Question 4

Ans

Stiff Differential equation

Error in the ~~soluti~~ approximation of solution of a I.V.P involve derivative terms. If the derivatives dominate the solution, the error can grow large giving terrible approximation.

These problems have a solution that contains a term of the form e^{-at} $a > 0$ & large.

nth derivative $a^n e^{-at}$ & this does not decay quickly.

example

1) Radioactive decay

$$\frac{dN}{dt} = -N\lambda$$

$$N = N_0 e^{-\lambda t}.$$

2) Damped Oscillator

$$\phi(x) = X_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi)$$

$$\ddot{x} + \frac{b}{m}\dot{x} = -\frac{k}{m}x \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

3) Chemical reactⁿ system

$A \xrightarrow{0.04} B$ (slow)

$B + B \xrightarrow{3 \times 10^7} C + B$ (fast)

$B + C \xrightarrow{10^4} A + C$ (fast)

Choice of Algo

Implicit Trapezoidal Method.

This method is A-stable.

$w_0 = \alpha$

$w_{i+1} = w_i + \frac{h}{2}\left[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)\right]$

Python Function

```
scipy.integrate.ode(f, jac).set_integrator('isoda',
    method='bdf')
```

Q7

To argue that the solution ~~is so~~ obtained numerically it is correct we need to show that the problem is well posed.

Theorem: Suppose $D = \{(t,y) \mid t \in [a,b] \; \& \; y \in [-\infty,\infty)\}$,

If f is continuous & satisfies a Lipschitz condition in variable y on the set D, then the initial value problem is well problem.

(a) $$y' = te^{3t} - 2y \qquad t \in [0,1]$$

$$y(0) = 0$$

$$f(t,y) = te^{3t} - 2y$$

$$\left|\frac{\partial f}{\partial y}\right| = |-2| = 2$$

$\Rightarrow$ f is Lipschitz in y ~~with~~ on D. with

Lipschitz constant $= 2$.

$\Rightarrow$ This problem is well posed.

$\Rightarrow$ Numerical solut^n is correct within errors.

(b) $$f(t,y) = 1 - (t-y)^2$$

$$t \in [2,3]$$

$$\left|\frac{\partial f}{\partial y}\right| = |0 - 2(t-y)(-1)|$$

$$= |2(t-y)|$$

Since $y \in [-\infty, \infty)$

$\left|\frac{\partial f}{\partial y}\right|$ can't be bounded & also

The theorem does not hold for this. Solut^n ~~to this is not converging~~.

(c) $f(y,t) = 1 + \frac{y}{t}$

$$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{1}{t}\right|$$

$\&\ t \in [1,2]$

$\left|\frac{\partial f}{\partial y}\right|$ is bounded by 1.

$\therefore$ f is Lipschitz ~~by~~ in domain $t \in [1,2]$ $y \in (-\infty,\infty)$ with constant 1.

$\Rightarrow$ ~~Solution~~ This problem is well posed & Numerical solut^n is correct within errors.

(d) $y' = \cos(3t) + \sin(3t)$ $\quad t \in [0,1]$

$$\left|\frac{\partial f}{\partial y}\right| = 0$$

$\Rightarrow$ f is Lipschitz in domain D with constant 0

$\Rightarrow$ This problem is well posed & Numerical solut^n is correct within errors.

Q8

To argue that the solutions obtained are correct we will use same arguments as done in 7.

$$y'' = -e^{-2y} \longrightarrow \frac{dz}{dx} = -e^{-2y} = f_1(z,y,x)$$

$$\frac{dy}{dx} = z = f_2(z,y,x)$$

$\frac{\partial f_1}{\partial z} = 0$ $\quad \frac{\partial f_2}{\partial y} = 0$ also $\frac{\partial f_1}{\partial y} = 2e^{-2y}$

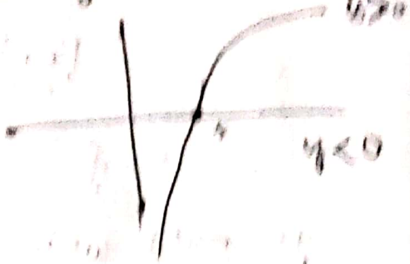
$\frac{\partial f_2}{\partial z} = 1$

all partial derivatives a bounded except

$$\frac{\partial f_1}{\partial y} = 2e^{-2y}$$

w/o y is restricted to +ve values only then partial derivative is bounded

which can also be seen from the the soln. $y = \ln(x)$



Theorem does not hold for this nothing can be said.

