Channel matrix

→ Table that shows how a communication channel behaves.

→ Chances of getting a certain o/p when you send a certain input.

→ Each row in the table is for a specific input (what you send).

→ Each column is for a specific output

→ Each entry in the table shows the Probability of a particular o/p given a specific input

→ It helps us to understand the behavior of noise (or) channel introduces

→ when you send "0" or "1" through a noisy line

0 → 90% → 0 and a chance 10% → 1

1 → 80% — 1 and a chance of 20% → 0.

$$P = \begin{bmatrix} 1-P & P \\ P & 1-P \end{bmatrix} = \begin{bmatrix} 90\% & 10\% \\ 0.2 & 0.8 \\ 20\% & 80\% \end{bmatrix}$$

Here Row 1 → Probabilities when '0' is sent

Row 2 → Probabilities when "1" is sent

Channel matrix for Binary Symmetric Channel (BSC)

In BSC

→ If you send a bit "0" or "1" there is a 10% chance of error

→ Calculate

→ The channel matrix

→ The Probability of receiving a "0" if "0" is sent

→ The Probability of receiving a "1" if "1" is sent

Channel matrix

$$P = \begin{bmatrix} P(0|0) & P(1|0) \\ P(0/1) & P(1|1) \end{bmatrix}$$

$P(0|0)$: Probability of receiving '0' when '0' is sent

$P(1|0)$: Probability of receiving "1" when "0" is sent

$P(0|1)$: Probability of receiving "0" when "1" is sent

$P(1|1)$: Probability of receiving "1" when "1" is sent

From the Problem

Probability of no error $= 1 - 0.1 = 0.9$

Probability of error $= 0.1$

$\therefore$ The channel matrix is

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$$

ii) Probability of receiving "0" if "0" is sent

$P(0|0) = 0.9$.

The probability of correctly receiving '0' when 0 is sent is 90%.

iii) $P(1|1) = 0.9$ i.e 90%.

Channel matrix $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$

BSC shows 10% errors while still having 90% reliability.

<u>Noisy communication channel</u>.

Suppose we have a channel with 3 inputs $X = \{x_1, x_2, x_3\}$ and 3 outputs $Y = \{y_1, y_2, y_3\}$.

If x_1 is sent $P(y_1|x_1) = 0.6$ $P(y_2|x_1) = 0.3$ $P(y_3|x_1) = 0.1$

If x_2 is sent $P(y_1|x_2) = 0.2$ $P(y_2|x_2) = 0.7$, $P(y_3|x_2) = 0.1$

x_3 is sent $P(y_1|x_3) = 0.3$, $P(y_2|x_3) = 0.3$, $P(y_3|x_3) = 0.4$

Channel matrix

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

→ what is the Probability of receiving y_2 if x_2 is sent?

$P(y_2 | x_2) = 0.7$ → 70%

$P(y_3 | x_1) = 0.1$ → 10%

Channel Point Probability matrix

→ It's a table that shows how likely it is to send a certain input ("0" or "1") and get a certain output ("0" or "1") together. x.

→ This is different from a normal channel matrix, which only shows the chance of getting a specific output after sending a specific input.

→ Input Probabilities : How likely you are to send each input ("0" or "1").

Channel matrix : The chance of receiving an output for a given input.

Formula $P(X, Y) = P(X) \times P(Y/X)$

$P(X, Y) = P(X) \times P(Y/X)$ is the fundamental rule of Probability. → Chain rule.

To calculate the joint Probability of two events X and Y, where

$P(X,Y)$: The joint probability that both X & Y happen together

$P(X)$: The probability of X

$P(Y|X)$: The conditional Probability that Y happens given that X has already happend.

Ex:-

Imagine tossing a coin and rolling a dice.

Find the Probability of

→ Tossing a Heads (X) and rolling "6" (Y).

1) Probability of tossing "Heads" $P(X)$: $P(X) = 0.5$

2) Probability of rolling a "6" given that

$P(Y|X) = \frac{1}{6} = 0.1667$

$P(X, Y) = P(X) \times P(Y|X)$

$= 0.5 \times 0.1667$

$= 0.0833$

So the Probability of tossing Heads and rolling a die 6 is 8.33%.

Cont'd: Point probability matrix

Assume bit (0, 1) over a noisy channel

→ Input Probabilities $P(0) = 0.7$ (70% chance you send "0")

$P(1) = 0.3$ (30% chance you send "1").

