Matrix Theory : Assignment 3

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***Abstract*—This problem is to demonstrate the way to prove the triangles are congruent and to prove a triangle as isosceles using matrix algebra.**

1 Problem

ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that

1) $\triangle$ABC $\cong$ $\triangle$ACF
2) AB = AC i.e, $\triangle$ABC is an isosceles triangle.

2 Solution

2.1 part 1

Let consider we have a triangle $\triangle$ABC. There are two altitudes BE and CF being drawn from the vertices B and C respectively.
In $\triangle$ABE, taking inner product of sides AE and EB we can write :

$$(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B}) = \|\mathbf{A}-\mathbf{E}\|\,\|\mathbf{E}-\mathbf{B}\|\cos\text{AEB} \tag{2.1.1}$$

$$\implies \cos\text{AEB} = \frac{(\mathbf{A}-\mathbf{E})^T(\mathbf{E}-\mathbf{B})}{\|\mathbf{A}-\mathbf{E}\|\,\|\mathbf{E}-\mathbf{B}\|} \tag{2.1.2}$$

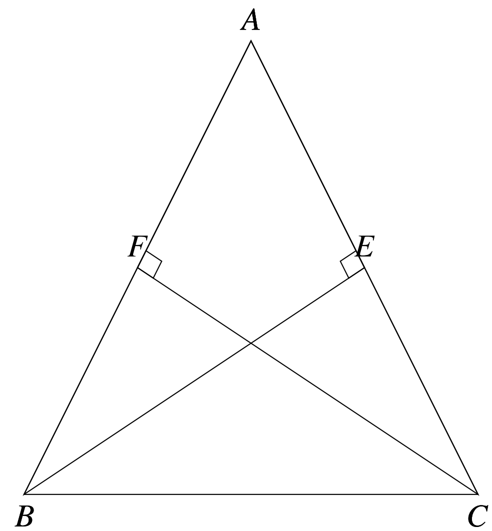
In $\triangle$ACF, taking inner product of sides AF and FC :

$$(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C}) = \|\mathbf{A}-\mathbf{F}\|\,\|\mathbf{F}-\mathbf{C}\|\cos\text{AFC} \tag{2.1.3}$$

$$\implies \cos\text{AFC} = \frac{(\mathbf{A}-\mathbf{F})^T(\mathbf{F}-\mathbf{C})}{\|\mathbf{A}-\mathbf{F}\|\,\|\mathbf{F}-\mathbf{C}\|} \tag{2.1.4}$$

In triangle $\triangle$ABC,

$$\cos\text{AFC} = \cos\text{AEB}\,(\text{CF} \perp \text{AB}\ \&\ \text{BE} \perp \text{AC}) \tag{2.1.5}$$



Given,

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{2.1.6}$$

$$\angle FAC = \angle EAB\,(\text{Common angle}) \tag{2.1.7}$$

We know that if the two angles of triangles are equal then the third angle will also be equal.Hence,

$$\angle FCA = \angle EBA \tag{2.1.8}$$

Hence by ASA (Angle - Side - Angle) We can say that ,

$$\triangle\text{ABC} \cong \triangle\text{ACF}. \tag{2.1.9}$$

2.2 part 2

we have given that ,

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{2.2.1}$$

Hence we know that if the two sides of the triangle are equal then angles opposite to them are also equal. So we can have

Let $\mathbf{m}_{AB}$ and $\mathbf{m}_{CF}$ are the direction vectors of AB and CF respectively. Since AB $\perp$ CF hence,

$$\mathbf{m}_{AB}\mathbf{m}_{CF} = 0 \tag{2.2.2}$$

$$(\mathbf{B}-\mathbf{E})^{\mathbf{T}}(\mathbf{A}-\mathbf{C})=\mathbf{0} \tag{2.2.3}$$

$$(\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{E})^{\mathbf{T}}(\mathbf{A}-\mathbf{C})=\mathbf{0} \tag{2.2.4}$$

$$(\mathbf{B}-\mathbf{A})^{\mathbf{T}}(\mathbf{A}-\mathbf{C})+\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{C})=\mathbf{0} \tag{2.2.5}$$

Similarly, AC $\perp$ BE hence,

$$\mathbf{m}_{AC}\mathbf{m}_{BE}=0 \tag{2.2.6}$$

$$(\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B})=\mathbf{0} \tag{2.2.7}$$

$$(\mathbf{C}-\mathbf{A}+\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B})=\mathbf{0} \tag{2.2.8}$$

$$(\mathbf{C}-\mathbf{A})^{\mathbf{T}}(\mathbf{A}-\mathbf{B})+\|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B})=\mathbf{0} \tag{2.2.9}$$

In $\triangle$ABC, taking inner product of sides AB and AC we can write :

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})=\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|\cos\mathrm{BAC} \tag{2.2.10}$$

$$\implies \cos\mathrm{BAC}=\frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{2.2.11}$$

and,

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{B})=\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{B}\|\cos\mathrm{CAB} \tag{2.2.12}$$

$$\implies \cos\mathrm{CAB}=\frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{B}\|} \tag{2.2.13}$$

From equation 2.2.11, and 2.2.13, we have ,

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C}) = (\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{B}) \tag{2.2.14}$$

using equation 2.2.14 in 2.2.5 and 2.2.9 we can write,

$$\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{C}-\mathbf{E})^T(\mathbf{A}-\mathbf{C})= \|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{2.2.15}$$

$$\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{C}-\mathbf{E})^T(\mathbf{A}-\mathbf{C})= \|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{2.2.16}$$

$$\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{C}-\mathbf{A}+\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C})= \|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{2.2.17}$$

$$\|\mathbf{A}-\mathbf{C}\|^2+\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})+(\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C})=\|\mathbf{A}-\mathbf{B}\|^2+\|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{A}-\mathbf{C})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) + (\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{2.2.18}$$

since BE $\perp$ AC and CF $\perp$ AB, hence :

$$(\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C})=\mathbf{0} \tag{2.2.19}$$

and,

$$(\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B})=\mathbf{0} \tag{2.2.20}$$

Now equation 2.2.18 become :

$$2\,\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})= 2\,\|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{A}-\mathbf{C})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{2.2.21}$$

Using equation 2.2.14 in equation 2.2.21,

$$\|\mathbf{A}-\mathbf{C}\|=\|\mathbf{A}-\mathbf{B}\| \tag{2.2.22}$$