

# Matrix Theory : Assignment 3

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**Abstract**—This problem is to demonstrate the way to prove a triangle as isosceles using matrix algebra.

## 1 PROBLEM

ABC is a triangle in which altitudes BE and CF to sides AC and AB, are equal. Show that

1)  $AB = AC$  i.e,  $\triangle ABC$  is an isosceles triangle.

## 2 SOLUTION

Let consider we have a triangle  $\triangle ABC$ . There are two altitudes BE and CF being drawn from the vertices B and C respectively. And we have ,

$$\|\mathbf{E} - \mathbf{B}\| = \|\mathbf{F} - \mathbf{C}\| \quad (2.0.1)$$

Let  $\mathbf{m}_{AB}$  and  $\mathbf{m}_{CF}$  are the direction vectors of AB and CF respectively. Since  $AB \perp CF$  hence,

$$\mathbf{m}_{AB}\mathbf{m}_{CF} = 0 \quad (2.0.2)$$

$$(\mathbf{B} - \mathbf{E})^T (\mathbf{A} - \mathbf{C}) = 0 \quad (2.0.3)$$

$$(\mathbf{B} - \mathbf{A} + \mathbf{A} - \mathbf{C} + \mathbf{C} - \mathbf{E})^T (\mathbf{A} - \mathbf{C}) = 0 \quad (2.0.4)$$

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{A} - \mathbf{C}) + \|\mathbf{A} - \mathbf{C}\|^2 + (\mathbf{C} - \mathbf{E})^T (\mathbf{A} - \mathbf{C}) = 0 \quad (2.0.5)$$

Similarly,  $AC \perp BE$  hence,

$$\mathbf{m}_{AC}\mathbf{m}_{BE} = 0 \quad (2.0.6)$$

$$(\mathbf{C} - \mathbf{F})^T (\mathbf{A} - \mathbf{B}) = 0 \quad (2.0.7)$$

$$(\mathbf{C} - \mathbf{A} + \mathbf{A} - \mathbf{B} + \mathbf{B} - \mathbf{F})^T (\mathbf{A} - \mathbf{B}) = 0 \quad (2.0.8)$$

$$(\mathbf{C} - \mathbf{A})^T (\mathbf{A} - \mathbf{B}) + \|\mathbf{A} - \mathbf{B}\|^2 + (\mathbf{B} - \mathbf{F})^T (\mathbf{A} - \mathbf{B}) = 0 \quad (2.0.9)$$

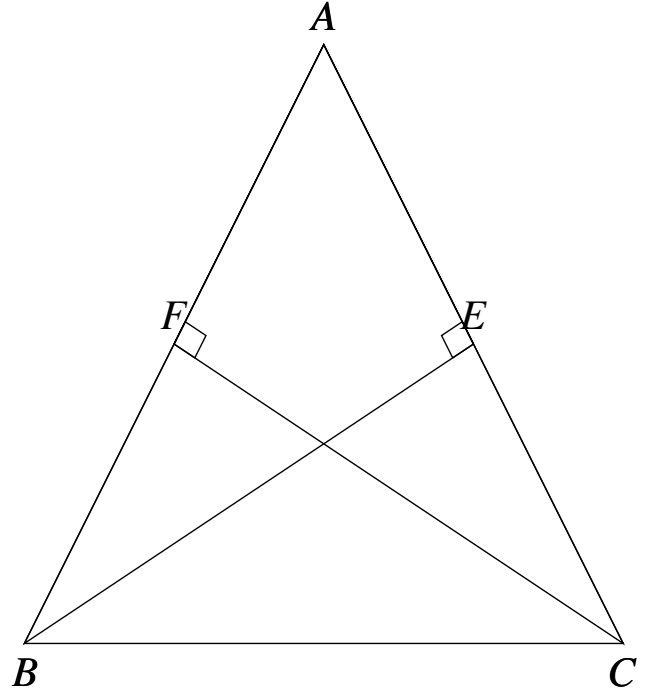


Fig. 1: isosceles triangle ABC

In  $\triangle ABC$ , taking inner product of sides AB and AC we can write :