→ Channel matrix (noise flips the bit)

$$P(y|x) = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}$$

Calculation

i) when sending 0 and receiving 0

$P(0,0) = P(0) \times P(0|0) = 0.7 \times 0.8 = 0.56$

ii) when sending 1 and receiving 1

$P(0,1) = P(0) \times P(1|0) = 0.7 \times 0.2 = 0.14$

Case iii) when sending 1 and receiving 0

$P(1,0) = P(1) \times P(0|1) = 0.3 \times 0.4 = 0.12$

iv) when sending 1 and receiving 1

$P(1,1) = P(1) \times P(1|1) = 0.3 \times 0.6 = 0.18$

$$\therefore P(x,y) = \begin{bmatrix} 0.56 & 0.14 \\ 0.12 & 0.18 \end{bmatrix}$$

Marginal Entropy :-

→ measures the average amount of uncertainty associated with a single random variable.

Discrete random Variable x with a Probability distribution $P(x)$, the marginal entropy $H(x)$ is defined as

$$H(x) = -\sum_{x \in X} P(x) \log_2 P(x)$$

$P(x)$: The Probability of each Possible value x of X

$\log_2$: logarithm of base 2

x : Each Possible Value of X

→ Marginal entropy gives the average no. of bits needed to encode the outcomes of X.

→ if all outcomes are equally likely, entropy is max
If " " " more likely, " " less.

Case : Coin Toss

$X = \{H, T\}$ $P(H) = 0.5$, $P(T) = 0.5$

Entropy : $H(X) = -[P(H)\log_2 P(H) + P(T)\log_2 P(T)]$

$H(X) = -[0.5 \log_2 0.5 + 0.5 \log_2 0.5]$

$= -[0.5(-1) + 0.5(-1)] = 1$ bit.

Special channels

→ Binary Symmetric Channel (BSC)

Definition:-

→ where each transmitted bit (0 or 1) has a fixed probability P of being received incorrectly (flipped) and a probability $1-P$ of being received correctly.

Explanation

→ It's like sending a message through a noisy wire

→ Errors happen randomly with a fixed chance P.

→ Sending 0 might result in the receiver hearing 1 because of noise in the system.

Example

Let $P = 0.1$ (10% error rate)

You send the bit 0: → 90% chance the receiver gets 0

→ 10% chance the receiver gets 1.

i.e $P(Y=0 \mid X=1) = P(Y=1 \mid X=0) = P$

$P(Y=1 \mid X=1) = P(Y=0 \mid X=0) = 1-P$.

Capacity of the BSC is $C = 1 - H(P)$

$H(P)$: binary entropy

Binary Erasure Channel

→ Channel where each bit is either received correctly or lost. The Probability of erasure is ϵ, and the Probability of correct reception is $1-\epsilon$.

Explanation

→ It's like sending a message that can sometimes get lost in transit.

→ Instead of flipping, the bit becomes unknown to the receiver.

→ If you send a bit and it gets erased, the receiver knows there's an error but doesn't know what the bit was.

Ex:- Let $\epsilon = 0.2$ (20% erasure rate)

i.e 80% chance the receiver gets 1

20% chance the receiver gets an erasure.

i.e A channel where each transmitted bit is either received correctly with a Probability $1-\epsilon$ or erased with a Probability ϵ

$P(Y=x) = 1-\epsilon$, $P(Y = \text{Erasure}) = \epsilon$

Capacity $C = 1-\epsilon$ i.e only $1-\epsilon$ bits are usable

Z channel

A Z channel where one input (eg 0) is always transmitted correctly but the other input (1) has a probability p of being flipped to 0.

Explanation

→ Think of this as a biased (or) one sided

→ 0 is safe and always received correctly

→ 1 is risky because it might flip to 0.

Ex:- It's like sending messages over a channel that favors one input over the other.

Let p = 0.3 (30% error rate for 1)

When you send 0: Always received as 0

1: 70% chance of being received as 1 & 30% chance of flipping to 0.

→ Commonly used in systems where noise affects one symbol more than the other.

Channel capacity.

→ It is defined as the maximum average mutual information between the input and the output of a channel, maximized over all possible input probability distributions.

$$C = \max_{P(x)} I(X;Y)$$

C - channel capacity, $I(X;Y)$ mutual information between input X and output Y.

$P(x)$: Input Probability distribution

→ It describes how much information can be sent through a channel without errors, despite noise or interference

Ex:-

for a Binary Symmetric Channel (BSC)

$C = 1 - H(P)$ H(P) binary entropy function

$$H(P) = -P\log_2 P - (1-P)\log_2(1-P)$$

Capacity of Special Channels

→ BSC is defined as with a crossover probability P which represents the probability of a bit being flipped.

