SOMAIYA
VIDYAVIHAR UNIVERSITY

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Course: AM-I
Experiment / assignment / tutorial No. 5
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Assignment - 1
Partial Differentiation

1 If $u = e^{xyz}$, Prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$

Sol^n $\frac{\partial u}{\partial z} = xy\, e^{xyz}$

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}(xy e^{xyz})$

$\frac{\partial^2 u}{\partial y \partial z} = xy \frac{\partial}{\partial y}(e^{xyz}) + e^{xyz} \frac{\partial}{\partial y}(xy)$

$= xy(xz)e^{xyz} + e^{xyz}(x)$

$= xy(xz)e^{xyz} + xe^{xyz}$

$= x^2yz \cdot e^{xyz} + xe^{xyz}$

$\frac{\partial^2 u}{\partial y \partial z} = (x^2yz + x)e^{xyz}$

$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial}{\partial x}\left[(x^2yz+x)e^{xyz}\right]$

$\frac{\partial^3 u}{\partial x \partial y \partial z} = (x^2yz+x)\frac{\partial}{\partial x}(e^{xyz}) + e^{xyz}\frac{\partial}{\partial x}(x^2yz+x)$

$= (x^2yz+x)yz \cdot e^{xyz} + e^{xyz}(2xyz+1)$

$= e^{xyz}(x^2y^2z^2 + xyz + 2xyz + 1)$

$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$

LHS = RHS

Hence Proved

2 If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}$

Sol^n $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

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$$\frac{\partial z}{\partial y} = x^2 \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - \left\{ y^2 \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{x}{y}\right)\right] + \tan^{-1}\left(\frac{x}{y}\right) \frac{\partial}{\partial y}(y^2) \right\}$$

$$= \frac{x^3}{x^2+y^2} - \left\{ \cancel{y^2} \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{-x}{\cancel{y^2}}\right) + 2y \tan^{-1}\left(\frac{x}{y}\right) \right\}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= 1 - 2\cancel{y} \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(\frac{1}{\cancel{y}}\right)$$

$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2} \quad \text{--- } ①$$

Also, we know that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ --- ②

By ① & ②

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}$$

Hence Proved

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3. If $u = f\left(\frac{x^2}{y}\right)$, prove that $x\left(\frac{\partial u}{\partial x}\right) + 2y\left(\frac{\partial u}{\partial y}\right) = 0$

And $x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 3xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + 2y^2\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$

Sol^n Proof 1:

$u = f\left(\frac{x^2}{y}\right)$

$\frac{\partial u}{\partial x} = \frac{2x}{y}$ $\qquad \frac{\partial u}{\partial y} = \frac{-x^2}{y^2}$

$x\left(\frac{\partial u}{\partial x}\right) + 2y\left(\frac{\partial u}{\partial y}\right) = x\left(\frac{2x}{y}\right) + 2\cancel{y}\left(\frac{-x^2}{y^{\cancel{2}}}\right)$

$= \frac{2x^2}{y} - \frac{2x^2}{y}$

$x\left(\frac{\partial u}{\partial x}\right) + 2y\left(\frac{\partial u}{\partial y}\right) = 0$

LHS = RHS

Hence Proved

Proof 2

From Proof 1,

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{y}\right) = +\frac{2}{y}$

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{-x^2}{y^2}\right) = +\frac{2x^2}{y^3}$

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x^2}{y^2}\right) = -\frac{2x}{y^2}$

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$$LHS = x^2 \frac{\partial^2 u}{\partial x^2} + 3xy \frac{\partial^2 u}{\partial x \partial y} + 2y^2 \frac{\partial^2 u}{\partial y^2} = 0$$

$$= x^2 \left(\frac{2}{y}\right) + 3x\cancel{y}\left(-\frac{2x}{y^{\cancel{2}}}\right) + 2\cancel{y^2}\left(\frac{2x^2}{\cancel{y^3}}\right)$$

$$= \frac{2x^2}{y} - \frac{6x^2}{y} + \frac{4x^2}{y}$$

$LHS = 0$

$LHS = RHS$

Hence Proved

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4\. If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, prove that $u_x + u_y + u_z = 0$

