'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2021

GE1-P1-MATHEMATICS

Time Allotted: 2 Hours Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains GE1, GE2, GE3, GE4 and GE5. Candidates are required to answer any *one* from the *five* courses and they should mention it clearly on the Answer Book.

GE1

CALCULUS, GEOMETRY AND DIFFERENTIAL EQUATION

GROUP-A

1. Answer any ***four*** questions from the following: $3\times4 = 12$

(a) Find the points of inflexion on the curve $(\theta^2-1)r = a\theta^2$. 3

(b) Find the envelopes of the lines $\frac{x}{a}+\frac{y}{b}=1$, where a and b are parameters related by $a+b=c$. 3

(c) Find the equation of the sphere through the circle $x^2+y^2+z^2=9$, $x+y-2z=4$ and the origin. 3

(d) Evaluate $\int_0^1 xe^{-\sqrt{x}}\,dx$ using reduction formula. 3

(e) Obtain the singular solution of the equation $(xp-y)^2 = p^2-1$, where $p=dy/dx$. 3

(f) Determine the nature of the quadric $5x^2-6xy+5y^2+22x-26y+29=0$. 3

GROUP-B

2. Answer any ***four*** questions from the following: $6\times4 = 24$

(a) If $y=\frac{\sin^{-1}x}{\sqrt{1-x}}$, $|x|<1$, then prove that $(1-x^2)y_{n+2}-(2n+3)xy_{n+1}-(n+1)^2y_n=0$. 6

(b) Find the asymptotes of the curve $x^3+2x^2y-4xy^2+8y^3-4x+8y-10=0$. 6

(c) Find the area of the region lying between the cissoid $y^2=\frac{x^3}{2a-x}$ and its asymptote. 6

(d) Solve: $y(2xy+1)\,dx+x(1+2xy+x^2y^2)\,dy=0$ 6

(e) Find a, b such that, $\lim\limits_{x\to 0}\frac{x\,(1+a\cos x)-b\sin x}{x^3}=1$. 6

(f) Find the volume of the solid generated by revolving the cardioid $r=a(1+\cos\theta)$ about initial line. 6

GROUP-C

Answer any *two* questions from the following 12×2 =24

3. (a) Find the range of values of x for which $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ is concave upward or downward. 6

   (b) Find the length of the arc of the cardioid $r = a(1 - \cos\theta)$ lying inside the circle $r = a\cos\theta$. 6

4. (a) Solve by using Bernoulli form $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$. 7

   (b) Solve: $(xy^2 - e^{1/x^3})\,dx - x^2 y\,dy = 0$ 5

5. (a) Reduce the equation $7x^2 - 2xy + 7y^2 - 16x + 16y - 8 = 0$ to its canonical form and hence determine the nature of the conic. 8

   (b) Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle. 4

6. (a) Find the value of $y_n(0)$, where $y = \log(x + \sqrt{1 + x^2})$. 6

   (b) If $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\,dx$, then show that $I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,\,n-1}$. 6

GE2

ALGEBRA

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

   (a) Apply Descarte's rule of signs to find the nature of the roots of the equation $x^4 + mx^2 + nx - p = 0$, where m, n, p are positive. 3

   (b) Prove that $\sqrt{i} + \sqrt{-i} = \sqrt{2}$. 3

   (c) Prove that the eigenvalues of a real skew symmetric matrix are purely imaginary or zero. 3

   (d) Find the sum of 99th power of the roots of the equation $x^7 - 1 = 0$. 3

   (e) Use Cayley-Hamilton theorem to find A^{-1} for the matrix 3

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

   (f) Find the quadratic equation whose roots are twice the roots of $2x^2 - 5x + 2 = 0$. 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

   (a) If $2\cos\theta = x + \frac{1}{x}$ and θ is real, prove that $2\cos n\theta = x^n + \frac{1}{x^n}$, n being an integer. 6

(b) Solve the equation $16x^4 - 64x^3 + 56x^2 + 16x - 15 = 0$ whose roots are in arithmetic progression. 6

(c) Find integers u and v satisfying $52u - 91v = 78$. 6

(d) Find all eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. 6

(e) For what values of λ the following system of equations are consistent? 6

$$x - y + z = 1$$
$$x + 2y + 4z = \lambda$$
$$x + 4y + 6z = \lambda^2$$

(f) Use Cayley-Hamilton theorem to find A^{100}, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. 6

GROUP-C

Answer any *two* questions from the following 12×2= 24

3. (a) If $\log\sin(\theta + i\varphi) = \alpha + i\beta$, then prove that $2e^{2\alpha} = \cosh 2\varphi - \cos 2\theta$. 6