$$\Rightarrow \cos BAC = \frac{(\mathbf{B} - \mathbf{A})^T (\mathbf{A} - \mathbf{C})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{A} - \mathbf{C}\|} \quad (2.0.10)$$

and,

$$(\mathbf{C} - \mathbf{A})^T (\mathbf{A} - \mathbf{B}) = \|\mathbf{C} - \mathbf{A}\| \|\mathbf{A} - \mathbf{B}\| \cos CAB \quad (2.0.11)$$

From equation 2.0.10, and 2.0.11, we have ,

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{A} - \mathbf{C}) = (\mathbf{C} - \mathbf{A})^T (\mathbf{A} - \mathbf{B}) \quad (2.0.12)$$

using equation 2.0.12 in 2.0.5 and 2.0.9 we can write,

$$\|\mathbf{A} - \mathbf{C}\|^2 + (\mathbf{C} - \mathbf{E})^T (\mathbf{A} - \mathbf{C}) = \|\mathbf{A} - \mathbf{B}\|^2 + (\mathbf{B} - \mathbf{F})^T (\mathbf{A} - \mathbf{B}) \quad (2.0.13)$$

$$\begin{aligned} \|\mathbf{A} - \mathbf{C}\|^2 + (\mathbf{C} - \mathbf{E})^T (\mathbf{A} - \mathbf{C}) = \\ \|\mathbf{A} - \mathbf{B}\|^2 + (\mathbf{B} - \mathbf{F})^T (\mathbf{A} - \mathbf{B}) \end{aligned} \quad (2.0.14)$$

$$\begin{aligned} \|\mathbf{A} - \mathbf{C}\|^2 + (\mathbf{C} - \mathbf{A} + \mathbf{A} - \mathbf{B} + \mathbf{B} - \mathbf{E})^T (\mathbf{A} - \mathbf{C}) = \\ \|\mathbf{A} - \mathbf{B}\|^2 + (\mathbf{B} - \mathbf{A} + \mathbf{A} - \mathbf{C} + \mathbf{C} - \mathbf{F})^T (\mathbf{A} - \mathbf{B}) \end{aligned} \quad (2.0.15)$$

$$\begin{aligned} \|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{A} - \mathbf{C}\|^2 + (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{C}) + (\mathbf{B} - \mathbf{E})^T \\ (\mathbf{A} - \mathbf{C}) = \|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{A} - \mathbf{B}\|^2 + (\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{B}) \\ + (\mathbf{C} - \mathbf{F})^T (\mathbf{A} - \mathbf{B}) \end{aligned} \quad (2.0.16)$$

since  $\mathbf{BE} \perp \mathbf{AC}$  and  $\mathbf{CF} \perp \mathbf{AB}$ , hence :

$$(\mathbf{B} - \mathbf{E})^T (\mathbf{A} - \mathbf{C}) = \mathbf{0} \quad (2.0.17)$$

and,

$$(\mathbf{C} - \mathbf{F})^T (\mathbf{A} - \mathbf{B}) = \mathbf{0} \quad (2.0.18)$$

Now equation 2.0.16 become :

$$\begin{aligned} 2 \|\mathbf{A} - \mathbf{C}\|^2 + (\mathbf{A} - \mathbf{B})^T (\mathbf{A} - \mathbf{C}) = \\ 2 \|\mathbf{A} - \mathbf{B}\|^2 + (\mathbf{A} - \mathbf{C})^T (\mathbf{A} - \mathbf{B}) \end{aligned} \quad (2.0.19)$$

Using equation 2.0.12 in equation 2.0.19,

$$\|\mathbf{A} - \mathbf{C}\| = \|\mathbf{A} - \mathbf{B}\| \quad (2.0.20)$$