Matrix Theory Assignment 14

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***Abstract*—This problem is all about to introducing the concept of characteristic polynomial over a filed.**

All the codes for this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_14

1 Problem

Let $\mathbf{V}$ be a real vector space and E an idempotent linear operator on $\mathbf{V}$, i.e., a projection. Prove that $\left(\mathbf{I}+\mathbf{E}\right)$ is invertible. Find $\left(\mathbf{I}+\mathbf{E}\right)^{-1}$.

2 solution

Since $\mathbf{E}$ is an idempotent matrix, that is :

$$\mathbf{E}^2 = \mathbf{E} \tag{2.0.1}$$

Hence it will satisfy the polynomial,

$$x^2 - x = 0 \tag{2.0.2}$$

Thus minimal polynomial will be,

$$m_{\mathbf{E}}(x) = x^2 - x = 0 \tag{2.0.3}$$
$$\implies m_{\mathbf{E}}(x) = x(x-1) = 0 \tag{2.0.4}$$

Hence minimal polynomial $m_{\mathbf{E}}(x)$, of $\mathbf{E}$ divides $x^2 - x$ that is,

$$m_{\mathbf{E}}(x) = x \tag{2.0.5}$$

or,

$$m_{\mathbf{E}}(x) = x - 1 \tag{2.0.6}$$

if

$$m_{\mathbf{E}}(x) = x \tag{2.0.7}$$
$$\implies \mathbf{E} = \mathbf{0} \tag{2.0.8}$$

if

$$m_{\mathbf{E}}(x) = x - 1 \tag{2.0.9}$$
$$\implies \mathbf{E} = \mathbf{I} \tag{2.0.10}$$

Hence, if $\mathbf{E}$ is idempotent then, minimal polynomial of $\mathbf{E}$ is product of distinct polynomial of degree one.Thus matrix $\mathbf{E}$ is similar to diagonal matrix with diagonal entries consisting of characteristic value 0 and 1.
Since $\mathbf{E}$ is diagonalizable, then there exist at least one basis such that :

$$ß = \{\beta_1, \beta_2 \ldots, \beta n\} \tag{2.0.11}$$

Such that

$$\mathbf{E}\beta_i = \beta_i, \forall i = 1, 2 \ldots k \tag{2.0.12}$$

and,

$$\mathbf{E}\beta_i = 0, \forall i = k+1, \ldots n. \tag{2.0.13}$$
$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_i = 2\beta_i, \forall i = 1, 2 \ldots k. \tag{2.0.14}$$

And,

$$\implies \left(\mathbf{I}+\mathbf{E}\right)\beta_i = \beta_i, \forall i = k+1 \ldots n. \tag{2.0.15}$$

In matrix form we can write it as:

$$\left(\mathbf{I}+\mathbf{E}\right) = \begin{pmatrix} 2\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_2} \end{pmatrix} \tag{2.0.16}$$

Where, $\mathbf{I_1}$ is $k \times k$ and $\mathbf{I_2}$ is $(n-k) \times (n-k)$ identity matrices, and each $\mathbf{0}$ represents the zero matrix of appropriate dimension.
From 2.0.16 we can calculate the determinant as:

$$\det\left(\mathbf{I}+\mathbf{E}\right) = 2^k \neq 0 \tag{2.0.17}$$

From 2.0.16 we can observe that, eigen values of $\left(\mathbf{I}+\mathbf{E}\right)$ are kth number of 2 and $(n-k)^{th}$ number of 1. hence none of the eigen value of the matrix is zero. hence it is invertible.
Since $\left(\mathbf{I}+\mathbf{E}\right)$ is combination of identity matrices. Hence, from (2.0.16), inverse of the matrix $\left(\mathbf{I}+\mathbf{E}\right)$ is given as :

Let,

$$A = I + E \tag{2.0.18}$$
$$\implies E = A - I \tag{2.0.19}$$

$$\implies E^2 = (A - I)(A - I) = A^2 - 2A + I^2 \tag{2.0.20}$$

from 2.0.1 $E^2 = E$),

$$\implies E = A^2 - 2A + I \tag{2.0.21}$$

Using (2.0.19) we have,

$$\implies A - I = A^2 - 2A + I = A^2 - 3A + 2I = 0 \tag{2.0.22}$$

$$\implies I = \frac{3A - A^2}{2} \tag{2.0.23}$$

multiplying A^{-1} both side,

$$A^{-1} = \frac{3I - A}{2} = \frac{3I - (I + E)}{2} \tag{2.0.24}$$
$$A^{-1} = \frac{2I - E}{2} \tag{2.0.25}$$

Using (2.0.19), we have,

$$(I + E)^{-1} = I - \frac{E}{2} \tag{2.0.26}$$

Hence,

$$\left(\mathbf{I} + \mathbf{E}\right)^{-1} = \mathbf{I} - \frac{1}{2}\mathbf{E} \tag{2.0.27}$$

3 Example

Let consider a matrix $\mathbf{E}$ as :

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.1}$$
$$\implies \mathbf{E}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.2}$$
$$\implies \mathbf{E}^2 = E. \tag{3.0.3}$$

Basis for this matix is

$$\beta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{and}, \beta_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.4}$$

We have,

$$\mathbf{E}\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta_1 \tag{3.0.5}$$

And,

$$\mathbf{E}\beta_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \beta_2 \tag{3.0.6}$$

Now,

$$\mathbf{I} + \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.7}$$
$$\implies \mathbf{I} + \mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.8}$$
$$\implies \left(\mathbf{I} + \mathbf{E}\right)\beta_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \tag{3.0.9}$$
$$\implies \left(\mathbf{I} + \mathbf{E}\right)\beta_1 = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\beta_1 \tag{3.0.10}$$

Similarly,

$$\implies \left(\mathbf{I} + \mathbf{E}\right)\beta_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.11}$$
$$\implies \left(\mathbf{I} + \mathbf{E}\right)\beta_1 = 2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \beta_2 \tag{3.0.12}$$
$$\implies \mathbf{I} + \mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\mathbf{I_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_1} \end{pmatrix} \tag{3.0.13}$$

Now let find the eigen value of matrix $\left(\mathbf{I} + \mathbf{E}\right)$:

$$\implies \begin{pmatrix} 2 - \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix} = 0 \tag{3.0.14}$$
$$\implies (2 - \lambda)(1 - \lambda) = 0 \tag{3.0.15}$$
$$\implies \lambda_1 = 2, \lambda_2 = 1 \tag{3.0.16}$$

The eigen values of the matrix $(\mathbf{I} + \mathbf{E})$ from (3.0.16) are 2 and 1. Since none of the eigen value is zero, hence matrix is invertible.

Inverse of the matrix (2.0.27) is :

$$\left(\mathbf{I} + \mathbf{E}\right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.17}$$