Matrix Theory Assignment 7

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***Abstract*—This problem demonstrate a method to find the solution of the given system of equation using linear algebra.**

All the codes for the figure in this document can be found at

https://github.com/Ritesh622/Assignment_EE5609/tree/master/Assignment_7

1 Problem

Consider the system of the equations

$$x_1 - x_2 + 2x_3 = 1 \tag{1.0.1}$$
$$x_1 - 0x_2 + 2x_3 = 1 \tag{1.0.2}$$
$$x_1 - 3x_2 + 4x_3 = 2 \tag{1.0.3}$$

Does this system have a solution ? If so describe explicitly all solutions.

2 Solution

Let $\mathbf{V}$ is the set of all $\begin{pmatrix} x_1, & x_2, & x_3 \end{pmatrix} \in \mathbb{R}^3$ which satisfy the (1.0.1), (1.0.2) and (1.0.3)
From equation (1.0.1) to (1.0.3) we can write,

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{2.0.1}$$
$$\implies \mathbf{A}\mathbf{x} = \mathbf{b} \tag{2.0.2}$$

Where,

$$\tag{2.0.3}$$
$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2.0.4}$$

Solving the matrix $\mathbf{A}$ for rank we get,

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 2 \\ 1 & -3 & 4 \end{pmatrix} \xleftrightarrow{R_2 = R_1 - 2R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix} \tag{2.0.5}$$
$$\xleftrightarrow{R_3 = R_3 - R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \tag{2.0.6}$$
$$\xleftrightarrow{R_3 = R_3 + R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.0.7}$$

Hence, rank ($\mathbf{A}$) = 2. Now solving the augmented matrix of (2.0.2) we get,

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_2 = R_1 - 2R_1} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \tag{2.0.8}$$
$$\xleftrightarrow{R_3 = R_3 - R_1} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{pmatrix} \tag{2.0.9}$$
$$\xleftrightarrow{R_3 = R_3 + R_2} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.0.10}$$

We have rank ($\mathbf{A}$) = rank ($\mathbf{A} : \mathbf{b}$) = $2 < n$, where $n = 3$. Hence we have infinite no of solutions for given system of equations.
Using Gauss - Jordan elimination method to getting the solution,

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \xleftrightarrow{R_2 = R_1 - 2R_1} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 1 & -3 & 4 & 2 \end{pmatrix} \tag{2.0.11}$$

$$\xleftrightarrow{R_3=R_3-R_1}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 2 & -2 & -1\\0 & -2 & 2 & 1\end{pmatrix} \tag{2.0.12}$$

$$\xleftrightarrow{R_2=\frac{R_2}{2}}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 1 & -1 & -\frac{1}{2}\\0 & -2 & 2 & 1\end{pmatrix} \tag{2.0.13}$$

$$\xleftrightarrow{R_3=R_3+2R_2}\begin{pmatrix}1 & -1 & 2 & 1\\0 & 1 & -1 & -\frac{1}{2}\\0 & 0 & 0 & 0\end{pmatrix} \tag{2.0.14}$$

$$\xleftrightarrow{R_1=R_1+R_2}\begin{pmatrix}1 & 0 & 1 & \frac{1}{2}\\0 & 1 & -1 & -\frac{1}{2}\\0 & 0 & 0 & 0\end{pmatrix} \tag{2.0.15}$$

$$\implies x_1 + x_3 = \frac{1}{2}, x_2 - x_3 = -\frac{1}{2} \tag{2.0.16}$$

$$\implies x_2 = -\frac{1}{2} + x_3, x_1 = \frac{1}{2} - x_3 \tag{2.0.17}$$

From equation (2.0.16) and (2.0.17)

$$\mathbf{x} = \begin{pmatrix}\frac{1}{2} - x_3\\-\frac{1}{2} + x_3\\x_3\end{pmatrix} \tag{2.0.18}$$

which can be written as,

$$\mathbf{x} = x_3\begin{pmatrix}-1\\1\\1\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\0\end{pmatrix} \tag{2.0.19}$$

from 2.0.19 we can say that for any value x_3, $\mathbf{V}$ will no be gives zero vector. Hence the given solution space will not span of the vector space $\mathbf{V}$