PROBABILITY DISTRIBUTION

1. Random Variable:

A variable used to denote the numerical value of the outcome of an experiment is called the random variable, abbreviated as r.v. A random variable is denoted by capital letters X, Y, Z, ……. and its values are denoted by $x_1, x_2, \dots y_1, y_2, \dots z_1, z_2, \dots$ etc.

2. Discrete Random Variable:

Definition : A random variable is called a **discrete random variable** if it takes discrete (distinct) values $x_1, x_2, \dots x_n \dots in\ (a, b)$.

3. Probability Distribution Of A discrete Random Variable:

Definition : Let X be a **discrete random variable** . Let $x_1, x_2, \dots x_n \dots$ be the possible values of X. with each possible outcome x_i we associate a number $p(x_i) = p(X = x_i) = p_i$ called the Probability of x_i. The numbers $p(x_i), i = 1, 2, \dots n \dots$ must satisfy the following conditions :

1. $p(x_i) \geq 0\ for\ all\ i$ **2.** $\sum_{i=1} p(x_i) = 1$.

The function p is called the **probability function** or **probability mass function (p.m.f.) or probability density function (p.d.f.)** of the random variable X and the set of pairs (x_i, p_i) is called the probability distribution of X.

The probability distribution of a discrete random variable X taking values $x_1, x_2, x_3 \dots\ x_n \dots$ with probabilities $p_1, p_2, p_3 \dots p_n \dots where\ p_1 \geq 0$ and $\sum p_i = 1$ can be given in tabular form as.

X	x_1	x_2	x_3………	x_n………..
$P(X = x_i)$	p_1	p_2	p_3………	p_n………..

4. Probability Distribution of a Function Of A Random Variable X:

Let X be a r.v. taking values $x_1, x_2, \dots x_n$ with probabilities $p_1, p_2 \dots p_n$.

Let Y $= g(X)$ be a function of X. Then when X take values $x_1, x_2 \dots x_n$ with probabilities $p_1, p_2 \dots p_n$, Y will take value $g(x_1), g(x_2), \dots g(x_n)$ with probabilities $p_1, p_2 \dots p_n$ Hence, we get the probabilities of y_i, $p(Y = y_i) = p_i$ for i= 1,2, … n where $p_i = p(X = x_i)$

However, it may happen that the function $Y = g(X)$ is such that several values of X lead to the same value of Y. Suppose X $= x_1, x_2, x_3$ …. lead the same value of Y $= y_k$ say i.e. $y_k = g(x_1) = g(x_2) = g(x_3) =$……..

Then, by the theorem of addition of probabilities we have

$$p(Y = y_k) = p(X = x_1) + p(X = x_2) + p(X = x_3)\dots\dots$$

5. Distribution Function of A Discrete Random Variable X:

Suppose, X is a random variable taking values $x_1, x_2 \dots x_n$ with probabilities $p(x_i), i = 1,2, \dots n \dots$ such that

(i) $p(x_i) \geq 0$ for all i , **(ii)** $\sum p(x_i) = 1$

and consider the following table.

X	x_1	x_2	x_3………	x_n………..
$F(x_i) = F(X = x_i)$	$P(x_1)$	$\sum_1^2 P(x_i)$	$\sum_1^3 P(x_i)$………	$\sum_1^n P(x_i)$………..

The function F is called the distribution function.

Definition: Let X be a discrete random variable taking values $x_1, x_2, \dots \dots$ such that $x_1 < x_2 < x_3 \dots \dots$ with probabilities $P(x_1), P(x_2), \dots$ *such that* $P(x_i) \geq 0$ *for all i and* $\sum P(x_i) = 1$.
Consider F defined by $F(x_i) = P(X \leq x_i), i = 1,2,3,\dots\dots$

i.e. $F(x_i) = P(x_1) + P(x_2) + \cdots + P(x_i)$

then the function F is called the **cumulative distribution function or simply distribution function** and the set of pairs $\{x_i, F(x_i)\}$ is called the **cumulative probability distribution.**

Properties of Distribution function:

The distribution function F of a random variable X has the following properties.

1. $0 \leq F(x) \leq 1$
2. $\mathrm{F(x)} = 0$ for $\mathrm{x} < a$ *and* $F(\mathrm{x}) = 1$ for $\mathrm{x} > b$ *where* $a < \mathrm{x}_1 < \mathrm{x}_2 < \cdots \mathrm{x_n} < b$.
3. F (x) is a step function

6. Continuous Random Variable:

Definition: A random variable is called a **continuous random variable** if it takes all values between an interval (a, b).

For example, age, height, weight are continuous random variables.

