POISSON DISTRIBUTION

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- Poisson distribution was discovered by the French Mathematician Poisson in 1837.
- Poisson distribution is the limiting case of the binomial distribution under the following conditions:

(i) n, the number of trials is infinitely large i.e $n \to \infty$

(ii) p, the probability of success in each trial is constant and infinitely small i.e $p \to 0$

(iii) np, the average success is finite say m, $i, e\ np = m$

DEFINITION

- A random variable X is said to follow Poisson distribution if probability of x is given by

$$P(X = x) = \frac{e^{-m}m^x}{x!}, x = 0,1,2 \ldots \ldots \ldots$$

and $m(> 0)$ is called the parameter of the distribution.

REMARKS

1. The sum of the probabilities is 1.

$$\sum_{x=0} P(X = x) = \sum_{x=0} \frac{e^{-m} m^x}{x!}$$

$$= e^{-m} \sum \frac{m^x}{x!}$$

$$= e^{-m} \left[1 + m + \frac{m^2}{2!} + \ \ldots . \right]$$

$$= e^{-m} . e^{m}$$

$$= 1$$

2. Poisson distribution occurs where the probability of occurance p is very small and the number of trials n is very large and where the probability of occurance only can be known

e.g the number of accidents, the number of deaths by a disease, the number of printing mistakes on a page etc.

In these cases we can only observe the number of successes but the number of failures cannot be observed. We can observe how many accidents occur; we cannot observe how many times accidents do not occur.

WHEN DO WE GET POISSON DISTRIBUTION

we get a Poisson distribution if the following conditions are satisfied.

1. The number of trials n is infinitely large i.e $n \to \infty$

2. A trial results in only two ways – success or failure

3. If p and q are probabilities of success and failure, then $p + q = 1$

4. These probabilities are mutually exclusive, exhaustive but not necessarily equally likely. (i.e. p is not necessarily = 1/2)

5. The probability p of success is very small i.e $p \to 0$

6. $n \to \infty$ and $p \to 0$ such that $np = m, (> 0)$ a constant.

USES

Poisson distribution is used in problems involving:

(i) The number of deaths due to a diseases such as heart attack, cancer etc.

(ii) The number of accidents during a week or a month etc.

(iii) The number of phone calls received at a particular telephone exchange during a period of time.

(iv) The number of cars passing a particular point on a road during a period of time.

(v) The number of printing mistakes on a page of book etc.

Note:

Since the Poisson distribution is the limiting case of Binomial distribution, we can calculate binomial probabilities approximately by using Possion distribution whenever n is large and p is small.

MEAN AND VARIANCE

$$E(x) = \sum p_i x_i$$

$$= \sum_{x=0} \left(\frac{e^{-m} m^x}{x!}\right) x$$

$$= \sum_{x=1} \frac{e^{-m} m^x}{(x-1)!}$$

$$= me^{-m} \sum \frac{m^{x-1}}{(x-1)!}$$

$$= me^{-m} \left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ \dots.\right]$$

$$= me^{-m}.e^m = m$$

$$\because \sum \frac{k^x}{x!} = 1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \cdots = e^k$$

Hence, $\boldsymbol{Mean \ = \ m}$

$$E(x^2) = \sum p_i {x_i}^2$$

$$= \sum_{x=0} \left(e^{-m}.\frac{m^x}{x!}\right).x^2$$

We can write $x^2 = x + x(x-1)$

$$= \sum_{x=0} \left(e^{-m}.\frac{m^x}{x!}\right).[x + x(x-1)] \ \mu_2' \ =$$

$$\sum_{x=0} e^{-m}.\frac{m^x x}{x!} + \sum_{x=0} e^{-m}.\frac{m^x}{x!} x.(x-1)$$

$$= me^{-m} \sum_{x=1} \frac{m^{x-1}}{(x-1)!} + m^2 e^{-m} \sum_{x=2} \frac{m^{x-2}}{(x-2)!}$$

$$\because \sum \frac{k^x}{x!} = 1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \cdots = e^k$$

$$= me^{-m}.e^m + m^2e^{-m}.e^m$$

$$= m + m^2$$

$$\therefore Variance \ = E(x^2) - [E(x)]$$

$$= (m + m^2) - m^2 = m$$

$$\boldsymbol{Variance \ = \ m}$$

Thus, **the mean and variance of the Poisson's distribution are both equal to m.**

MODE OF POISSON DISTRIBUTION

- If m is not an integer then the mode is the integer between $m - 1$ and m.
- If m is an integer then there are two modes, m and $m - 1$.

