SOMAIYA
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Experiment / assignment / tutorial No. 9
Grade: Signature of the Faculty with date

Tutorial 9 : Complex Numbers

Q1) If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2 \cdot 2^{n/2} \cos n\pi/4$, Hence deduce $\alpha^8 + \beta^8 = 32$

Sol^n $x^2 - 2x + 2 = 0$

$\therefore x = 1+i \; , \; 1-i$

Let, $\alpha = 1+i \; ; \; \beta = 1-i$

$\therefore$ α & β can be written as

$$\alpha = \sqrt{2} \cos \frac{\pi}{4} + i\sqrt{2} \sin \frac{\pi}{4}$$

$$\beta = \sqrt{2} \cos \frac{\pi}{4} - i\sqrt{2} \sin \frac{\pi}{4}$$

Hence, for $\alpha^n + \beta^n = \left[\sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4}\right]^n + \left[\sqrt{2}\cos\frac{\pi}{4} - i\sqrt{2}\sin\frac{\pi}{4}\right]^n$

By solving further,

$$= \left[\sqrt{2}^n \cos\frac{\pi}{4}n + i\sqrt{2}^n \sin\frac{\pi}{4}n\right] + \left[\sqrt{2}^n \cos\frac{\pi}{4}n - i\sqrt{2}^n \sin\frac{\pi}{4}n\right]$$

$$= 2\left(\sqrt{2}^n \cos\frac{n\pi}{4}\right)$$

$$\alpha^n + \beta^n = 2 \cdot 2^{n/2} \cdot \cos\frac{n\pi}{4}$$

For, $\alpha^8 + \beta^8 = 2 \cdot 2^{8/2} \cos\frac{8\pi}{4}$

$$= 2 \cdot 2^4 \cos 2\pi$$

$$\boxed{\alpha^8 + \beta^8 = 32} \qquad [\cos 2\pi = 1]$$

Q2) Find all the values of $(1+i)^{3/4}$ and find continued product of all the roots.

Sol^n $(1+i)^{3/4} = \left(\sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\sin\frac{\pi}{4}\right)^{3/4}$

$$= (\sqrt{2})^{3/4}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^{3/4}$$

$$= (\sqrt{2})^{3/4}\left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{3/4}$$

$$= (\sqrt{2})^{3/4}\left[\cos\left(\frac{8k\pi + \pi}{4}\right) + i\sin\left(\frac{8k\pi + \pi}{4}\right)\right]^{3/4}$$

$$= (\sqrt{2})^{3/4}\left[\cos\left(\frac{24k\pi + 3\pi}{16}\right) + i\sin\left(\frac{24k\pi + 3\pi}{16}\right)\right]$$

For the values,

$K = 0 \rightarrow x_0 = (\sqrt{2})^{3/4}\left[\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right]$

$K = 1 \rightarrow x_1 = (\sqrt{2})^{3/4}\left[\cos\frac{27\pi}{16} + i\sin\frac{27\pi}{16}\right]$

$K = 2 \rightarrow x_2 = (\sqrt{2})^{3/4}\left[\cos\frac{51\pi}{16} + i\sin\frac{51\pi}{16}\right]$

$K = 3 \rightarrow x_3 = (\sqrt{2})^{3/4}\left[\cos\frac{75\pi}{16} + i\sin\frac{75\pi}{16}\right]$

$$\therefore x_0 x_1 x_2 x_3 = (\sqrt{3})^3 \cdot \left[\cos\left(\frac{3\pi + 27\pi + 51\pi + 75\pi}{16}\right) + i\sin\left(\frac{3\pi + 27\pi + 51\pi + 75\pi}{16}\right)\right]$$

$$\boxed{\therefore x_0 x_1 x_2 x_3 = 2 \cdot \sqrt{2}\left[\cos\left(\frac{156\pi}{16}\right) + i\sin\left(\frac{156\pi}{16}\right)\right]}$$

Q3) If $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$. Prove that

(i) $\cosh u = \sec\theta$ (ii) $\sinh u = \tan\theta$

(iii) $\tanh u = \sin\theta$ (iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

