EE7330: Network Information Theory **2021**

Lecture Notes 2: Convexity

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. Please email the course instructor in case of any errors.*

2.1 Open set

A set $\mathcal{A}$ is said to be an open set if for every $\underline{x} \in \mathcal{A}$, we can find $\epsilon > 0$ such that,

$$B_\epsilon(\underline{x}) \subseteq \mathcal{A} \tag{2.1}$$

Where $B_\epsilon(\underline{x})$ is an n-dimensional ball with radius ϵ and center $\underline{x}$. No matter which point we are taking , it always lies within the set. Open sets are typically denoted as $(\mathcal{A})$. In general we denote open interval using (). For example (a, b) is open interval and $(4, 5) \cup (6, 8)$ is an open set.

2.2 Closed set

A set $\mathcal{A}$ is said to be closed set if:

- Compliment of the set is an open set.
- Every limit point of the set lies inside the set.
- Closed interval is denoted as [] and $[a, b] \cup [p, q]$ is closed set. But $(a, b) \cup [p, q]$ is neither closed set neither open set.

2.3 Convex set

$\mathcal{A}$ is said to be a convex set if, $\forall\ \underline{x}, \underline{y} \in \mathcal{A}$ and $\forall\ 0 \leqslant \alpha \leqslant 1$

$$\alpha\underline{x} + (1 - \alpha)\underline{y} \in \mathcal{A} \tag{2.2}$$

Geometrically, if we consider any two points inside the set, then every point that lies on the line that joins these two points must lie inside the set.

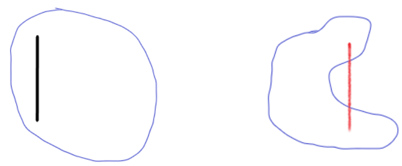

Figure 2.1: (a)Convex set (b) Not a convex set

Example : Achievable rate regions are convex: a time-sharing argument : Consider a multi-user system where we have some L-transmitter and L-receiver and they want to communicate through the channel described in figure 2.2. We want to find the achievable rate over this channel.

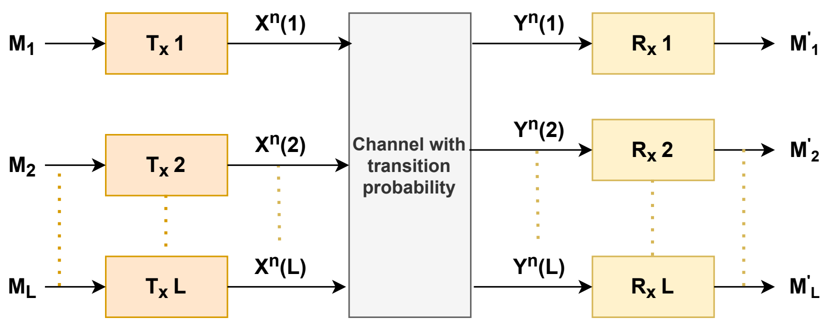

Figure 2.2: (a) Transceiver for L-transmitter and Receiver setting

A rate tuple $(R_1, R_2 \dots R_L)$ is achievable if $\exists$ pair of encoder and decoder.
$(ENC_n(1), ENC_n(2), \dots ENC_n(L), DEC_n(1), DEC_n(2) \dots, DEC_n(L))$ pair can be defined as :
$ENC_n(i) : \{0,1\}^{K_n(i)} \longrightarrow \mathcal{X}^n(l)$ and , $DEC_n(i) : \mathcal{Y}^n(i) \longrightarrow \{0,1\}^{K_n(i)}$ such that :

$$\limsup_{n \to \infty} \frac{K_n(i)}{n} = R_i \tag{2.3}$$

$$\limsup_{n \to \infty} Pr\left[M_i' \neq M_i\right] = P_e = 0 \tag{2.4}$$

And we want to find what kind of rate can be achievable. Consider the tuple (R_1, R_2) i.e (L = 2) which is achievable. For single T$_X$ - R$_X$ setting (i.e $L = 1$), according to the Shannon coding theorem, any rate less than the capacity of the channel is achievable. So, rate region for the single $T_X - R_X$ is a closed interval which is $[0, C]$.

