Question 4

Ans

Stiff Differential equation

Error in the soluti approximation of solution of a I.V.P involve derivative terms. If the derivatives dominate the solution, the error can grow large giving terrible approximation.

These problems have a solution that contains a term of the form e^{-at} $a > 0$ & large.

nth derivative $a^n e^{-at}$ & this does not decay quickly.

example

1) Radioactive decay

$$\frac{dN}{dt} = -N\lambda$$

$$N = N_0 e^{-\lambda t}.$$

2) Damped Oscillator

$$\phi(x) = X_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi)$$

$$\ddot{x} + \frac{b}{m}\dot{x} = -\frac{k}{m}x \qquad \omega' = \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}}$$

3) Chemical reactn system

$$A \xrightarrow{0.04} B \quad \text{(slow)}$$

$$B + B \xrightarrow{3\times10^7} C + B \quad \text{(fast)}$$

$$B + C \xrightarrow{10^4} A + C \quad \text{(fast)}$$

Choice of Algo

Implicit Trapezoidal Method.

This method is A-stable.

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)\right]$$

Python Function

```
scipy.integrate.ode(f, jac).set_integrator('lsoda', method='bdf')
```

Q7

To argue that the solution ~~is~~ so obtained numerically it is correct we need to show that the problem is well posed.

Theorem: Suppose $D = \{(t,y) \mid t \in [a,b] \text{ \& } y \in [-\infty, \infty)\}$.

If f is continuous & satisfies a Lipschitz condition in variable y on the set D, then the initial value problem is well problem.

(a) $y' = te^{3t} - 2y \quad t \in [0,1]$

$y(0) = 0$

$f(t,y) = te^{3t} - 2y$

$\left|\frac{\partial f}{\partial y}\right| = |-2| = 2$

$\Rightarrow$ f is Lipschitz in y ~~with~~ on D. with

Lipschitz constant $= 2$.

$\Rightarrow$ This problem is well posed.

$\Rightarrow$ Numerical solut^n is correct within errors.

(b) $f(t,y) = 1 - (t-y)^2$

$t \in [2,3]$

$\left|\frac{\partial f}{\partial y}\right| = |0 - 2(t-y)(-1)|$

$= |2(t-y)|$

Since $y \in [-\infty, \infty)$

$\left|\frac{\partial f}{\partial y}\right|$ can't be bounded & also

The theorem does not hold for this. ~~Solutn to this is not converging.~~

(c) $f(y,t) = 1 + \frac{y}{t}$

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{1}{t}\right|$$

$\& \; t \in [1, 2]$

$\left|\frac{\partial f}{\partial y}\right|$ is bounded by 1.

$\therefore$ f is Lipschitz ~~by~~ in domain $t \in [1,2]$

$y \in (-\infty, \infty)$ with constant 1.

$\Rightarrow$ ~~Solution~~ This problem is well posed & Numerical solut^n is Correct within errors.

(d) $y' = \cos(3t) + \sin(3t)$ $\quad t \in [0,1]$

$$\left|\frac{\partial f}{\partial y}\right| = 0$$

$\Rightarrow$ f is Lipschitz in domain D with constant 0

$\Rightarrow$ This problem is well posed & Numerical solut^n is correct within errors.

Q8

To argue that the solutions obtained are correct we will use same arguments as done in 7.

$$y'' = -e^{-2y} \longrightarrow \quad \frac{dz}{dx} = -e^{-2y} = f_1(z, y, x)$$

$$\frac{dy}{dx} = z = f_2(z, y, x)$$

$\frac{\partial f_1}{\partial z} = 0$ $\quad \frac{\partial f_2}{\partial y} = 0$ also $\frac{\partial f_1}{\partial y} = 2e^{-2y}$

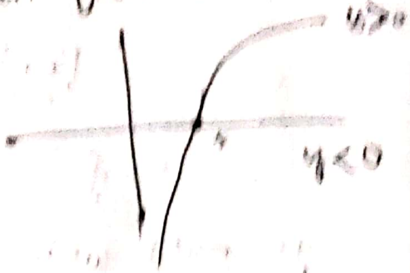
$\frac{\partial f_2}{\partial z} = 1$

all partial derivatives are bounded except

$$\frac{\partial f_1}{\partial y} = 2e^{-2y}$$

If y is restricted to +ve values only then partial derivative is bounded

which can also be seen from the the soln $y = \ln(x)$



Theorem does not hold for this nothing can be said.

(b) $y'' = y'\cos x - y\ln y$

$y' = z$ $\quad \frac{dy}{dx} = z = f_1$

$$\frac{dz}{dx} = z\cos x - y\ln y = f_2$$

$\frac{\partial f_1}{\partial y} = 0$ $\quad \frac{\partial f_2}{\partial y} = -\ln y - 1 \longrightarrow$ not bounded

$\frac{\partial f_1}{\partial z} = 1$ $\quad \left|\frac{\partial f_2}{\partial z}\right| = |\cos(x)| \longrightarrow$ bounded

Theorem does not hold for this, but unique soln exists.

(c)

$$\frac{dz}{dx} = -(2(z)^3 + y^2 z)\sec x = f_1$$

$$\frac{dy}{dx} = z = f_2$$

$$\left.\begin{aligned} \frac{\partial f_1}{\partial z} &= -\sec(x)(6z^2 + y^2) \\ \frac{\partial f_2}{\partial y} &= 2y\sec x \end{aligned}\right\} \longrightarrow \text{not bounded}$$

This system of linear equation is not Lipschitz but unique soln exists.

(d) $y'' = \frac{1}{2} - \frac{y^2}{2} - \frac{y\sin(x)}{2}$

$y' = z \quad = f_2$

$z' = \frac{1}{2} - \frac{z^2}{2} - \frac{y\sin(x)}{2} = f_1$

$\left|\frac{\partial f_1}{\partial z}\right| = |z| \longrightarrow$ not bounded.

$\left|\frac{\partial f_1}{\partial y}\right| = \left|\frac{\sin(x)}{2}\right|$

$\left|\frac{\partial f_2}{\partial z}\right| = 1 \quad \left|\frac{\partial f_2}{\partial y}\right| = 0$

$\therefore$ this is not well posed but solution exist.