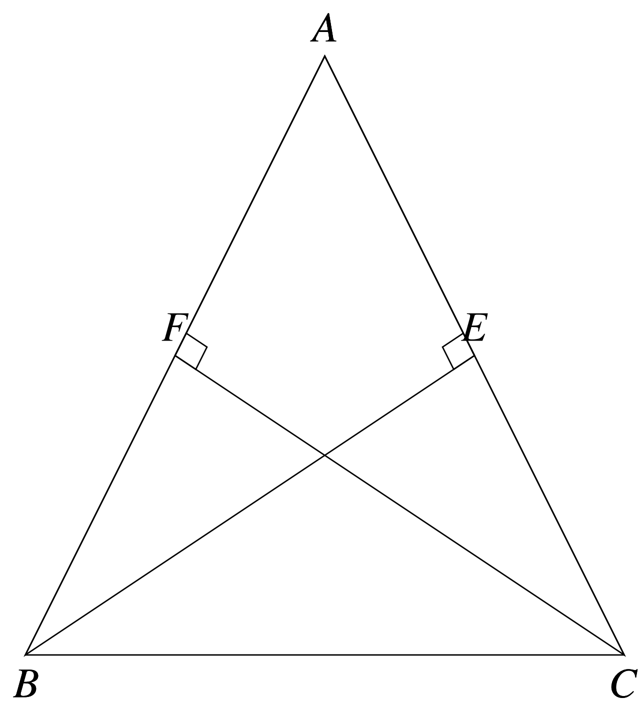

Fig. 1: Isosceles triangle ABC

In $\triangle$ABC, taking inner product of sides AB and AC we can write :

$$\implies \cos BAC = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{2.0.10}$$

and,

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{B}) = \|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{B}\|\cos CAB \tag{2.0.11}$$

From equation 2.0.10, and 2.0.11, we have ,

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C}) = (\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{B}) \tag{2.0.12}$$

using equation 2.0.12 in 2.0.5 and 2.0.9 we can write,

$$\|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{C}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{2.0.13}$$

$$\|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{C}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \quad (2.0.14)$$

$$\|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{C}-\mathbf{A}+\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \quad (2.0.15)$$

$$\|\mathbf{A}-\mathbf{C}\|^2+\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})+(\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{A}-\mathbf{C})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) + (\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \quad (2.0.16)$$

since BE $\perp$ AC and CF $\perp$ AB, hence :

$$(\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \mathbf{0} \quad (2.0.17)$$

and,

$$(\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) = \mathbf{0} \quad (2.0.18)$$

Now equation 2.0.16 become :

$$2\|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) = 2\|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{A}-\mathbf{C})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \quad (2.0.19)$$

Using equation 2.0.12 in equation 2.0.19,

$$\|\mathbf{A}-\mathbf{C}\| = \|\mathbf{A}-\mathbf{B}\| \quad (2.0.20)$$