EE7330: Network Information Theory **2021**

Lecture Notes 2: Convexity

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. Please email the course instructor in case of any errors.*

2.1 Open set

A set $\mathcal{A}$ is said to be an open set if for every $\underline{x} \in \mathbb{R}^n$, we can find $\epsilon > 0$ such that,

$$B_\epsilon(\underline{x}) \subseteq \mathcal{A} \tag{2.1}$$

Where $B_\epsilon(\underline{x})$ is an n-dimensional ball with radius ϵ and center $\underline{x}$. No matter which point we are taking , it always lies within the set. open sets are typically denoted as $(\mathcal{A})$. In general we denote open interval suing (). For example (a, b) is open interval and $(4, 5) \cup (6, 8)$ is an open set.

2.2 Closed set

A set $\mathcal{A}$ is said to be closed set if:

- If compliment of the set is an open set.
- If every limit point of the set lies inside the set.
- Close interval denoted as [] and $[a, b] \cup [p, q]$ is closed set. But $(a, b) \cup [p, q]$ is neither closed set neither open set.

2.3 Convex set

$\mathcal{A}$ is said to be a convex set if, $\forall\ \underline{x}, \underline{y} \in \mathcal{A}$ and $\forall\ 0 \leqslant \alpha \leqslant 1$

$$\alpha\underline{x} + (1 - \alpha)\underline{y} \in \mathcal{A} \tag{2.2}$$

Geometrically, if we consider any two points inside the set, then every point that lies on the line that joins these two points must lie inside the set.

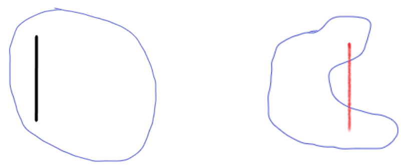

Figure 2.1: (a)Convex set (b) Not a convex set

Example : Achievable rate regions are convex: a time-sharing argument : Consider a multi-user system where we have some L-transmitter and L-receiver and they want to communicate through the channel described in figure 2.2. We want to find the achievable rate over this channel.

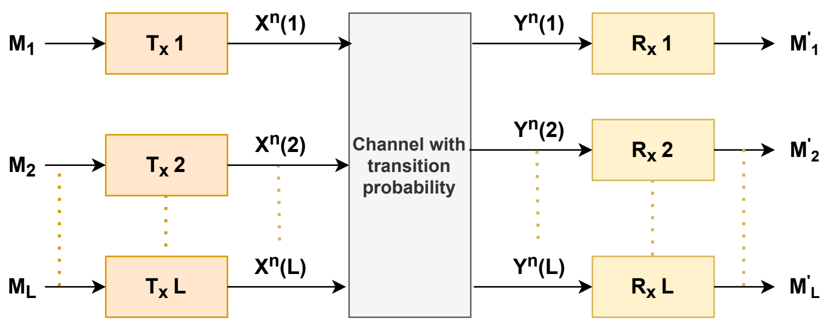

Figure 2.2: (a) Transceiver for L-transmitter and Receiver setting

A rate tuple $(R_1, R_2 \ldots R_L)$ is achievable if $\exists$ pair of encoder and decoder.
$(ENC_n(1), ENC_n(2), \ldots ENC_n(L), DEC_n(1), DEC_n(2) \ldots, DEC_n(L))$ pair can be define as :
$ENC_n(i) : \{0,1\}^{K_n(i)} \longrightarrow \mathcal{X}^n(l)$ and , $DEC_n(i) : \mathcal{Y}^n(i) \longrightarrow \{0,1\}^{K_n(i)}$ such that :

$$\limsup_{n \to \infty} \frac{K_n(i)}{n} = R_i \tag{2.3}$$

$$\limsup_{n \to \infty} Pr\left[M'_i \neq M_i\right] = P_e = 0 \tag{2.4}$$

And we want to find what kind of rate can be achievable. Consider the tuple (R_1, R_2) i.e (L = 2) which is achievable. For single T$_X$ - R$_X$ setting (i.e $L = 1$), according to the Shannon coding theorem, any rate less than the capacity of the channel is achievable. So, rate region for the single $T_X - R_X$ is a closed interval which is $[0, C]$.

For 2 user Case :
Theorm : Every rate region are convex. i.e if $(R_1, R_2)\,\&\,(R'_1, R'_2)$ are achievable and it lie in the convex set then $\forall$ $0 \leqslant \alpha \leqslant 1$, $\quad (\alpha R_1 + (1-\alpha)R'_1, \alpha R'_2 + (1-\alpha)R_2,)$ is achievable.
Proof : Consider a time sharing argument (R_1, R_2) and (R'_1, R'_2) for ENC and DEC pair such that : $K_n(1) \geqslant n\,(R_1 - \epsilon)$ and $K_n(2) \geqslant n\,(R_2 - \epsilon)$ for (ENC, DEC). $K'_n(1) \geqslant n\,(R'_1 - \epsilon)$ and $K'_n(2) \geqslant n\,(R'_2 - \epsilon)$ for (ENC', DEC')
We have n-channel uses and consider αn times channel used by first $(ENC - DEC)$ pair and $(1-\alpha)\,n$ for $(ENC' - DEC')$ pair. Hence