→ The capacity of the BSC is given by

$$C = 1 - H(P)$$

$$H(P) = -P\log_2(P) - (1-P)\log_2(1-P)$$

$P=0$, (no noise), $C=1$ bit/use (Perfect communication)

$P=0.5$ (max noise), $C=0$ (no reliable communication)

if $P = 0.1$

$$H(0.1) = -(0.1\log_2(0.1) + (0.9)\log_2(0.9))$$

$$= 0.469$$

$$C = 1 - 0.469 \simeq 0.531 \text{ bits per use.}$$

BEC

$$C = 1 - \epsilon$$

if $\epsilon = 0$ (no Erasure) $C = 1$ bit/use

if $\epsilon = 1$ (all bits erased), $C = 0$.

If $\epsilon = 0.2$ $C = 1 - 0.2 = 0.8$ bits per use.

Z channel is an asymmetric channel. 0 - correct
1 - flip to 0.

* Comparision of two sources

$R_1 = r_1 . h$
$R_2 = r_2 . h$

If $r_1 > r_2$
$R_1 > R_2$
The first source generates more information per se-cond.

Verify the equation,

$I(n_i; x_j) = I(x_i) + (x_j)$
If x_i & x_j are independent

I)
$I(x,y)$ — mutual information
$I(x)$, $I(y)$ — Information measures for x & y

$p(x,y) = p(x) . p(y) \rightarrow x, y$ independent

$$I(x,y) = \log \frac{1}{p(x,y)}$$
$$= \log \frac{1}{p(x) . p(y)}$$

$\log(ab) = \log a + \log b$

$$\log \frac{1}{p(x) . p(y)} = \log \frac{1}{p(x)} + \log \frac{1}{p(y)}$$

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$$\therefore I(x,y) = \log \frac{1}{p(x)} + \log \frac{1}{p(y)}$$

$$\log \frac{1}{p(x)} = I(x)$$

$$\log \frac{1}{p(y)} = I(y)$$

$$\boxed{\therefore I(x,y) = I(x) + I(y)}$$

Mutual Information measures dependence b/w 2 random variables
If x & y are independent, $I(x,y) = 0$

Channel Model

Input	Channel
Output	Noiseless Channel
Distortion	Noisy Channel
Attenuation	Additive White Gaussian
Interference	Noise (AWGN)
Noise	

* Joint Entropy

$H(x,y)$ i.e. rolling a die or flipping a coin
$x = 6$, $Y = 2$

$$H(x,y) = -\sum_{x=1}^{m}\sum_{y=1}^{n} P(x,y) \log_2 P(x,y)$$

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Properties

i) $H(x,y) \geq \max(H(x), H(y))$

ii) $H(x,y) = H(x) + H(y)$

Eg. Imagine X is rolling a 6 sided die, Y is flipping a coin.
If the die is fair, $P(x) = 1/6$ $P(y) = 1/2$

$$P(X,Y) = P(X) . P(Y)$$
$$= \frac{1}{6} . \frac{1}{2}$$
$$= \frac{1}{12}$$

$$H(x,y) = -\sum_{x=1}^{6} \sum_{y=1}^{2} \frac{1}{12} \log_2 \left(\frac{1}{12}\right)$$

$$= -\sum\sum(-0.298)$$
$$= \sum_{1}^{6} \sum_{1}^{2} 0.298$$

* **Conditional Probability**

$$H(X/Y) = \sum_{x=1}^{m} \sum_{y=1}^{n} P(x_i y_j) \log_2 \left[\frac{1}{P(x/y)}\right]$$

$$H(Y/X) = \sum_{x=1}^{m} \sum_{y=1}^{n} P(xy) \log_2 \left[\frac{1}{P(y|x)}\right]$$