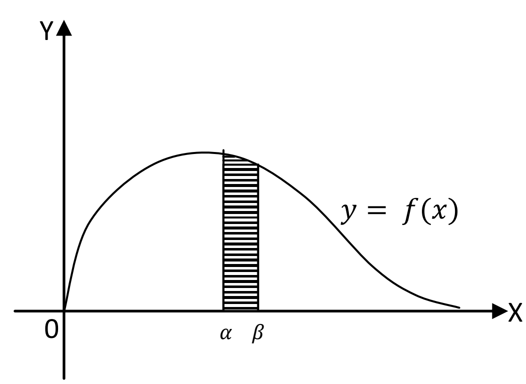
7. Probability Density Function Of A Continuous Random Variable:

Let y $= f(x)$ be a continuous function of x such that the area f(x) δx represents the probability that X will lie in the interval $\left(x - \frac{\delta x}{2}, x + \frac{\delta x}{2}\right)$. Symbolically, $P\left(x - \frac{\delta x}{2} \leq X \leq x + \frac{\delta x}{2}\right) = f_x(x)\delta x$

Where, $f_x(x)$ denotes the value of f (x) at x.

The adjoining figure denotes the curve $y = f(x)$ and the area under the curve in the interval $\left(x - \frac{\delta x}{2}, x + \frac{\delta x}{2}\right)$

The function satisfying certain conditions giving the probability that X will lie between certain limits is called **Probability density function**, or simply density function of a continuous random variable X and is abbreviated as p.d.f.



The curve given by y $= f(x)$ is called the **probability density curve** or simply **probability curve**.

The expression $f(x)dx$ is usually denoted by $df(x)$ and is known as **probability differential**

Definition: A continuous function $y = f(x)$ such that

(i) $f(x)$ is integrable. **(ii)** $f(x) \geq 0$ **(iii)** $\int_a^b f(x)dx = 1 \; if \; X \; lies \; in \; [a, b]$ and

(iv) $\int_\alpha^\beta f(x)dx = P(\alpha \leq X \leq \beta) \; where \; a < \alpha < \beta < b$

is called **probability desnity function** of a continuous random variable X.

Properties Of Probability Density Function:

The probability density function f(x) has the following properties

(i) $f(x) \geq 0, -\infty < x < \infty$

(i.e. the curve y = f(x) lies above the x – axis in the first and second quadrants only)

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$ (i.e. the total area under the curve and the x – axis is one).

(iii) The probability that $\alpha \leq X \leq \beta$ is given by $P(\alpha \leq X \leq \beta) = \int_\alpha^\beta f(x)dx$

Remark: You know that for discrete random variable the probability at $X = C$ may not be zero. But, in a continuous random variable $P(X = C)$ is always zero because $P(X = C) = \int_C^C f(x)dx$ and this definite integral is zero. Hence, for a continuous random variable X

$P(\alpha \leq X \leq \beta) = P(\alpha < X < \beta) = P(\alpha < X \leq \beta) = P(\alpha \leq X < \beta)$

In other words we may include or may not include the end points in the interval.

8. Continuous Distribution Function:

Probability distribution of X or the probability density function of X helps us to find the probability that X will be within a given interval $[a, b]$ i. e. $P(a \leq X \leq b) = \int_a^b f(x)dx$, other conditions being satisfied.

However, sometimes we need to know that probability that X will be less that a given value x. For a continuous random variable X, this probability is obtained by integrating f(x) from $-\infty$ (or the lower limit of the interval) to x. The function so obtained is called distribution function.

Definition : If X is a continuous random variable X, having the probability density function f(x) then the function $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt, -\infty < x < \infty$

is called **distribution function** or **cumulative distribution function** of the random variable X.

Properties of Distribution Function F(x) of a Continuous Random Variable

1. The function F(x) is defined for every real number x.
2. Since F(x) denotes probability and probability of X lies between 0 and 1, $0 \leq F(x) \leq 1$.
3. F(x) is a non – decreasing function which means if $x_1 \leq x_2$, $F(x_1) \leq F(x_2)$.
4. The derivative of F(x) is equal to the probability density function f(x)

   F' (x) $= \frac{d}{dx}F(x) = f(x) \geq 0$ provided the derivative exists.

5. If F (x) is a distribution function of a continuous random variable then $P(a \leq X \leq b) = F(b) - F(a)$.

9. Expectation of a Random Variable:

Definition: If a discrete random variable X assumes values $x_1, x_2 \dots\ x_n \dots$ with probability $P_1, P_2 \dots\ P_n \dots$ respectively then the mathematical expectation of X denoted by E (x) (if it exists) is defined by

$\boldsymbol{E(X) = P_1X_1 + P_2X_2 + \cdots + P_nX_n} \dots \dots .. = \sum \boldsymbol{P_iX_i}$ where $\sum P_i = 1$.

if $\sum P_iX_I$ is absolutely convergent. This value is also referred to as **mean** value of X.

Definition: Let X be a continuous random variable with probability density function f (x). Then the mathematical expectation of X, denoted by E(X) (if it exists), is defined by

$$\boldsymbol{E(X) = \int_{-\infty}^{\infty} x.f(x)dx} \quad \text{where, } \int_{-\infty}^{\infty} f(x)dx = 1$$

if the integral is absolutely convergent.

