EXPONENTIAL DISTRIBUTION

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- Many scientific experiments involve the measurement of the duration of time X between an initial point of time and occurrence of some phenomenon of interest.
- For example X is the life time of a light bulb which is turned on and left until it burns out.

DEFINITION

- The continuous random variable X having the probability density function
- $f(x) = \begin{cases} \lambda e^{-\lambda x}, \text{if } x \geq 0 \\ 0, \qquad \text{if } x < 0 \end{cases}$ is said to have an exponential distribution.
- Here the only parameter of the distribution is λ which is greater than zero.

EXAMPLES:

Examples of random variables modeled as exponential are

a) (inter arrival) time between two successive job arrivals

b) duration of telephone calls

c) life time (or time to failure) of a component or a product

d) service time at a server in a queue

e) time required for repair of component

- The exponential distribution occurs most often in application **Reliability Theory** and **Queuing Theory** because of the memoryless property and relation to the (discrete) Possion Distribution.
- Exponential distribution can be obtained from the Poisson distribution by considering the inter arrival times rather than the number of arrivals

MEAN AND VARIANCE

For any $r \geq 0$, $\quad E(X^r) = \int_0^\infty x^r f(x)\,dx = \int_0^\infty x^r \lambda e^{-\lambda x}\,dx$

Put $\lambda x = t, x = \frac{t}{\lambda}, dx = \frac{1}{\lambda} dt$ Then

$$E(X^r) = \int_0^\infty \left(\frac{t}{\lambda}\right)^r \cdot \lambda \cdot e^{-t} \frac{1}{\lambda} dt = \frac{1}{\lambda^r} \int_0^\infty e^{-t} t^r\,dt = \frac{\Gamma(r+1)}{\lambda^r}$$

$\therefore E(X^r) = \frac{\Gamma(r+1)}{\lambda^r}$

In particular with $r = 0$, $\int_0^\infty f(x)\,dx = \Gamma(1) = 1$ (i.e., $f(x)$ is probability density function)

With $r = 1$, mean $=$ $\mu = E(X) = \frac{\Gamma(2)}{\lambda} = \frac{1}{\lambda}$

With $r = 2$, variance $=$ $\sigma^2 = E(X^2) - \{E(X)\}^2$

$$= \frac{\Gamma(3)}{\lambda^2} - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Note: Both the mean and standard deviation of the exponential distribution are equal to $\frac{1}{\lambda}$

CUMULATIVE DISTRIBUTION FUNCTION

$F(x) = \int_0^x f(t)\,dt$

$= \int_0^x \lambda e^{-\lambda t}\,dt = \left.\frac{\lambda e^{-\lambda t}}{-\lambda}\right|_{t=0}^{x}$

$= 1 - e^{-\lambda x}$ for $x \geq 0$

and $F(x) = 0$ when $x < 0$

$F(x)$ gives the probability that the “system” will “die” before x units of time have passed

PROBABILITY CALCULATIONS

For any $a \geq 0$,

$$P(X \geq a) = P(X > a) = 1 - F(a) = e^{-\lambda a}$$

$$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X < b)$$
$$= P(a < X \leq b)$$
$$= F(b) - F(a) = e^{-\lambda a} - e^{-\lambda b}$$

Corollary: $P\left(X > \frac{1}{\lambda}\right) = e^{-\lambda\frac{1}{\lambda}} = e^{-1} = 0.368 < \frac{1}{2}$

SURVIVAL FUNCTION

It gives the probability that the "system" survives more than x units of time and is given by

$$P(X > x) = 1 - F(x) = \begin{cases} 1, & \text{if } x < 0 \\ e^{-\lambda x}, & \text{if } x \geq 0 \end{cases}$$

EX 1. Let the mileage (In thousand of miles) of a particular tire be a random variable X having the probability density $f(x) = \begin{cases} \frac{1}{20} e^{-x/20}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$

Find the probability that one of these tires will last

(a) at most 10,000 miles

(b) anywhere from 16,000 to 24,000 miles

(c) at least 30,000 miles

(d) find the mean

(e) find the variance of the given probability density

Solution:

(a) probability that a tire will last at most 10,000 miles

$$= P(X \le 10) = \int_0^{10} f(x)\,dx = \int_0^{10} \frac{1}{20} e^{-x/20}\,dx$$

$$= \frac{1}{20} \cdot e^{-x/20} \cdot \left(\frac{-20}{1}\right)\Big|_0^{10} = 1 - e^{-\frac{1}{2}} = 0.3934$$

