

## Lecture Notes 2: Convexity

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**Disclaimer:** *These notes have not been subjected to the usual scrutiny reserved for formal publications. Please email the course instructor in case of any errors.*

## 2.1 Open set

A set  $\mathcal{A}$  is said to be an open set if for every  $\underline{x} \in \mathcal{A}$ , we can find  $\epsilon > 0$  such that,

$$B_\epsilon(\underline{x}) \subseteq \mathcal{A} \quad (2.1)$$

Where  $B_\epsilon(\underline{x})$  is an n-dimensional ball with radius  $\epsilon$  and center  $\underline{x}$ . No matter which point we are taking, it always lies within the set. Open sets are typically denoted as  $(\mathcal{A})$ . In general we denote open interval using  $()$ . For example  $(a, b)$  is open interval and  $(4, 5) \cup (6, 8)$  is an open set.

## 2.2 Closed set

A set  $\mathcal{A}$  is said to be closed set if:

- Compliment of the set is an open set.
- Every limit point of the set lies inside the set.
- Closed interval is denoted as  $[]$  and  $[a, b] \cup [p, q]$  is closed set. But  $(a, b) \cup [p, q]$  is neither closed set neither open set.

## 2.3 Convex set

$\mathcal{A}$  is said to be a convex set if,  $\forall \underline{x}, \underline{y} \in \mathcal{A}$  and  $\forall 0 \leq \alpha \leq 1$

$$\alpha \underline{x} + (1 - \alpha) \underline{y} \in \mathcal{A} \quad (2.2)$$

Geometrically, if we consider any two points inside the set, then every point that lies on the line that joins these two points must lie inside the set.

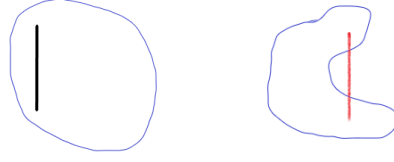


Figure 2.1: (a) Convex set (b) Not a convex set

**Example :** Achievable rate regions are convex: a time-sharing argument : Consider a multi-user system where we have some L-transmitter and L-receiver and they want to communicate through the channel described in figure 2.2. We want to find the achievable rate over this channel.

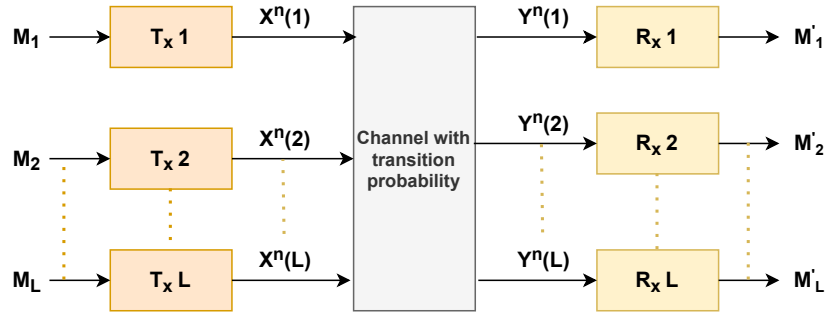


Figure 2.2: (a) Transceiver for L-transmitter and Receiver setting

A rate tuple  $(R_1, R_2 \dots R_L)$  is achievable if  $\exists$  pair of encoder and decoder.

$(ENC_n(1), ENC_n(2), \dots, ENC_n(L), DEC_n(1), DEC_n(2) \dots, DEC_n(L))$  pair can be defined as :

$ENC_n(i) : \{0, 1\}^{K_n(i)} \rightarrow \mathcal{X}^n(i)$  and  $DEC_n(i) : \mathcal{Y}^n(i) \rightarrow \{0, 1\}^{K_n(i)}$  such that :

$$\limsup_{n \rightarrow \infty} \frac{K_n(i)}{n} = R_i \quad (2.3)$$

$$\limsup_{n \rightarrow \infty} Pr [M'_i \neq M_i] = P_e = 0 \quad (2.4)$$

And we want to find what kind of rate can be achievable. Consider the tuple  $(R_1, R_2)$  i.e  $(L = 2)$  which is achievable. For single  $T_X - R_X$  setting (i.e  $L = 1$ ), according to the Shannon coding theorem, any rate less than the capacity of the channel is achievable. So, rate region for the single  $T_X - R_X$  is a closed interval which is  $[0, C]$ .

**For the 2 user case :**

**Theorem :** Every rate region are convex. i.e if  $(R_1, R_2)$  &  $(R'_1, R'_2)$  are achievable and they lie in the rate region then  $\forall 0 \leq \alpha \leq 1$ ,  $(\alpha R_1 + (1 - \alpha)R'_1, \alpha R_2 + (1 - \alpha)R'_2)$  is achievable.