Note: If m is an integer then the mean and one mode coincide.

ADDITIVE PROPERTY OF INDEPENDENT POISSON DISTRIBUTIONS

(1) If two independent variates have a Poisson distribution with means m_1 and m_2 then their sum is also a Poisson distribution with mean $m_1 + m_2$

(2) The sum of two Poisson variates is a Poisson variate, but the difference between two Poisson variates is not a Poisson variates.

(3) If X_1 and X_2 are independent Poisson variates with parameter m_1, m_2 then $Y = a_1X_1 + a_2X_2$ is not a Poisson variate.

RECURRENCE RELATION OF POISSON DISTRIBUTION

For Poisson distribution

$$p(x) = \frac{e^{-m}.m^{x}}{x!}$$

$$\therefore p(x+1) = \frac{e^{-m}.m^{x+1}}{(x+1)!}$$

$$\therefore \frac{p(x+1)}{p(x)} = \frac{m^{x+1}}{(x+1)!}.\frac{x!}{m^{x}} = \frac{m}{x+1}$$

$$\therefore \boldsymbol{p(x+1) = \frac{m}{x+1}.p(x)}$$

EX 1. The mean and the variance of a probability distribution is 2. Write down the distribution.

SOLUTION : Here given Mean and Variance of distribution is 2.

Therefore the distribution is Poisson distribution.

$$m = 2$$

The distribution is given by

$$p(x) = \frac{e^{-m}.m^{x}}{x!}$$

$$p(x) = \frac{e^{-2}2^{x}}{x!}$$

EX 2. In a Poisson distribution the probability $P(x=3)$ is $2/3\ of\ P(X=4)$. Find the mean and the Standard deviation.

Solution: Let the parameter of the Poisson distribution be m

$$\therefore P(x)=\frac{e^{-m}\times m^{x}}{x!}$$

We are given that $P(X=3)=\frac{2}{3}P(X=4)$

$$\therefore \frac{e^{-m}\times m^{3}}{3!}=\frac{2e^{-m}\times m^{4}}{3\times 4!}$$

$$\therefore m=6$$

$\therefore$ The mean and variance $=6$

The standard deviation $=\sqrt{6}$

EX 3. If the mean of the Poisson distribution is 4, find $P(m - 2\sigma < X < m + 2\sigma)$

Solution:

For Poisson distribution mean $=$ variance $= m$

Hence, $m = 4$ and $\sigma = 2$

$$\therefore P(m - 2\sigma < X < m + 2\sigma)$$

$$= P(0 < X < 8) = P[X = 1, 2, 3, \ldots \ldots, 7]$$

But $P(X) = e^{-m} \frac{m^x}{x!} = e^{-4} \frac{4^x}{x!}$

$\therefore$ Required probability

$$= e^{-4} \left[\frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!}\right]$$

$$= 0.93$$

EX 4. If a random variable X follows Poisson distribution such that $P(X=2)=9\,P(X=4)+90P(X=6)$, find the mean and the variance of X.

Solution: Let m be the mean of X

$$P(X=x)=e^{-m}\frac{m^x}{x!}$$

By data $e^{-m}\frac{m^2}{2!}=9\cdot e^{-m}\frac{m^4}{4!}+90\cdot e^{-m}\frac{m^6}{6!}$

$$\therefore \frac{1}{2}=\frac{3m^2}{8}+\frac{m^4}{8}$$

$$\therefore m^4+3m^2-4=0$$

$$\therefore (m^2+4)(m^2-1)=0$$

$\therefore m^2=-4$ or $m^2=1$

$\therefore$ The mean and variance is 1 since $m>0$

EX 5. In a Poisson distribution the probability p(x) for $x = 0$ is 20 percent. Find the mean of the distribution.

EX 6. The probability that a Poisson variable X takes a positive value is $1 - e^{-1.5}$. Find the variance and the probability that X lies between -1.5 and 1.5 .

EX 7. If X and Y are independent Poisson variates such that $P(X=1)=P(X=2)\ and\ P(Y=2)=P(Y=3)$, find the variance of $3X-4Y$.