(b) probability that one of these tires will last anywhere from 16,000 to 24,000 miles

$$P(16 \le X \le 24) = \int_{16}^{24} f(x)\,dx = \int_{16}^{24} \frac{1}{20} e^{-x/20}\,dx = -e^{-x/20}\Big|_{16}^{24}$$

$$= e^{-\frac{4}{5}} - e^{-\frac{6}{5}} = 0.148$$

(c) probability that a tire will last at least 30,000 miles

$$P(X \ge 30) = \int_{30}^{\infty} f(x)\,dx = \int_{30}^{\infty} \frac{1}{20} e^{-x/20}\,dx = -e^{-x/20}\Big|_{30}^{\infty} = e^{-\frac{3}{2}}$$

$$= 0.2231$$

Solution:

(d) Mean $= \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} x \cdot \frac{1}{20} e^{-x/20}\,dx$

$$= -\int_0^{\infty} x \cdot d\left(e^{-\frac{x}{20}}\right)$$

$$= -xe^{-\frac{x}{20}} - 20e^{-\frac{x}{20}}\Big|_0^{\infty} = 0 - (-20) = 20 = \frac{1}{\lambda}$$

(e) Variance $= \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

Consider $\int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2 \frac{1}{20} e^{-\frac{x}{20}}\,dx$

$$= -x^2 e^{-\frac{x}{20}}\Big|_0^{\infty} + 2.20 \cdot \int_0^{\infty} \frac{1}{20} e^{-\frac{x}{20}}\,dx$$

$$= 0 + 2.20 \cdot \mu = 2.20.20 = 2.20^2$$

Then $\sigma^2 = \int_0^{\infty} x^2 f(x)\,dx - \mu^2 = 2.20^2 - 20^2 = 20^2 = \frac{1}{\lambda^2}$

EX 2. If a random variable X has the exponential distribution with mean $\mu = \frac{1}{\lambda} = \frac{1}{2}$ calculate the probabilities that

(a) X will lie between 1 and 3

(b) X is greater than 0.5

(c) X is at most 4

Solution: Given $\lambda = 2$

The P.d.f of X is $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) $P(1 < X < 3) = \int_1^3 2e^{-2x}\,dx = -[e^{-2x}]_1^3 = e^{-2} - e^{-6} = 0.1328$

(b) $P(X > 0.5) = \int_{0.5}^{\infty} 2e^{-2x}\,dx = 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty} = -(0 - e^{-1}) = 0.3678$

(c) $P(X < 4) = \int_{-\infty}^{4} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^4 2e^{-2x}\,dx = 2\left[\frac{e^{-2x}}{-2}\right]_0^4 = -(e^{-8} - e^0) = 1 - e^{-8} = 0.999$

EX 3. The length of time for one person to be served at a cafeteria is a random variable X having an exponential distribution with a mean of 4 minutes. Find the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days

Solution: The probability that a person served at a cafeteria in less than 3 minutes is

$$P(T < 3) = 1 - P(T \geq 3)$$

Since the mean $\mu = \frac{1}{\lambda} = 4$ or $\lambda = \frac{1}{4}$, the exponential distribution is $\frac{1}{4} e^{-\frac{t}{4}}$

Now, $\quad P(T < 3) = 1 - P(T \geq 3) = 1 - \int_3^\infty \frac{1}{4} e^{-\frac{t}{4}} dt$

$$= 1 - \frac{1}{4} e^{-\frac{t}{4}} \cdot \left(\frac{-4}{1}\right)\Bigg|_3^\infty = 1 - e^{-\frac{3}{4}} = 0.5276$$

Let X represent the number of days on which a person is served in less than 3 minutes.

Then using the binomial distribution, the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days is

$$P(X \geq 4) = \sum_{x=4}^{6} {}^6C_x \, (0.5276)^x (0.4724)^{6-x} = 0.3968$$

EX 4. Let T be the time (in years) to failure of certain components of a system. The random variable T has exponential distribution with mean time to failure $\beta = 5$. If 5 of these component are in different system, find the probability that at least 2 are still functioning at the end of 8 years.

