EE7330: Network Information Theory **2021**

Lecture Notes 16: Joint Typicality Lemma

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16.1 Joint typicality lemma

Consider a distribution P_{XYZ} such that $Z^n \sim$ i.i.d $P_{Z|XY}$, if X^n and Y^n are typical, then Z will also be typical. If Z is not drawn from the distribution $P_{Z|XY}$ but it's drawn from another distribution i.e Z $\sim P_{Z|X}$ that is Z is independent to Y.

Lemma 16.1. *Suppose we have sequences x^n and y^n, then for any (x^n, y^n),and $Z^n \sim P^n_{Z|X}(.|x^n)$, then for any $\epsilon > 0$,*

$$Pr\left[(x^n, y^n, z^n) \in T_\epsilon^{(n)}(P_{XYZ})\right] \leqslant 2^{-nI(Y;Z|X)(1+\delta(\epsilon))} \tag{16.1}$$

for some $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.

It gives an upper bound on the probability of 3 sequences being jointly typical. For the upper bound $x^n$and y^n may jointly typical or not. In some cases, we also use a lower bound on the probability that they are jointly typical. Hence similar to the upper bound, we have a lower bound on that as follows

Lemma 16.2. *For any $\epsilon > \epsilon'$,*

$$Pr\left[(x^n, y^n, z^n) \in T_{\epsilon'}^{(n)}(P_{XYZ})\right] > 2^{-nI(Y;Z|X)(1+\delta(\epsilon))} \tag{16.2}$$

for some $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.

For the lower bound, we have to assume that, x^n and y^n are jointly typical. If they are not jointly typical then, we may not have this lower bound.

Proof.

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n(P_{XYZ}))\right] = \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} \prod_{i=1}^{n} P_{Z|X}(z_i|x_i) \tag{16.3}$$

For every $z^n \in T_\epsilon^n\left(P_{XZ|x^n}\right)$, we have

$$Pr_{Z|X}(z^n|x^n) \leqslant 2^{-nH(Z|X)(1-\epsilon)} \tag{16.4}$$

In R.H.S. of equation (16.3), we are summing all the possible sequences in the different typical set (which is not conditional typical set). There we are considering z^n for which we are summing over the is $T_\epsilon^n(P_{XYZ})$ set. Hence the equation (16.3) can be bounded as :

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] = \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} \prod_{i=1}^{n} P_{Z|X}\left(z_i|x_i\right)$$
$$\leqslant \left|T_\epsilon^n\left(P_{XYZ}|x^n, y^n\right)\right| 2^{-nH(Z|X)(1-\epsilon)} \quad (16.5)$$

The size of the typical set is bounded as :

$$\left|T_\epsilon^n\left(P_{XYZ}|x^n, y^n\right)\right| \leqslant 2^{nH(Z|XY)(1+\epsilon)} \quad (16.6)$$

Now using equation (16.4) and equation (16.6) in (16.5),

$$\begin{aligned} Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] &\leqslant 2^{nH(Z|XY)(1+\epsilon)} \times 2^{-nH(Z|X)(1-\epsilon)} \\ &\leqslant 2^{-n[H(Z|X)-H(Z|XY)](1-\epsilon)} \\ &= 2^{-n[I(Z;Y|X)(1-\epsilon)]} \end{aligned}$$

Hence,

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] \leqslant 2^{-n[I(Z;Y|X)(1-\epsilon)]} \quad (16.7)$$

□

Now coming the proof of lemma 1.2

Proof. For the lower bound, we have to assume that, x^n and y^n are jointly typical i.e $((x^n, y^n) \in T_{\epsilon'}^n)$ and $Z^n \sim P_{Z|X}^n\left(.|x^n\right)$

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] = \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} P_{Z|X}\left(z^n|x^n\right) \quad (16.8)$$

$$\sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} P_{Z|X}\left(z^n|x^n\right) \geqslant \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} 2^{-nH(Z|X)(1+\epsilon)} \quad (16.9)$$
$$= 2^{-nH(Z|X)(1+\epsilon)} \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} 1 \quad (16.10)$$

For this condition the size of the typical set is bounded as :

$$\left|T_\epsilon^n\left(P_{XYZ|x^n,y^n}\right)\right| \leqslant 2^{nH(Z|XY)(1+\epsilon)} \quad (16.11)$$

Now using equation (16.11) in equation (16.10) we have,

$$\begin{aligned} Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] &\geqslant 2^{-nH(Z|XY)(1+\epsilon)} \times 2^{nH(Z|X)(1+\epsilon)} \\ &= 2^{-n[H(Z|X)-H(Z|XY)](1+\epsilon)} \\ &= 2^{-n[I(Z;Y|X)(1+\epsilon)]} \\ &= 2^{-n[I(Z;Y|X)+\delta(\epsilon)]} \end{aligned}$$

finally,

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] \geqslant 2^{-n[I(Z;Y|X)+\delta(\epsilon)]} \quad (16.12)$$

This concludes the proof of lemmas 1.1 and 1.2 □