Solution: Let the parameter of X and Y be m_1 and m_2

$\therefore P(X=1)=P(X=2)$ gives $\frac{e^{-m_1}m_1}{1}=\frac{e^{-m_1}m_1{}^2}{2!}$

$$\therefore 2e^{-m_1}m_1-e^{-m_1}m_1{}^2=0 \qquad \therefore e^{-m_1}m_1(2-m_1)=0$$

$\therefore m_1=2$

$P(Y=2)=P(Y=3)$ gives $\frac{e^{-m_2}m_2{}^2}{2!}=\frac{e^{-m_2}m_2{}^3}{3!}$

$$\therefore 3e^{-m_2}m_2{}^2-e^{-m_2}m_2{}^3=0 \qquad \therefore e^{-m_2}m_2{}^2(3-m_2)=0$$

$$\therefore \text{Var.}(X)=m_1=2;\ \ \text{Var.}(Y)=m_2=3$$

Since, X and Y are independent

$V(3X-4Y)=9V(X)+16V(Y)=9(2)+16(3)=66$

EX 8. If X_1, X_2, X_3 are three independent Poisson variates with parameters $m_1 = 1, m_2 = 2, m_3 = 3$ respectively, find **(i)** $P[(X_1 + X_2 + X_3) \geq 3]$

(ii) $P[(X_1 + X_2 + X_3) \leq 3]$

(iii) $P[X_1 = 1/(X_1 + X_2 + X_3) = 3]$.

Solution: By additive property of Poisson distribution $Z = X_1 + X_2 + X_3$ is also a Poisson distribution with parameter $m = m_1 + m_2 + m_3 = 6$

(i) $\therefore P[(X_1 + X_2 + X_3) \geq 3] = P(Z \geq 3)$

$= 1 - P(Z \leq 2) \qquad = 1 - \sum_{Z=0}^{2} \frac{e^{-6} 6^Z}{Z!}$

$= 1 - 25e^{-6} = 1 - 25(0.002478) = 0.938$

(ii) $P(X_1 + X_2 + X_3 \leq 3) = P(Z \leq 3) = \sum_{z=0}^{3} \frac{e^{-6} 6^z}{z!}$

$= \frac{e^{-6} 6^0}{0!} + \frac{e^{-6} 6^1}{1!} + \frac{e^{-6} 6^2}{2!} + e^{-6} \frac{6^3}{3!}$

$= e^{-6}(1 + 6 + 18 + 36) = 61e^{-6} = 0.1512$

(iii) By definition of conditional probability

$$P[X_1 = 1/(X_1 + X_2 + X_3) = 3] = \frac{P(X_1 = 1 \text{ and } X_2 + X_3 = 2)}{P[(X_1 + X_2 + X_3) = 3]}$$

Now, X_1 is a Poisson variate with parameter $m_1 = 1$, $X_2 + X_3$ is a Poisson variate with parameter $m_2 + m_3 = 2 + 3 = 5$, $X_1 + X_2 + X_3$ is a Poisson variate with parameter $m_1 + m_2 + m_3 = 1 + 2 + 3 = 6$

$$\therefore P[X = 1/(X_1 + X_2 + X_3) = 3] = \frac{\left(e^{-1} \cdot \frac{1}{1}\right)\left(e^{-5} \cdot \frac{5^2}{2!}\right)}{e^{-6} \cdot \frac{6^3}{3!}} = \frac{25}{72}$$

EX 9. If the mean of the Poisson distribution is 2. Find the probabilities of $x = 1, 2, 3, 4,$ from the recurrence relation of Poisson distribution.

Solution: We have $P(x) = e^{-m}\frac{m^x}{x!}$

Since variance $= m = 2$ by data $P(x) = e^{-2}\frac{2^x}{x!}$

when $x = 0, P(0) = e^{-2}$

Now, the recurrence relation is $P(x+1) = \frac{m}{x+1}P(x)$

Putting $x = 0, P(1) = \frac{2}{1}P(0) = 2e^{-2}$

Putting $x = 1, P(2) = \frac{2}{2}P(1) = \frac{2}{2}\cdot 2e^{-2} = 2e^{-2}$

Putting $x = 2, P(3) = \frac{2}{3}P(2) = \frac{4}{3}e^{-2}$

Putting $x = 3, P(4) = \frac{2}{4}P(3) = \frac{2}{3}e^{-2}$

EX 10. Using Poisson distribution, find the approximate value of

$$^{300}C_2(0.03)^2(0.97)^{298} + {}^{300}C_3(0.03)^3(0.97)^{297}$$

Solution: Clearly the above probabilities are the probabilities of Binomial distribution.