10. Expectation of a Function of a Random Variable X:

Definition: Let X be a discrete random variable taking values $x_{1,}\ x_2, \dots\ x_n$ with probabilities $P_1, P_2 \dots\ P_n$ and g (X) be a function of X then mathematical expectation of g(X) (if it exists) is defined by

$\boldsymbol{E[g(X)] = \sum P_i g(x_i)}$

Definition: Let X be a continuous random variable with p.d.f. (x),let g(X) be a function such that g(X) is a random variable then E$[g(X)]$ (if it exists) is defined by, $\boldsymbol{E[g(X)] = \int_{-\infty}^{\infty} g(x).f(x)dx}$

11. Laws of Expectation:

1. If X is a discrete random variable such that $x_i \geq 0$ for all i, then $\boldsymbol{E(X) \geq 0}$
2. If X is a discrete (or continuous) random variable, a and b are constants then

   $\boldsymbol{E(aX + b) = aE(X) + b}$
3. Putting $a = 0$. $E(b) = b$ i.e. expectation of a constant is the constant itself.
4. Putting $b = 0$. $E(ax) = aE(X)$ i.e. for calculations the constant can be taken out.
5. Putting $a = 1, b = -\bar{X}$, $E(X - \bar{X}) = 0$.
6. **Theorem of Addition:** The expectation of the sum (or difference) of two (discrete or continuous) variates is equal to the sum (or difference) of their expectations. In symbols, $\boldsymbol{E(X \pm Y) = E(X) \pm (Y)}$
7. **Theorem of Multiplication**: The expectation of the product of **two independent** variates (discrete or continuous) is equal to the product of their expectations if the expectation exist.

   In symbols, $\boldsymbol{E(XY) = E(X).E(Y)}$

Note: It should be noted that the converse of the above theorem is not. If $E(XY) = E(X).E(Y)$ then it does not mean that X, Y are independent.

14. Mean, Variance And Other Measures:

(a) Arithmetic Mean:

μ_1' =Mean = E(X) = $\sum P_i x_i$ **or** $\mu_1' = \int_{-\infty}^{\infty} x f(x) dx$

We then find $\mu_2' = E(X^2) = \sum P_i x_i^2$ **or** $\mu_2' = \int_{-\infty}^{\infty} x^2 f(x) dx$

(b) Variance:

Var (X) $= E(X - \bar{X})^2$

$= E[X - E(X)]^2 = E[X^2 - 2XE(X) + \{E(X)\}^2]$

$= E(X^2) - 2E(X).E(X) + [E(X)]^2$

Var (X) $= \boldsymbol{E(X^2) - [E(X)]^2}$

But $E(X^2) = \mu_2'$ *and* $E(X) = \mu_1'$

∴ **Var (X)** $= \boldsymbol{\mu_2' - \mu_1'^2}$

Properties of Variance:

1. Variance of a constant is zero, V(C) = 0.

2. If X is a random variate and a, b are constants then $\boldsymbol{V(aX + b) = a^2 V(X)}$

3. $\boldsymbol{V(aX) = a^2 V(X)}$

4. $\boldsymbol{V(X + b) = V(X)}$

Note: Note that although $E(aX + b) = aE(X) + b$ we do not have V $(aX + b) = aV(X) + b$.

Instead, we have V$(aX + b) = a^2 V(X)$.

5. $\boldsymbol{V(a_1 X_1 + a_2 X_2) = a_1^2 V(X_1) + a_2^2 V(X_2)}$ Where X_1 *and* X_2 are independent random variates.

6. If $a_1 = 1, a_2 = 1$, we get $\boldsymbol{V(X_1 + X_2) = V(X_1) + V(X_2)}$ &

If $a_1 = 1, a_2 = -1$, we get $\boldsymbol{V(X_1 - X_2) = V(X_1) + V(X_2)}$

(c) Median:

Since the median is the size of the item which lies at the middle, for a continuous distribution the median M divides the area under the curve from x = a to x = b in to two equal parts.

Thus, if M is the median then $\int_a^M f(x)\, dx = \int_M^b f(x) dx = \frac{1}{2}$

Thus, by solving any one of the equations $\int_a^M f(x)\, dx = \frac{1}{2}, \int_M^b f(x)\, dx = \frac{1}{2}$ We can get the median M.

(d) Mode:

Since mode is the size of the item having maximum frequency, using the theory of maxima, mode is obtained by solving the equation $\frac{dy}{dx} = 0$ i.e. f ’(x) = 0

with the condition that $\frac{d^2y}{dx^2} < 0\ i.e. f''(x) < 0$ and that x lies in the interval $[a, b]$ of X.

15. Covariance:

Definition: If X and Y are discrete random variates then the covariance between them denoted by $cov(X, Y)$ is defined by $\boldsymbol{cov(X, Y) = E[\{X - E(X)\}.\{Y - E(Y)\}]}$