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Relationship

$$H(X,Y) = H(X/Y) + H(Y)$$

$$= H(Y/X) + H(X)$$

* Marginal Entropy

Consider two independent events - X & Y

1) For every X, there are m symbols
$X = [x_1, x_2, x_3, x_4 \ldots]$

2) For every Y, there are ~~m~~n symbols

Marginal Entropy

$$H(x) = \sum_{i=1}^{m} P(x_i) \log_2 \frac{1}{P_{x_i}}$$

or

$$H(x) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i)$$ = Avg uncertainity in input

$$H(Y) = -\sum_{i=1}^{n} P(y_i) \log_2 P(y_i)$$ = Avg uncertainity in output

* Shannon's formula

$$C = B \log_2 (1 + SNR)$$

C : Channel capacity
B : Bandwidth
SNR : Signal to Noise Ratio

$$SNR = \frac{S}{N}$$

B = ~~1~~ MHz = $\times 10^6$ Hz
SNR = db

Eg. $B = 1 MHz$

$SNR = 10 db$

$C = 1 \times 10^6 \log_2 (1+10) = 3.459$

Eg. $B = 3 kHz$

$SNR = -20 dB = -(10^{20/10})$

$= -(10)^2$

$= -100$

$C = B \log_2 (1 + SNR)$

$= 3 \times 10^3 \log_2 (1 + (-100))$

$= 19888$

→ Numerical

Let (X,Y) have the following Joint distribution

Q Let (X,Y) have the following Joint Distribution

Y \ X →	1	2	3	4
$P(x,y)=1$ → 1	1/8	1/16	1/32	1/32
2	1/16	1/8	1/32	1/32
3	1/16	1/16	1/16	1/16
4	1/4	0	0	0
	$P(y, \hat{x})=1$	$P(y, \hat{x})=2$		

Marginal Distribution of X $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$

Marginal Distribution of Y $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

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Sol^n ① find H(X)

$$H(X) = -\sum_{i=1}^{4} P(x_i) \log_2 P(x_i)$$

$$= -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{4}\log_2\frac{1}{4} + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{8}\log_2\frac{1}{8}\right)$$

$$= -\left(\frac{-7}{4}\right)$$

$$H(X) = \frac{7}{4} \text{ bits}$$

Similarly,

② $H(Y) = 2$ bits

③ $H(X/Y) = \sum_{i=1}^{4} P(Y=i)\, H(X/Y=i)$

$$= P(Y=1)\,H(X/Y=1) + P(Y=2)\,H(X/Y=2) + P(Y=3)\,H(X/Y=3) + P(Y=4)\,H(X/Y=4)$$

$\because P(x,y) = P(y)\,P(x,y)$

if we take $P(y=1)$ common from $P(x,y)=1$

Then, by taking $\frac{1}{4}$ common, we get;

$$P(x/y=1) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

Similarly,

$P(x/y=2) = (1/4, 1/2, 1/8, 1/8)$

$P(x/y=3) = (1/4, 1/4, 1/4, 1/4)$

$P(x/y=4) = (1, 0, 0, 0)$

$\therefore H(X/Y) = 1/4\, H(1/2, 1/4, 1/8, 1/8) + 1/4\, H(1/4, 1/2, 1/8, 1/8) + 1/4\, H(1/4, 1/4, 1/4, 1/4) + 1/4\, H(1, 0, 0, 0)$

$= \frac{1}{4} \times \frac{7}{4} + \frac{1}{4} \times \frac{7}{4} + \frac{1}{4} \times 2 + \frac{1}{4} \times 0$

$= \frac{11}{8}$ bits

④ find $H(Y/X)$

$$H(Y/X) = \sum_{i=1}^{4} P(x=i)\, H(Y/x=i)$$

$$= P(X=1)\, H(Y/X=1) + P(X=2)\, H(Y/X=2) + P(X=3)\, H(Y/X=3) + P(Y=4)\, H(Y/X=4)$$

$$= \frac{1}{2} H\left(\tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{2}\right) + \frac{1}{4} H\left(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}, 0\right) + \frac{1}{8} H\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, 0\right) + \frac{1}{8} H\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, 0\right)$$

$$H\left(\tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{2}\right) = -\left(\frac{1}{4}\log_2\frac{1}{4} + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{8}\log_2\frac{1}{8} + \frac{1}{2}\log_2\frac{1}{2}\right)$$

$$= \frac{7}{4}$$

$$\therefore H(Y/X) = \frac{1}{2} \times \frac{7}{4} + \frac{1}{4} \times \frac{3}{2} + \frac{1}{8} \times \frac{3}{2} + \frac{1}{8} \times \frac{3}{2}$$

$= \frac{13}{8}$ bits

⑤ find $H(x,y)$

$$H(x,y) = H(X) + H(Y/X) = \frac{7}{4} + \frac{13}{8} = \frac{14+13}{8} = \frac{27}{8} \text{ bits}$$

$$= H(Y) + H(X/Y) = 2 + \frac{11}{8} = \frac{16+11}{8} = \frac{27}{8} \text{ bits}$$