Comparing them with $P(X = x) = {}^nC_x\, p^x\, q^{n-x}$

We see that $n = 300, p = 0.03, q = 0.97, x = 2$ and 3

Now, Binomial distribution is related to Poisson distribution where $m = np$

Hence, $m = 300 \times 0.03 = 9$

$\therefore$ Corresponding Poisson distribution is given by

$$P(X = x) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-9} 9^x}{x!}$$

$$P(X = 2) + P(X = 3) = \frac{e^{-9} \cdot 9^2}{2} + \frac{e^{-9} \cdot 9^3}{6}$$

$$= 0.0199$$

EX 11. A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1.5. Calculate the proportion of days on which **(i)** neither car is used,

(ii) some demand is refused.

Solution: We have $P(x) = e^{-m}\frac{m^x}{x!} = \frac{e^{-1.5}\cdot(1.5)^x}{x!}, \quad x = 0, 1, 2, \ldots\ldots$

(i) Probability that there is no demand is $P(X = 0) = e^{-1.5}\frac{(1.5)^0}{0!} = 0.2231$

(ii) Probability that some demand is refused means there was demand for more than two cars

$$\therefore P(X > 2) = P(X = 3) + P(X = 4) + \cdots$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[e^{-1.5}\frac{(1.5)^0}{0!} + e^{-1.5}\frac{(1.5)^1}{1!} + e^{-1.5}\frac{(1.5)^2}{2!}\right]$$
$$= 1 - [0.2231 + 0.3347 + 0.2510] = 0.1912$$

$\therefore$ Proportion of days on which

(i) neither car is used is 0.2231

(ii) some demand is refused is 0.1912

EX 12. A hospital switch board receives an average of 4 emergency calls in a 10 minutes interval. What is the probability that

(i) there are at least 2 emergency calls

(ii) There are exactly 3 emergency call an interval of 10 minutes?

Solution: We have $P(x) = \frac{e^{-m} \cdot m^x}{x!}$. Here, $m = 4$

(i) $P(X \geq 2) = 1 - P(X < 2)$

$$= 1 - \{P(X = 0) + P(X = 1)\}$$

$$= 1 - \frac{e^{-4} \cdot 4^0}{0!} - \frac{e^{-4} \cdot 4^1}{1!}$$

$= 1 - 5e^{-4} = 0.9084$

(ii) $P(X = 3) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-4} \cdot 4^3}{3!}$

$$= 0.195$$

EX 13. An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that no more than two of its clients are involved in such accident next year?

Solution: We have $p = \frac{0.01}{100} = 0.0001, n = 1000$

$$m = np = 1000 \times 0.0001 = 0.1$$

$$\therefore P(X = x) = e^{-0.1} \cdot \frac{(0.1)^x}{x!}$$

$$\therefore P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= e^{-0.1} \left[\frac{(0.1)^0}{0!} + \frac{(0.1)^1}{1!} + \frac{(0.1)^2}{2!} \right]$$

$$= 0.9998$$

EX 14. Find the probability that at most 4 defective bulbs will be found in a box of 200 bulbs if it is know that 2 percent of the bulbs are defective.

Solution: Since the probability of a defective bulbs is small we can use Poisson distribution

We have, $\therefore m = np = 200 \times 0.02 = 4$

$$\therefore P(X = x) = \frac{e^{-4} \times 4^x}{x!}$$

$$\therefore P(X \leq 4) = p(0) + p(1) + p(2) + p(3) + p(4)$$

$$= \frac{e^{-4} \times 4^0}{0!} + \frac{e^{-4} \times 4^1}{1!} + \frac{e^{-4} \times 4^2}{2!} + \frac{e^{-4} \times 4^3}{3!} + \frac{e^{-4} \times 4^4}{4!}$$

$$= e^{-4}\left[1 + \frac{4}{1!} + \frac{16}{2!} + \frac{64}{3!} + \frac{256}{4!}\right]$$

$$= e^{-4} \times \frac{103}{3} = 0.0183 \times \frac{103}{3} = 0.6283$$

EX 15. In sampling a large number of parts manufactured by a machine, the mean numbers of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would you expect to contain 3 defectives using

(i) Binomial distribution, **(ii)** Poisson distribution.