(b) Find the relation among the coefficients of the equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, so that the second term and the fourth term may be removed by the transformation $x = y + h$. 6

4. (a) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, find the equation whose roots are $\beta + \gamma - 2\alpha$, $\gamma + \alpha - 2\beta$, $\alpha + \beta - 2\gamma$. 6

(b) Determine all values of $(1 + i\sqrt{3})^{3/4}$ and show that their product is 8. 4+2

5. (a) Solve the equation $3x^3 + 5x^2 + 5x + 3 = 0$, which has three distinct roots of equal moduli. 6

(b) If roots of $ax^3 + bx^2 + cx + d = 0$ are in arithmetic progression. Show that $2b^3 - 9abc + 27a^2d = 0$. 6

6. (a) Determine the conditions for which the system of equation has 2+2+2
   (i) only one solution
   (ii) no solution
   (iii) infinitely many solution.

$$x + 2y + z = 1$$
$$2x + y + 3z = b$$
$$x + ay + 3z = b + 1$$

(b) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ with ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ is given by 6

$$\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$$

Find the matrix of T relative to the ordered basis $\{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$ of $\mathbb{R}^3$.

GE3

DIFFERENTIAL EQUATION AND VECTOR CALCULUS

GROUP-A

1. Answer any ***four*** questions from the following: $3\times 4 = 12$

(a) Show that the function $f(x, y) = xy^2$ does not satisfy the Lipschitz condition on the strip $|x| \le 1,\ |y| < \infty$.

(b) Find the Wronskian of $\{1,\ 1+x,\ 1+x+x^2+x^3\}$.

(c) Define Lipschitz constant. Find Lipschitz constant for the function $f(x, y) = x^2 y^2$ defined on $|x| \le 1,\ |y| \le 1$.

(d) Solve: $\dfrac{d^5 y}{dx^5} - 2\dfrac{d^4 y}{dx^4} + \dfrac{d^3 y}{dx^3} = 0$

(e) Examine whether the vector valued function $\vec{r} = t^3\hat{i} + e^t\hat{j} + \dfrac{1}{t+3}\hat{k}$ is continuous at $t = -3$ or not.

(f) Evaluate: $\lim\limits_{t \to 1} \left[\dfrac{t^3 - 1}{t-1}\hat{i} + \dfrac{t^2 - 3t + 2}{t^2 + t - 2}\hat{j} + (t^2+1)e^{t-1}\hat{k}\right]$

GROUP-B

2. Answer any ***four*** questions from the following: $6\times 4 = 24$

(a) (i) If y_1 and y_2 are two independent solutions of the linear equation $\dfrac{d^2 y}{dx^2} + p\dfrac{dy}{dx} + qy = 0$, then show that the Wronskian $W(y_1, y_2) = Ae^{-\int p\,dx}$, where A is a constant. 3+3

(ii) Show that the functions $\{e^{2x}, e^{2x}\cos 4x,\ e^{2x}\sin 4x\}$ are linearly independent.

(b) Show that linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the conditions $y(0) = 2,\ y'(0) = -3$. 6

(c) Solve: $x^3\dfrac{d^3 y}{dx^3} + 2x^2\dfrac{d^2 y}{dx^2} + 2y = 10(x + x^{-1})$ 6

(d) Solve: $(D^3 - 1)y = x\sin x$, $D \equiv \dfrac{d}{dx}$ 6

(e) (i) Find the co-ordinates of the point where the line $\vec{r} = t\hat{i} + (1+2t)\hat{j} - 3t\hat{k}$ intersects the plane $3x - y - z = 2$. 3+3

(ii) Show that the graph of $\vec{r}(t) = t\hat{i} + \dfrac{1+t}{t}\hat{j} + \dfrac{1-t^2}{t}\hat{k}$, $t > 0$ lies on the plane $x - y + z + 1 = 0$.

(f) (i) Find the domain of the vector function $h(t)\,F(t)$, where $h(t) = \sin t$ and $F(t) = \dfrac{1}{\cos t}\hat{i} + \dfrac{1}{\sin t}\hat{j} + \dfrac{1}{\tan t}\hat{k}$. 3+3

(ii) Find $(F \times G)(t)$ if $F(t) = t^2\hat{i} + t\hat{j} - (\sin t)\hat{k}$ and $G(t) = t^2\hat{i} + \dfrac{1}{t}\hat{j} + 5\hat{k}$.

