* Notations

$n \rightarrow$ number of customers in the system

$P_n \rightarrow$ probability of n customers in the system

$\lambda \rightarrow$ average customer arrival rate

$\mu \rightarrow$ average service rate

$\frac{\lambda}{\mu} = \rho = \frac{\text{Average service completion time } (1/\mu)}{\text{Average interarrival time } (1/\lambda)}$

$\rightarrow$ traffic intensity / server utilization factor

$P_0 \rightarrow$ probability of no customer in the system

$s \rightarrow$ number of service channels

$N \rightarrow$ maximum number of customers allowed in the system

$L_q \rightarrow$ average no. of customers (in queue)
$L_s \rightarrow$ average no. of customers (in queue + service)

$W_q \rightarrow$ average waiting time (in queue)
$W_s \rightarrow$ average waiting time (in queue + service)

P_w Probability that an arriving customer has to wait (system being busy), $1 - P_0 = (\lambda / \mu)$

For achieving a steady state cond^n, it is necessary that $\lambda / \mu < 1$

$\lambda < \mu$

Probability of being in system (waiting & being served) longer than time t is given by

$$P(W_s > t) = e^{-(\mu-\lambda)t}$$

$$P(W_s \le t) = 1 - P(W_s > t)$$

Probability of only waiting for service longer than time t is given by

$$P(W_q > t) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$$

Probability that the system, n exceeds a given number, r is given by

$$P(n > r) = \left(\frac{\lambda}{\mu}\right)^{r+1}$$

→ $\{(a|b|c) : (d|e)\}$ (a: Arrival, b: Service, d: system capacity)

→ <u>Single server queuing theory</u>

Model 1 : $\{(M/M/1) : (\infty / FCFS)\}$

Exponential Service – Unlimited Queue

* Formulae for model – 1

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \rho^n (1-\rho); \quad \rho < 1 \quad (\lambda < \mu)$$

Expected No. of customers in the system (queue + being served)

$$L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}; \quad \rho = \frac{\lambda}{\mu}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$W_s = \frac{1}{\mu - \lambda}$$

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The variance (fluctuation) length of queue

$$Var(n) = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

Probability that the queue is non empty

$$P(n>1) = 1 - P_0 - P_1$$

$$= 1 - \left(1-\frac{\lambda}{\mu}\right) - \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)$$

$$= \left(\frac{\lambda}{\mu}\right)^2$$

$$P(n>r) = \left(\frac{\lambda}{\mu}\right)^{r+1}$$

$$P(n \geq r) = \left(\frac{\lambda}{\mu}\right)^{r}$$

Expected length of non-empty queue:

$$L = \frac{\mu}{\mu-\lambda}$$

Idle time = 8 hours − busy time

$= 8 - 5$

$= 3$

Busy Time $= 8 \times \frac{\lambda}{\mu}$

$= 8 \times \frac{5/4}{2}$

$= 5$ hours

(c) $W_q > 3$

$$\frac{\lambda'}{\mu(\mu-\lambda')} > 3$$

$$\frac{\lambda'}{\mu(\mu-\lambda')} = 3$$

$$\lambda' = 0.16$$

Increase $= \lambda' - \lambda = 0.06/\text{min}$

Sol^n

$\lambda = 4 / hour$

$\mu = 6 / hour$

Existing total hourly cost = wages + op. cost

$= (6 \times 3) + (Ls \times 20)$

→ Model 2 : $\{(M/M/1) : (N/FCFS)\}$

Exponential service - Finite Queue

* Performance measures for model 2

$$P_0 = \begin{cases} \left(\dfrac{1-\rho}{1-\rho^{N+1}}\right) & \dfrac{\lambda}{\mu} \neq 1 \\[2ex] \dfrac{1}{N+1} & \dfrac{\lambda}{\mu} = 1 \end{cases}$$

$$P_n = \begin{cases} \left(\dfrac{1-\rho}{1-\rho^{N+1}}\right)\rho^n \; ; \; n \leq N \; ; & \dfrac{\lambda}{\mu} \neq 1 \\[2ex] \dfrac{1}{N+1} \; ; & \dfrac{\lambda}{\mu} = 1 \end{cases}$$

$\cancel{\rho_n = \rho_0}$

$\boxed{P_n = P_0 \times \rho^n}$

1. Expected no. of customers in the system

$$L_s = \sum_{n=0}^{N} n P_n$$

$$L_s = \begin{cases} \dfrac{\rho}{1-\rho} - \dfrac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} & \rho \neq 1 \ (\lambda \neq \mu) \\[2ex] \dfrac{N}{2} & \rho = 1 \ (\lambda = \mu) \end{cases}$$

2. Expected Numbers of customers waiting in queue

$$L_q = L_s - (1 - P_0)$$

All formulas are dependent on L_s

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3. Expected waiting time of customer in the system (waiting + service)

$$W_s = \frac{L_s + 1}{\mu}$$

4. Expected waiting time of a customer in the queue

$$W_q = W_s - \frac{1}{\mu} = \frac{L_s}{\mu}$$

5. Potential customer lost

$P_N = P_0 \rho^N$

Effective arrival rate, $\lambda_{eff} = \lambda(1 - P_N)$

Effective traffic intensity, $\rho_{eff} = \frac{\lambda_e}{\mu}$