Solution: P.d.f. for T is $f(t) = \begin{cases} \frac{1}{5} e^{-t/5}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$P(\text{component is functioning at the end of 8 years}) = P(T > 8)$$

$$= \int_8^\infty f(t)\, dt = \int_8^\infty \frac{1}{5} e^{-t/5}\, dt = \frac{1}{5}\left(\frac{e^{-t/5}}{-1/5}\right)_8^\infty = -\left(0 - e^{-8/5}\right) = 0.2018$$

Let X represent the no. of component are functioning at the end of 8 years

Then using the binomial distribution, $p = 0.2018, q = 0.7982, n = 5$

Required probability

$$= P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - [{}^5C_0 p^0 q^5 + {}^5C_4 p^1 q^4] = 1 - [(0.7982)^5 + 5(0.2018)(0.7982)^4]$$

$$= 0.2666$$

EX 5. The life (in years) of a certain electrical switch has an exponential distribution with an average life of $\frac{1}{\lambda} = 2$. If 100 of these switches are installed in different system, find the probability that at most 30 fail during the first year

Solution: Given $\lambda = \frac{1}{2}$

$\therefore$ P.d.f for T is $f(t) = \begin{cases} \frac{1}{2} e^{-t/2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$P(\text{switch falis during the 1st year})$

$= P(T < 1) = \int_{-\infty}^{1} f(t)\, dt$

$= \frac{1}{2} \int_{0}^{1} e^{-t/2}\, dt = -\left[e^{-t/2}\right]_0^1 = -\left(e^{-1/2} - 1\right) = 0.3934$

Let X represents the no. of switches which fail during the 1st year Then using the binomial distribution,

$p = 0.3934, q = 0.6066, n = 100$

Required probability $= P(X \leq 30)$

$= \sum_{r=0}^{30} {}^{n}C_r p^r q^{n-r} = \sum_{r=0}^{30} {}^{100}C_r (03934)^r (0.6066)^{100-r}$

EX 6. The time X (seconds) that it takes a certain online computer terminal (the elapsed time between the end of user's inquiry and the beginning of the system's response to that inquiry) has an exponential distribution with expected time 20 seconds. Compute the probabilities

(a) $P(X \leq 30)$ **(b)** $P(X \geq 20)$ **(c)** $P(20 \leq X \leq 30)$

(d) For what value of t is $P(X \leq t) = 0.5$

Solution: Given $\lambda = \frac{1}{20}$

$\therefore$ P.d.f for X is $f(x) = \begin{cases} \frac{1}{20} e^{-x/20}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(a) $P(X \leq 30) = \int_0^{30} \frac{1}{20} e^{-\frac{1}{20}x}\, dx$

$$= -\left[e^{-\frac{1}{20}x}\right]_0^{30} = 1 - e^{-\frac{3}{2}} = 0.7768$$

(b) $P(X \geq 20) = \int_{20}^{\infty} \frac{1}{20} e^{-\frac{1}{20}x}\, dx = -\left[e^{-\frac{x}{20}}\right]_{20}^{\infty}$

$$= -(0 - e^{-1}) = \frac{1}{e} = 0.3678$$

(c) $P(20 \leq X \leq 30) = \int_{20}^{30} \frac{1}{20} e^{-\frac{x}{20}}\, dx$

$= -\left[e^{-\frac{x}{20}}\right]_{20}^{30} = e^{-1} - e^{-\frac{3}{2}} = 0.1447$

(d) $P(X \leq t) = 0.5$

$\therefore \int_{0}^{t} \frac{1}{20} e^{-\frac{x}{20}}\, dx = 0.5$

$\therefore -\left[e^{-\frac{x}{20}}\right]_{0}^{t} = 0.5$

$\therefore -e^{-\frac{t}{20}} + e^{0} = 0.5$

$\therefore e^{-\frac{t}{20}} = 1 - 0.5 = 0.5$

$\therefore e^{\frac{t}{20}} = 2$

$t = 20 \log 2 = 13.8629$

EX 7. Suppose the life length X (in hours) X of a fuse has exponential distribution with mean $\frac{1}{\lambda}$. Fuses are manufactured by two different processes. Process I yield an expected life length of 100 hours and Process II yeild an expected life length of 150 hours. Cost of production of a fuse by process I is Rs C while by the process II it is Rs $2C$. A fine of Rs K is levied if a fuse lasts less than 200 hours. Determine which process should be preferred?

Answer: Prefer Process I if $C > 0.13K$

Hint: $c_1 = c$ if $X > 200$

$= c + k$ if $x \leq 200$

$E(c_1) = c \cdot P(X > 200) + (c+k)P(X \leq 200)$

$= c \cdot e^{-\frac{1}{100} \cdot 200} + (c+k)\left(1 - e^{-\frac{1}{100} \cdot 200}\right) = k(1 - e^{-2}) + c$

$E(c_2) = k\left(1 - e^{-4/3}\right) + 2c,$

$E(c_2) - E(c_1) = c - 0.13k$