GROUP-C

Answer any *two* questions from the following 12×2 =24

3\. (a) (i) Solve by the method of variation of parameters $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\log x$. 5+3+4

(ii) Evaluate: $\frac{1}{D^2-3D+2}xe^{3x}$

(iii) Solve: $\frac{d^2y}{dx^2}+16y=1$, $y(0)=1$, $y'(0)=2$

(b) (i) Solve by using the method of undetermined coefficient 6+6
$(D^2+D-6)y=10e^{2x}-18e^{3x}-6x-11$

(ii) Solve: $(D^4+2D^3-3D^2)y=x^2+3e^{2x}+4\sin x$

(c) (i) Solve the equations 6+6

$$\begin{cases}\dfrac{dx}{dt}=-wy\\ \dfrac{dy}{dt}=wx\end{cases}$$

and show that the point $(x,\ y)$ lies on a circle.

(ii) Solve the system of equations

$$\begin{cases}\dfrac{dx}{dt}=-x+6y\\ \dfrac{dy}{dt}=x-2y\end{cases}$$

(d) (i) Find the slope of the line in R^2 for the vector equation 4+1+4+3
$\vec{r}(t)=(1-2t)\hat{i}-(2-5t)\hat{j}$

(ii) Define continuity of a vector valued function.

(iii) Show that the vector function $\vec{r}(t)=\begin{cases}\frac{\sin t}{t}\hat{i}+t\hat{j}+t^2\hat{k} & ,\ t\neq 0\\ \hat{i} & ,\ t=0\end{cases}$

is continuous at $t=0$.

(iv) Find a vector function F whose graph is the curve of intersection of the hemisphere $z=\sqrt{4-x^2-y^2}$ and the curve $y=x^2$.

GE4

GROUP THEORY

GROUP-A

1\. Answer any ***four*** questions from the following: 3×4= 12

(a) Let $(S,\ \circ)$ be a semigroup. If for all $x,\ y\in S$, $x^2\circ y=y=y\circ x^2$, prove that $(S,\ \circ)$ is an abelian group. 3

(b) Suppose H, K are subgroups of index 2 in a group G. Prove that $H\cap K$ is a normal subgroup of G. 3

(c) Let $G=\langle a\rangle$ be a cyclic group of order n. Prove that every subgroup of G is of the form $\langle a^m\rangle$, where m is a divisor of n. 3

(d) Find all elements of order 10 in the group $(\mathbb{Z}_{30},\ +)$. 3

(e) Show that there does not exist an onto homomorphism from the group $(\mathbb{Z}_6, +)$ to $(\mathbb{Z}_4, +)$. 3

(f) Prove or disprove: $(\mathbb{Q}, +)$ is isomorphic to $(\mathbb{Q}^+, \cdot)$ 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) Let S be the set of all permutations on the set $\{1, 2, 3\}$. Show that S forms a non-abelian group with respect to multiplication. 6

(b) Suppose that the order of an element a in a group $(G, \circ)$ is n. Show that $O(a^m) = \frac{n}{d}$, where $d = \gcd(m, n)$. Find the order of $\overline{n-1}$ in $(\mathbb{Z}_n, +)$. 4+2

(c) (i) Let H be a subgroup of a group G and $a, b \in G$. Prove that $b \in Ha$ iff $ba^{-1} \in H$. 2+4

(ii) Let H be a subgroup of a group G. Show that the set of all distinct left cosets of H in G and the set of all distinct right cosets of H in G have the same cardinality.

(d) (i) If H is a subgroup of G and N is a normal subgroup of G, then show that $H \cap N$ is a normal subgroup of H. 4+2

(ii) Prove that N is a normal subgroup of G iff $gNg^{-1} = N$ for every $g \in G$.

(e) (i) Let $(G, \circ)$ be a group and H, K be subgroups of $(G, \circ)$. Show that HK forms a subgroup of $(G, \circ)$ iff $HK = KH$. 4+2

(ii) Check whether the union of two subgroups of a group $(G, \circ)$ is a subgroup of $(G, \circ)$ or not?

(f) (i) Let $(G, \circ)$ be a group and a mapping $\varphi : G \to G$ is defined by $\varphi(x) = x^2$, $x \in G$. Prove that φ is a homomorphism iff G is commutative. 4+2

(ii) Prove that $(\mathbb{Z}_4, +)$ and "Klein's 4-group" are not isomorphic.

GROUP-C

Answer any *two* questions from the following 12×2 =24

3. (a) (i) Let $G = S_3$, $G' = (\{-1, 1\}, \cdot)$ and $\varphi : G \to G'$ is defined by 4+2+2

$$\varphi(\alpha) = \begin{cases} 1 & , \quad \alpha \text{ be an even permutation in } S_3 \\ -1 & , \quad \alpha \text{ be an odd permutation in } S_3 \end{cases}.$$

Then, (I) Show that φ is homomorphism.