(b) $y'' = y'\cos x - y\ln y$.

$y' = z$ $\quad \frac{dy}{dx} = z \quad = f_1$

$$\frac{dz}{dx} = z\cos x - y\ln y = f_2$$

$$\frac{\partial f_1}{\partial y} = 0 \quad \frac{\partial f_2}{\partial y} = -\ln y - 1 \longrightarrow \text{not bounded}$$

$$\frac{\partial f_1}{\partial z} = 1 \quad \left|\frac{\partial f_2}{\partial z}\right| = |\cos(x)| \longrightarrow \text{bounded}$$

Theorem does not hold for this, but unique soln exists.

(c)

$$\frac{dz}{dx} = -(2(z)^3 + y^2 z)\sec x = f_1$$

$$\frac{dy}{dx} = z = f_2$$

$$\left.\begin{array}{l} \frac{\partial f_1}{\partial z} = -\sec(x)(6z^2 + y^2) \\ \frac{\partial f_2}{\partial y} = 2y\sec x \end{array}\right\} \longrightarrow \text{not bounded}$$

This system of linear equation is not Lipschitz but unique soln exists.

(d) $y'' = \frac{1}{2} - \frac{y^2}{2} - \frac{y\sin(x)}{2}$

$y' = z \quad = f_2$

$z' = \frac{1}{2} - \frac{z^2}{2} - \frac{y\sin(x)}{2} \quad = f_1$

$\left|\frac{\partial f_1}{\partial z}\right| = |z| \longrightarrow$ not bounded.

$\left|\frac{\partial f_1}{\partial y}\right| = \left|\frac{\sin(x)}{2}\right|$

$\left|\frac{\partial f_2}{\partial z}\right| = 1 \quad \left|\frac{\partial f_2}{\partial y}\right| = 0$

$\therefore$ this is not well posed but solution exist.

Q12

$$y' = f(x, y) \quad y(x_0) = y_0$$

Let y_{n+1} be approximation to $y(x_{n+1}) = y(x_n + h)$

: seven stage order 6 method for one step takes the form -

$$y_1 = y_0 + h \sum_{i=1}^{7} k_i b_i \quad —(1)$$

$$Y_i = y\left(x_0 + h \sum_{j=1}^{7} (a_{ij})\right) + O(h^2)$$

according to taylor series

$$Y_i = y(x_0) + h \sum_{i=1}^{7} a_{ij} k_i \quad —(2)$$

$$k_1 = f(x_0, y_0)$$

$$k_2 = f(x_0 + c_2 h, y_0 + h a_{21} k_1)$$

$$k_3 = f(x_0 + c_3 h, y_0 + h(a_{31} k_1 + a_{32} k_2))$$

$$k_4 = f(x_0 + c_4 h, y_0 + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3))$$

$$k_5 = f(x_0 + c_5 h, y_0 + h(a_{51} k_1 + a_{52} k_2 + a_{53} k_3 + a_{54} k_4))$$

$$k_6 = f(x_0 + c_6 h, y_0 + h(a_{61} k_1 + a_{62} k_2 + a_{63} k_3 + a_{64} k_4 + a_{65} k_5))$$

$$k_7 = f(x_0 + c_7 h, y_0 + h(a_{71} k_1 + a_{72} k_2 + a_{73} k_3 + a_{74} k_4 + a_{75} k_5 + a_{76} k_6))$$

$$y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7)$$

where from ① & ②

$$\sum_{k=1}^{j-1} a_{jk} = c_j \quad j = 2, 3, \dots 7$$

$$\sum b_i = 1$$

now

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{1}{2} h^2 y''(x_0) + \dots$$
$$+ \frac{1}{6!} h^6 y^{(6)}(x_0) + O(h^7) \quad \text{—③}$$

$$y'' = f_y f$$

$$y''' = f_{yy} f f + f_y f_y f$$

$$y^{(4)} = f_{yyy} f f f + 3 f_{yy} f_y f f + f_y f_{yy} f f + f_y f_y f_y f$$

Similary higher derivations can be obtained.