For the 2 user case :
Theorm : Every rate region are convex. i.e if $(R_1, R_2) \,\&\, (R_1', R_2')$ are achievable and they lie in the rate region then $\forall\ 0 \leqslant \alpha \leqslant 1,\quad (\alpha R_1 + (1-\alpha)R_1', \alpha R_2' + (1-\alpha)R_2,)$ is achievable.
Proof : Consider a time sharing argument (R_1, R_2) and (R_1', R_2') for ENC and DEC pair such that :
$K_n(1) \geqslant n\,(R_1 - \epsilon)$ and $K_n(2) \geqslant n\,(R_2 - \epsilon)$ for (ENC, DEC). $K_n'(1) \geqslant n\,(R_1' - \epsilon)$ and $K_n'(2) \geqslant n\,(R_2' - \epsilon)$ for (ENC', DEC')
We have n channel uses and consider αn channel uses by first $(ENC - DEC)$ pair and $(1-\alpha)\,n$ for $(ENC' - DEC')$ pair. Hence

$$\alpha K_n(1) \geqslant \alpha n\,(R_1 - \epsilon) \tag{2.5}$$

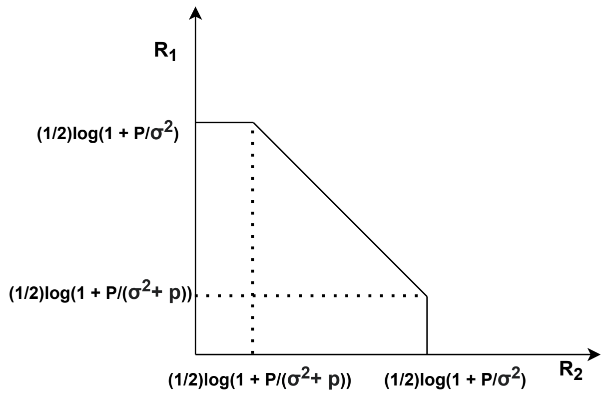

Figure 2.3: 2 user rate tuple case (2 user Gaussian multiple-access channel (MAC) only)

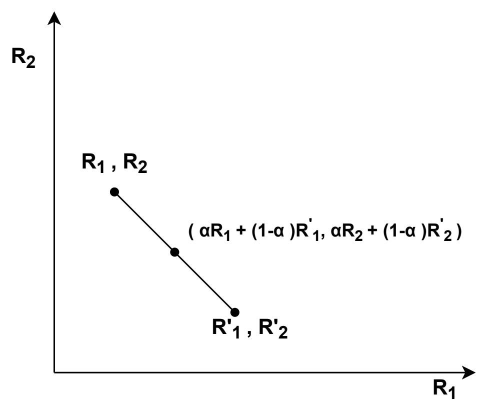

Figure 2.4: 2 user rate case

$$\alpha K_n(2) \geqslant \alpha n\left(R_2 - \epsilon\right) \tag{2.6}$$

and,

$$(1-\alpha)\,K_n'(1) \geqslant (1-\alpha)\,n\left(R_1' - \epsilon\right) \tag{2.7}$$

$$(1-\alpha)\,K_n'(2) \geqslant (1-\alpha)\,n\left(R_2' - \epsilon\right) \tag{2.8}$$

Combining (2.5), (2.7) and (2.6), (2.8), we have

$$\alpha K_n(1) + (1-\alpha)\,K_n'(1) \geqslant \alpha n\left(R_1 - \epsilon\right) + (1-\alpha)\,n\left(R_1' - \epsilon\right) \tag{2.9}$$

$$\alpha K_n(2) + (1-\alpha)\,K_n'(2) \geqslant \alpha n\left(R_2 - \epsilon\right) + (1-\alpha)\,n\left(R_2' - \epsilon\right) \tag{2.10}$$

And it holds $\forall\ \epsilon > 0$.
Now consider, we have pair of encoder and decoder (ENC, DEC) and (ENC', DEC'). First pair of encoder and decoder transmit $\alpha \mathrm{k}_1$ bits in αn channel use. And receiver received αn symbols. Similarly second pair

of encoder and decoder transmit $(1-\alpha)\mathrm{k}_2$ bits in $(1-\alpha)\mathrm{n}$ channel use. And receiver received $\alpha\mathrm{n}$ symbols. Overall rate of this setting is give as :

$$\text{Rate} = \frac{\alpha\mathrm{k}_1 + (1-\alpha)\mathrm{k}_2}{\mathrm{n}} \tag{2.11}$$

Suppose the probability of error for channel first pair of ENC - DEC is P_{e1} and for the second pair is P_{e2}, then For this the overall probability of error is :

$$P_e = P_{e1} + P_{e2} - (P_{e1} \cap P_{e2}) \tag{2.12}$$

Since both are independent.

$$P_e = P_{e1} + P_{e2} - P_{e1}.P_{e2} \tag{2.13}$$

$$\Rightarrow P_e \leqslant P_{e1} + P_{e2} \tag{2.14}$$

Where $P_{e1} = P_e^{\alpha\mathrm{n}}$, $P_{e2} = P_e^{(1-\alpha)\mathrm{n}}$ As we know, if n is sufficiently large, P_{e1} and P_{e2} asymptotically approaches to zero. Hence from 2.12, we can observe the overall probability of error will be zero.