$$\alpha K_n(1) \geqslant \alpha n\,(R_1 - \epsilon) \tag{2.5}$$

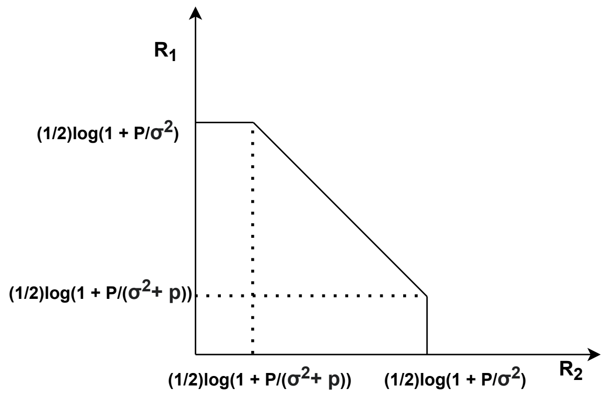

Figure 2.3: 2 user rate tuple case (2 user Gaussian multiple-access channel (MAC) only)

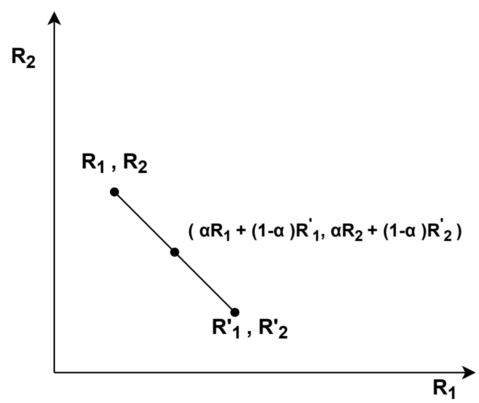

Figure 2.4: 2 user rate case

$$\alpha K_n(2) \geqslant \alpha n\left(R_2 - \epsilon\right) \tag{2.6}$$

and,

$$(1-\alpha)\, K_n'(1) \geqslant (1-\alpha)\, n\left(R_1' - \epsilon\right) \tag{2.7}$$

$$(1-\alpha)\, K_n'(2) \geqslant (1-\alpha)\, n\left(R_2' - \epsilon\right) \tag{2.8}$$

Combining (2.5), (2.7) and (2.6), (2.8), we have

$$\alpha K_n(1) + (1-\alpha)\, K_n'(1) \geqslant \alpha n\left(R_1 - \epsilon\right) + (1-\alpha)\, n\left(R_1' - \epsilon\right) \tag{2.9}$$

$$\alpha K_n(2) + (1-\alpha)\, K_n'(2) \geqslant \alpha n\left(R_2 - \epsilon\right) + (1-\alpha)\, n\left(R_2' - \epsilon\right) \tag{2.10}$$

And it holds $\forall\ \epsilon > 0$.
Using Fano's inequality we can prove the equation (2.4). We can have,

$$K_n \leqslant H_2(P_e) + P_e log_2(2^{M^{K_n}}) + I(M^{K_n} : {M'}^{K_n}) \tag{2.11}$$

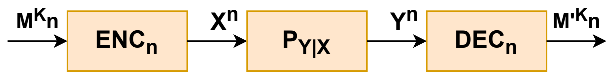

Figure 2.5: Single user transmission system

Where, $\mathrm{P}_e = \limsup_{n \to \infty} Pr\left[M_i' \neq M_i\right]$

$$K_n \leqslant H_2(P_e) + P_e log_2(2^{K_n}) + I(X^n : Y^n) \text{ (using data processing inequality)} \tag{2.12}$$

$$K_n \leqslant H_2(P_e) + P_e K_n + I(X^n : Y^n) \tag{2.13}$$

$$K_n \leqslant H_2(P_e) + P_e K_n + nC \Rightarrow \frac{H_2(P_e)}{n} \geqslant \frac{K_n(1-P_e)}{n} - C$$
$$\lim_{n \to \infty} \left(\frac{K_n(1-P_e)}{n} - C\right) \leqslant \lim_{n \to \infty} \frac{H_2(P_e)}{n}$$

Let $\frac{K_n}{n} = R$,

$$\lim_{n \to \infty} P_e \geqslant \frac{R-C}{R} \tag{2.14}$$

$$\limsup_{n \to \infty} P_e \geqslant \frac{R-C}{R} \tag{2.15}$$

If we operate at the rate more than capacity of the channel say R = C + ϵ

$$\limsup_{n \to \infty} \; P_e \geqslant \frac{\epsilon}{R} \tag{2.16}$$

That is probability or error is always non-zero.But if we operate the channel below the capacity and for large n, in (2.15) we can observe P_e approaches to 0.