Substituting derivatives into ③

we get

$$y_1 = y_0 + \left(\sum_{i=1}^{7} b_i\right) hf + \left(\sum_{i=2}^{7} b_i c_i\right) h^2 f_y f$$
$$+ \frac{1}{2}\left(\sum_{i=2}^{7} b_i c_i^2\right) h^3 f_{yy} f f + \dots$$

Constraints are

$$\sum_{i=1}^{7} b_i = 1$$

$$\sum_{i=2}^{7} b_i c_i^3 = \frac{1}{4}$$

$$\sum_{i=2}^{7} b_i c_i = \frac{1}{7}$$

$$\sum_{i=2}^{7} b_i c_i^4 = \frac{1}{5}$$

$$\sum_{i=2}^{7} b_i c_i^2 = \frac{1}{3}$$

$$\sum_{i=2}^{7} b_i c_i^5 = \frac{1}{6}$$

$$\sum_{i=3}^{7} \sum_{j=i-1}^{2} a_{ij} c_j = \frac{1}{3)}$$

$$\sum_{i=3}^{7} \sum_{j=i-1}^{2} a_{ij} c_j^2 = \frac{1}{12}$$

$$\sum_{i=3}^{7} \sum_{j=i-1}^{2} a_{ij} c_j^3 = \frac{1}{20}$$

$$\sum_{i=3}^{7} \sum_{j=i-1}^{2} a_{ij} c_j^3 = \frac{1}{30}$$

$$\sum_{i=4}^{7} b_i \left[\sum_{k=i-2}^{2} \left(\sum_{j=i-1}^{k+1} a_{ij} a_{jk} \right) c_k \right] = \frac{1}{24}$$

$$\sum_{i=4}^{7} b_j \left[\sum_{j=i-2}^{2} \left(\sum_{j=i-1}^{k+1} a_{ij} a_{jk} \right) c_k^2 \right] = \frac{1}{60}$$

$$\sum_{i=4}^{7} b_i \left[\sum_{k=i-2}^{2} \left(\sum_{j=i-1}^{k+1} a_{ij} a_{jk} \right) c_k^3 \right] = \frac{1}{120}$$

Solving these we get

$K_1 = f(y_n)$

Solving these the solution we get is

$$k_1 = h f(y_n, x)$$

$$k_2 = h f\left(x+\frac{h}{6}, y_n + \frac{k_1}{6}\right)$$

$$k_3 = h f\left(x+\frac{4h}{15}, y_n + \frac{4k_1}{75} + \frac{16k_2}{75}\right)$$

$$k_4 = h f\left(x+\frac{2h}{3}, y_n + \frac{5k_1}{6} - \frac{8k_2}{3} + \frac{5}{2}k_3\right)$$

$$k_5 = h f\left(x+\frac{5h}{6}, y_n + \frac{55}{6}k_2 - \frac{165}{64}k_1 - \frac{425}{64}k_3 + \frac{85}{96}k_4\right)$$

$$k_6 = h f\left(t+h, y_n + \frac{12}{5}k_1 - 8k_2 + \frac{4015}{612}k_3 - \frac{11}{36}k_4 + \frac{88}{255}k_5\right)$$

$$k_7 = h f\left(t+\frac{h}{15}, y_n + \frac{124}{75}k_2 - \frac{8263}{15000} - \frac{643}{680}k_3 - \frac{81}{250}k_4 + \frac{2484}{10625}k_5\right)$$

$$k_8 = h f\left(t+h, y_n + \frac{3501}{1720}k_1 - \frac{300k_2}{43} + \frac{297275}{52632}k_3 - \frac{319}{2322}k_4 + \frac{24068}{84065}k_5 + \frac{3850}{26703}k_7\right)$$

$$y_{n+1} = y_n + \left(\frac{3}{40}k_1 + \frac{875}{2244}k_3 + \frac{23}{72}k_4 + \frac{264}{1955}k_5 + \frac{125}{11592}k_7 + \frac{43}{616}k_8\right).$$