EE7330: Network Information Theory **2021**

Lecture Notes 16: Jointly Typical Lemma

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16.1 Jointly typical lemma

Consider a distribution P_{XYZ} such that Z $\sim$ i.i.d $P_{Z|XY}$, if X and Y are typical, then Z will also be typical. If Z is not driven from the distribution $P_{Z|XY}$ but it's driven from another distribution i.e Z $\sim$ $P_{Z|X}$ that is Z is independent to Y.
Lemma1.1: Suppose we have sequences x^n and y^n, then for any (x^n, y^n),and $Z^n \sim P^n_{Z|X}(.|x^n)$, then for any $\epsilon > 0$,

$$Pr\left[(x^n, y^n, z^n) \in T_\epsilon^{(n)}(P_{XYZ})\right] \leqslant 2^{-nI(Y;Z|X)(1+\delta(\epsilon))} \tag{16.1}$$

for some $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.
It gives an upper bound on the probability of 3 sequences being jointly typical. For the upper bound $x^n$and y^n may jointly typical or not. In some cases, we also use a lower bound on the probability that they are jointly typical. Hence similar to the upper bound, we have a lower bound on that as follows
lemma 1.2 For any $\epsilon > \epsilon'$,

$$Pr\left[(x^n, y^n, z^n) \in T_\epsilon^{(n)}(P_{XYZ})\right] > 2^{-nI(Y;Z|X)(1+\delta(\epsilon))} \tag{16.2}$$

for some $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.
For the lower bound, we have to assume that, x^n and y^n are jointly typical. If they are not jointly typical then, we may not have this lower bound.

Proof.

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n(P_{XYZ}))\right] = \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} \prod_{i=1}^n P_{Z|X}(z_i|x_i) \tag{16.3}$$

$\prod_{i=1}^n P_{Z|X}(z_i|x_i)$ belong to conditional typical set which is $z^n \in T_\epsilon^n\left(P_{XZ|x^n}\right)$ and for condtional typical set, the probability of nay z^n in the set is $\approx 2^{-nH(Z|X)(1-\epsilon)}$, that is

$$Pr_{Z|X}(z^n|x^n) \leqslant 2^{-nH(Z|X)(1-\epsilon)} \tag{16.4}$$

In R.H.S. of equation (16.3), we are summing all the possible sequences in the different typical set (which is not conditional typical set). There we are considering z^n for which we are summing over the is $T_\epsilon^n(P_{XYZ})$ set. Hence the equation (16.3) can be bounded as :

$$\begin{aligned} Pr\left[(x^n, y^n, z^n \in T_\epsilon^n(P_{XYZ}))\right] &= \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} \prod_{i=1}^n P_{Z|X}(z_i|x_i) \\ &\leqslant |T_\epsilon^n(P_{XYZ})|2^{-nH(Z|X)(1-\epsilon)} \end{aligned} \tag{16.5}$$

The size of the typical set is bounded as :

$$|T_\epsilon^n\left(P_{XYZ}\right)| \leqslant 2^{nH(Z|X)(1+\epsilon)} \tag{16.6}$$

Now using equation (16.4) and equation (16.6) in (16.5) ,

$$\begin{aligned} Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] &\leqslant 2^{nH(Z|X)(1+\epsilon)} \times 2^{-nH(Z|X)(1\epsilon)} \\ &\leqslant 2^{-n[H(Z|X)-H(Z|XY)-\epsilon(H(Z|X)-H(Z|XY))]} \\ &= 2^{-n[I(Z;Y|X)(1-\epsilon)]} \end{aligned}$$

Hence,

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] \leqslant 2^{-n[I(Z;Y|X)(1-\epsilon)]} \tag{16.7}$$

Now coming the proof of lemma 1.2
For the lower bound, we have to assume that, x^n and y^n are jointly typical i.e $((x^n, y^n) \in T_{\epsilon'}^n)$ and $Z^n \sim P_{Z|X}^n\left(.|x^n\right)$

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] = \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} P_{Z|X}\left(z^n|x^n\right) \tag{16.8}$$

$$\sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} P_{Z|X}\left(z^n|x^n\right) \geqslant \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} 2^{-nH(Z|X)(1+\epsilon)} \tag{16.9}$$

$$= 2^{-nH(Z|X)(1+\epsilon)} \sum_{z^n:((x^n,y^n,z^n)\in T_\epsilon^n(P_{XYZ}))} 1 \tag{16.10}$$

For this condition the size of the typical set is bounded as :

$$|T_\epsilon^n\left(P_{XYZ}\right)| \leqslant 2^{nH(Z|X)(1-\epsilon)} \tag{16.11}$$

Now using equation (16.11) in equation (16.10) we have,

$$\begin{aligned} Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] &\geqslant 2^{-nH(Z|X)(1+\epsilon)} \times 2^{nH(Z|X)(1-\epsilon)} \\ &= 2^{-n[H(Z|X)-H(Z|XY)-\epsilon(H(Z|X)-H(Z|XY))]} \\ &= 2^{-n[I(Z;Y|X)(1-\epsilon)]} \\ &= 2^{-n[I(Z;Y|X)+\delta(\epsilon)]} \end{aligned}$$

finally,

$$Pr\left[(x^n, y^n, z^n \in T_\epsilon^n\left(P_{XYZ}\right))\right] \geqslant 2^{-n[I(Z;Y|X)+\delta(\epsilon)]} \tag{16.12}$$

This concludes the proof of lemmas 1.1 and 1.2 $\square$