Sol^n $e^{y-z} = l$

$e^{z-x} = m$

$e^{x-y} = n$

$\frac{\partial l}{\partial x} = 0 \qquad \frac{\partial l}{\partial y} = e^{y-z} \qquad \frac{\partial l}{\partial z} = -e^{y-z}$

$\frac{\partial m}{\partial x} = -e^{z-x} \qquad \frac{\partial m}{\partial y} = 0 \qquad \frac{\partial m}{\partial z} = e^{z-x}$

$\frac{\partial n}{\partial x} = e^{x-y} \qquad \frac{\partial n}{\partial y} = -e^{x-y} \qquad \frac{\partial n}{\partial z} = 0$

$u = f(l, m, n)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l} \cdot 0 + \frac{\partial u}{\partial m}(-e^{z-x}) + \frac{\partial u}{\partial n}(e^{x-y})$$

$$= \frac{\partial u}{\partial m}(-e^{z-x}) + \frac{\partial u}{\partial n}(e^{x-y})$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial n}(e^{x-y}) - \frac{\partial u}{\partial m}(e^{z-x}) \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \frac{\partial m}{\partial y} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}(e^{y-z}) + 0 + \frac{\partial u}{\partial n}(-e^{x-y})$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}(e^{y-z}) - \frac{\partial u}{\partial n}(e^{x-y}) \quad \text{— (2)}$$

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$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}(-e^{y-z}) + \frac{\partial u}{\partial m}(e^{z-x}) + 0$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial m}(e^{z-x}) - \frac{\partial u}{\partial l}(e^{y-z}) \quad \text{— (3)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial n}(e^{x-y})} - \cancel{\frac{\partial u}{\partial m}(e^{z-x})} \qquad \text{[from (1), (2) \& (3)]}$$

$$+ \cancel{\frac{\partial u}{\partial l}(e^{y-z})} - \cancel{\frac{\partial u}{\partial n}(e^{x-y})}$$

$$+ \cancel{\frac{\partial u}{\partial m}(e^{z-x})} - \cancel{\frac{\partial u}{\partial l}(e^{y-z})}$$

$$u_x + u_y + u_z = 0$$

Hence Proved

5. If $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

Prove that, $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

Solⁿ $x = r\cos\theta$, $y = r\sin\theta$

Differentiating partially wrt θ,

$\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

Differentiating partially wrt r,

$\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

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$$\frac{\partial z}{\partial r} = \cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y} \quad \text{—(1)}$$

Similarly,

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y} \quad \text{—(2)}$$

$$\text{RHS} = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \left(\cos\theta \frac{\partial z}{\partial x} + \sin\theta \frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta \frac{\partial z}{\partial x} + r\cos\theta \frac{\partial z}{\partial y}\right)$$

... [From (1) & (2)]

$$= \cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + 2\sin\theta\frac{\partial z}{\partial y}.\cos\theta\frac{\partial z}{\partial y} + \sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 + \frac{1}{r^2}$$

$$\left[r^2\cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2 - 2\, r\cos\theta\frac{\partial z}{\partial y}.r\sin\theta\frac{\partial z}{\partial x} + r^2\sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2\right]$$

$$= \cos^2\theta\left(\frac{\partial z}{\partial x}\right)^2 + 2\sin\theta\frac{\partial z}{\partial y}\cos\frac{\partial z}{\partial x} + \sin^2\theta\left(\frac{\partial z}{\partial y}\right)^2 + \cos^2\theta\left(\frac{\partial z}{\partial y}\right)^2$$

$$- 2\cos\theta\frac{\partial z}{\partial y}\sin\theta\frac{\partial z}{\partial x} + \sin^2\theta\left(\frac{\partial z}{\partial x}\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2(\cos^2\theta + \sin^2\theta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \text{LHS}$$

LHS = RHS

Hence Proved