(II) Find $\ker \varphi$.

(III) Deduce that A_3 is a normal subgroup of S_3.

(ii) Prove that a finite cyclic group of order n is isomorphic to $(\mathbb{Z}_n, +)$. 4

(b) (i) Let H be a normal subgroup of G. Prove that the quotient group G/H is abelian iff $xyx^{-1}y^{-1} \in H$ for all $x, y \in G$. 4+4+4

(ii) Suppose that a subgroup H of a group G has the property that $x^2 \in H$ for every $x \in G$. Prove that H is normal in G and G/H is abelian.

(iii) Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\}$ and $H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$. Show that H is a normal subgroup of G.

(c) Let M be the set of all real matrices $\left\{\begin{pmatrix} a & a \\ b & b \end{pmatrix} : a+b \neq 0\right\}$. Prove that 4+4+4

(i) $(M, \circ)$ is a semi-group under matrix multiplication.

(ii) there is no left identity in the semi-group.

(iii) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is a right identity.

(d) (i) Let $(G, \circ)$, $(G', *)$ be two groups and $\varphi : (G, \circ) \to (G', *)$ be an onto homomorphism. Then prove that $G/\ker\varphi \simeq G'$. 6+6

(ii) Let G be a cyclic group of order 10 and G' be a cyclic group of order 5. Show that there exists a homomorphism φ of G onto G' with $o(\ker\varphi) = 2$.

GE5

NUMERICAL METHODS

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) If $f(x) = 4\cos x - 6x$, find the relative percentage error in $f(x)$ for $x = 0$, if the error in $x = 0.005$. 3

(b) Deduce the iterative procedure $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ for evaluating $\sqrt{a}$ using Newton-Raphson method. 3

(c) Prove that $\left(\frac{\Delta^2}{E}\right)x^3 = 6xh^2$ where the notations used have their usual meanings. 3

(d) Show that $\nabla y_{n+1} = h\,[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \cdots\cdots]\,Dy_n$, where D is the differential operator. 3

(e) Write down the convergence of bisection method. 3

(f) What is the geometrical significance of Simpson's one-third rule? 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) If a number is connected to n significant figures and the first significant figure of the number is k, then prove that the relative error $\varepsilon_r < \dfrac{1}{k \cdot 10^{n-1}}$. 6

(b) Find the positive root of the equation $x^3 + x - 1 = 0$ by fixed point iteration method correct upto three decimal places. 6

(c) Find a real root of the equation $x^x + 2x - 2 = 0$ correct upto five decimal places using bisection method. 6

(d) Define backward difference operator ∇ and shifting operator E. Show that 6

$$\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$$

(e) Use Runge-Kutta method of order two to find $y(0\cdot 1)$ and $y(0\cdot 2)$ correct upto four decimal places given $\frac{dy}{dx}=y-x$, $y(0)=2$. 6

(f) Explain Gauss-Seidel method for solving a system of linear equations. Obtain the sufficient condition for convergence of Gauss-Seidel method. 6

GROUP-C

Answer any *two* questions from the following 12×2 =24

3. (a) Evaluate $\int_0^{\pi/2} \sqrt{1-0.162\sin^2\theta}\, d\theta$, by Simpson's $\frac{1}{3}$rd rule, correct upto 4 decimal places taking 12 points. 6

(b) Given $\frac{dy}{dx}=\frac{-y}{1+x}$, $y\,(0.3)=2$. Compute $y(1)$ by Euler's method, correct upto four decimal places, taking step length h = 0.1. 6

4. (a) Solve the system of equations by Gauss-elimination method 6

$$\begin{aligned}3x+9y-2z&=11\\4x+2y+13z&=24\\4x-2y+z&=-8\end{aligned}$$

correct upto 2 decimal places.

(b) Using Newton-Raphson method find a positive root of the equation $e^x-3x=0$ correct upto four decimal places. 6

5. (a) Find $f(x)$ as a polynomial in x by using the following table: 6

x	0	2	4	6	8
$f(x)$	2.51881	2.53148	2.54407	2.55630	2.56820

(b) Obtain the missing terms in the following table: 6

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	*	64	*	216	343	512

6. (a) Explain the method of fixed point iteration with the condition of convergence for numerical solution of an equation of the form $x=\phi(x)$. 6

(b) What is interpolation? Establish Lagrange's polynomial interpolation formula. 6

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