**Proof :** Consider a time sharing argument  $(R_1, R_2)$  and  $(R'_1, R'_2)$  for  $ENC$  and  $DEC$  pair such that :  $K_n(1) \geq n(R_1 - \epsilon)$  and  $K_n(2) \geq n(R_2 - \epsilon)$  for  $(ENC, DEC)$ .  $K'_n(1) \geq n(R'_1 - \epsilon)$  and  $K'_n(2) \geq n(R'_2 - \epsilon)$  for  $(ENC', DEC')$

We have  $n$  channel uses and consider  $\alpha n$  channel uses by first  $(ENC - DEC)$  pair and  $(1 - \alpha)n$  for  $(ENC' - DEC')$  pair. Hence

$$\alpha K_n(1) \geq \alpha n(R_1 - \epsilon) \quad (2.5)$$

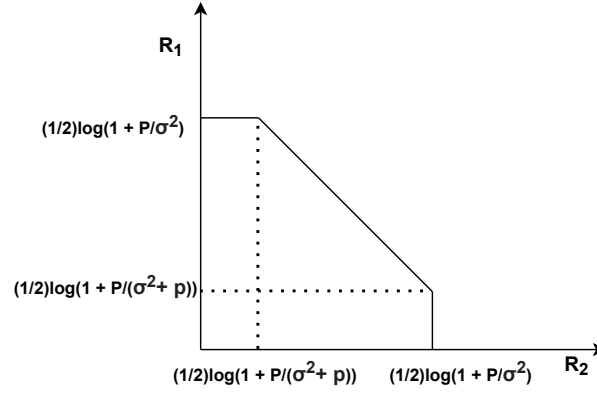


Figure 2.3: 2 user rate tuple case ( 2 user Gaussian multiple-access channel (MAC) only)

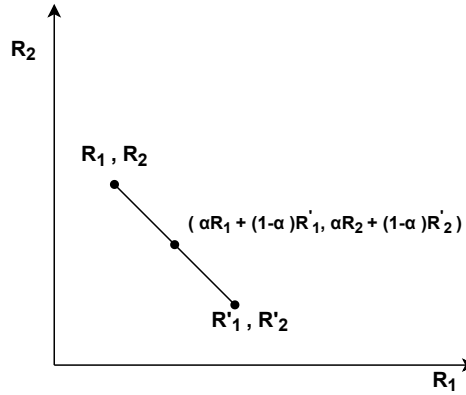


Figure 2.4: 2 user rate case

$$\alpha K_n(2) \geq \alpha n (R_2 - \epsilon) \quad (2.6)$$

and,

$$(1 - \alpha) K'_n(1) \geq (1 - \alpha) n (R'_1 - \epsilon) \quad (2.7)$$

$$(1 - \alpha) K'_n(2) \geq (1 - \alpha) n (R'_2 - \epsilon) \quad (2.8)$$

Combining (2.5), (2.7) and (2.6), (2.8), we have

$$\alpha K_n(1) + (1 - \alpha) K'_n(1) \geq \alpha n (R_1 - \epsilon) + (1 - \alpha) n (R'_1 - \epsilon) \quad (2.9)$$

$$\alpha K_n(2) + (1 - \alpha) K'_n(2) \geq \alpha n (R_2 - \epsilon) + (1 - \alpha) n (R'_2 - \epsilon) \quad (2.10)$$

And it holds  $\forall \epsilon > 0$ .

Now consider, we have pair of encoder and decoder  $(ENC, DEC)$  and  $(ENC', DEC')$ . First pair of encoder and decoder transmit  $\alpha k_1$  bits in  $\alpha n$  channel use. And receiver received  $\alpha n$  symbols. Similarly second pair

of encoder and decoder transmit  $(1 - \alpha)k_2$  bits in  $(1 - \alpha)n$  channel use. And receiver received  $\alpha n$  symbols. Overall rate of this setting is give as :

$$\text{Rate} = \frac{\alpha k_1 + (1 - \alpha)k_2}{n} \quad (2.11)$$

Suppose the probability of error for channel first pair of ENC - DEC is  $P_{e1}$  and for the second pair is  $P_{e2}$ , then For this the overall probability of error is :

$$P_e = P_{e1} + P_{e2} - (P_{e1} \cap P_{e2}) \quad (2.12)$$

Since both are independent.

$$P_e = P_{e1} + P_{e2} - P_{e1} \cdot P_{e2} \quad (2.13)$$

$$\Rightarrow P_e \leq P_{e1} + P_{e2} \quad (2.14)$$

Where  $P_{e1} = P_e^{\alpha n}$ ,  $P_{e2} = P_e^{(1-\alpha)n}$  As we know, if  $n$  is sufficiently large,  $P_{e1}$  and  $P_{e2}$  asymptotically approaches to zero. Hence from 2.12, we can observe the overall probability of error will be zero.