Solution:

(i) Binomial Distribution: We have $p = \frac{2}{20} = 0.1$

$$q = 1 - p = 0.9, n = 20$$

$$\therefore P(X = x) = {}^{n}C_x\, p^x\, q^{n-x} = {}^{20}C_x\, (0.1)^x\, (0.9)^{20-x}$$

$$P(X = 3) \qquad = {}^{20}C_3\, (0.1)^3\, (0.9)^{17}$$

$$= 0.1901$$

$\therefore$ out of 1000 such samples, $1000 \times 0.1901 = 190$ samples are expected to have 3 defectives.

(ii) Poisson Distribution: We have $m = 2$

$$\therefore P(X = x) = e^{-m} \cdot \frac{m^x}{x!} = e^{-2} \cdot \frac{(2)^x}{x!}$$

$$\therefore P(X = 3) = e^{-2} \cdot \frac{(2)^3}{3!}$$

$$= 0.1804$$

$\therefore$ out of 1000 such samples, $1000 \times 0.1804 =$ 180 samples are expected to have 3 defectives.

EX 16. In a certain factory turning out blades, there is a small chance $1/500$ for any blade to be defective. The blades are supplied in packets of 10. Use the Poisson distribution to calculate the approximate number of packets containing no defective, one defective, two defective blades in a consignment of 10,000 packets (Given $e^{-0.02} = 0.9802$)

Solution: We have, $n = 10, p = \frac{1}{500}$

$\therefore m = np = 10 \times \frac{1}{500} = 0.02$

$\therefore P(X = x) = \frac{e^{-0.02} \times (0.02)^x}{x!}$

$\therefore P(X = 0) = \frac{e^{-0.02} \times (0.02)^0}{0!} = e^{-0.02} = 0.9802$

$\therefore P(X = 1) = \frac{e^{-0.02} \times (0.02)^1}{1!} = e^{-0.02} \times 0.02 = 0.0196$

$\therefore P(X = 2) = \frac{e^{-0.02} \times (0.02)^2}{2!} = e^{-0.02} \times 0.0002 = 0.0002$

$\therefore$ Expected freq. of no defective $= 10000 \times 0.9802 = 9802$

Expected freq. of one defective $= 10000 \times 0.0196 = 196$

Expected freq. of two defective $= 10000 \times 0.0002 = 2$

EX 17. If m is the mean of a Poisson distribution, show that the probability that the value of the variate taken at random will be even or odd are $\mathrm{e^{-m}} \cos \mathrm{hm}$ & $e^{-m} \sin \mathrm{hm}$.

Solution: We have $P(X=x)=e^{-m}\frac{m^x}{x!}$

$$P(\text{Even number}) = P(X=0,2,4,6,\ldots\ldots)$$
$$= P(X=0)+P(X=2)+P(X=4)+\cdots$$
$$= e^{-m}\frac{m^0}{0!}+e^{-m}\frac{m^2}{2!}+e^{-m}\frac{m^4}{4!}+\cdots$$
$$= e^{-m}\left[1+\frac{m^2}{2!}+\frac{m^4}{4!}+\cdots\right]=e^{-m}\cosh m$$
$$P(\text{Odd number}) = P(X=1,3,5,7,\ldots.)$$
$$= P(X=1)+P(X=3)+P(X=5)+\cdots$$
$$= e^{-m}\frac{m^1}{1!}+e^{-m}\frac{m^3}{3!}+e^{-m}\frac{m^5}{5!}+\cdots$$
$$= e^{-m}\left[m+\frac{m^3}{3!}+\frac{m^5}{5!}+\cdots\right]=e^{-m}\sinh m$$

EX 18. Fit a Poisson distribution to the following data.

No of heads	0	1	2	3	4
Frequency	123	59	14	3	1

Solution: Fitting Poisson distribution means finding expected frequencies of $X = 0, 1, 2, 3, 4$

Now mean $= \frac{\sum f_i x_i}{\sum f_i} = m = \frac{123(0)+59(1)+14(2)+3(3)+1(4)}{200}$

$$= \frac{100}{200} = 0.5$$

$\therefore$ Poisson distribution of X is $P(X = x) = \frac{e^{-m} \times m^x}{x!} = \frac{e^{-0.5} \times (0.5)^x}{x!}$

Expected frequency $= N \times p(x) = 200 \times \frac{e^{-0.5} \times (0.5)^x}{x!}$

Putting $x = 0, 1, 2, 3, 4$ we get the expected frequencies as $121, 61, 15, 2, 1$