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Sol^n

$$u = \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{(Property of log)}$$

$$e^u = \frac{1 + \tan \theta/2}{1 - \tan \theta/2} \quad \text{—— ①}$$

$$e^{-u} = \frac{1 - \tan \theta/2}{1 + \tan \theta/2} \quad \text{—— ②}$$

let $\cosh u = \dfrac{e^u + e^{-u}}{2}$

$$= \frac{\left(\frac{1 + \tan \theta/2}{1 - \tan \theta/2} + \frac{1 - \tan \theta/2}{1 + \tan \theta/2}\right)}{2} \quad [\text{from ① \& ②}]$$

$$= \left[\frac{(1+\tan \theta/2)^2 + (1 - \tan \theta/2)^2}{2 - 2\left(\tan^2 \frac{\theta}{2}\right)}\right]$$

$$= \left(\frac{1+2\tan \theta/2 + \tan^2 \theta/2 + 1 - 2\tan\theta/2 + \tan^2\theta/2}{2 - 2\tan^2 \theta/2}\right)$$

$$= \frac{2 + 2\tan^2 \theta/2}{2 - 2\tan^2 \theta/2}$$

$$\cosh u = \frac{1}{\cos\theta}$$

$\therefore$ We get, $\cosh u = \sec\theta$ —— ③

We know that,

$$\sinh u = \sqrt{\cosh^2 u - 1}$$

$$\sinh u = \sqrt{\sec^2\theta - 1} \quad [\text{from ③}]$$

$$\sinh u = \sqrt{\tan^2\theta}$$

$$\sinh u = \tan\theta \quad \text{—— ④}$$

$$\tanh u = \frac{\sinh u}{\cosh u}$$

$$= \frac{\tan\theta}{\sec\theta} \quad \text{[from ③ \& ④]}$$

$$= \frac{\sin\theta \cdot \cos\theta}{\cos\theta}$$

$$\tanh u = \sin\theta \quad \text{—— ⑤}$$

considering,

$$\tanh\frac{u}{2} = \frac{\sinh u/2}{\cosh u/2}$$

Multiplying by $2\cosh u/2$

$$= \frac{\sinh u}{2\cosh^2 u/2}$$

$$= \frac{\sinh u}{1+\cosh u}$$

$$= \frac{\tan\theta}{1+\sec\theta} \quad \text{[from ③ \& ④]}$$

$$= \frac{\sin\theta/\cos\theta}{(\cos\theta+1)/\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta+1}$$

$$= \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$= \frac{\sin\theta/2}{\cos\theta/2}$$

$$\tan\frac{u}{2} = \tan\frac{\theta}{2} \quad \text{—— ⑥}$$

Hence Proved

Q4) If $\cos(x+iy) = \cos\alpha + i\sin\alpha$, prove that

(i) $\sin\alpha = \pm\sin^2 x = \pm\sinh^2 y$

(ii) $\cos 2x + \cosh 2y = 2$

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Sol^n ① $\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$

$\cos\alpha + i\sin\alpha = \cos x \cosh y - i \sin x \sinh y$

separate,

$\cos\alpha = \cos x \cosh y$

$\sin\alpha = -\sin x \sinh y$

Square & add,

$\cos^2\alpha + \sin^2\alpha = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$1 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$1 = (1+\sinh^2 y)(1-\sinh^2 y) + \sin^2 x . \sinh^2 y$

$1 = 1 - \sin^2 x + \sinh^2 y - \sinh^2 y . \sin^2 y + \sin^2 x \sinh^2 y$

$0 = -\sinh^2 y - \sin^2 x$

$\therefore \pm \sinh y = \pm \sin^2 x$ — ①

Hence Proved

② $\cos 2x + \cosh^2 y = (1 - \sin^2 x . 2) + (1 + 2\sinh^2 y)$

$= 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

but, $\sin^2 x = \sinh^2 y$ [from ①]

$\therefore \cos 2x + \cosh^2 y = 2$

Q5) Prove that $\cosh^{-1}(\sqrt{1+x^2}) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

Sol^n consider

L.H.S $= \cosh^{-1}\sqrt{1+x^2} = a$

$\therefore \sqrt{1+x^2} = \cosh a$

$1+x^2 = \cosh^2 a$

But, $1 + \sinh^2 a = \cosh^2 a$

$\therefore x = \sinh a$ — ①

Consider,

$\frac{x}{\sqrt{1+x^2}} = \frac{\sinh a}{\cosh a}$

$\therefore \frac{\sinh a}{\cosh a} = \tanh a$

$\therefore \tan^{-1}\left(\frac{\sin ha}{\cos ha}\right) = a$

$\therefore \tan^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = a$

$\therefore \cosh^{-1}(\sqrt{1+x^2}) = \tan^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

LHS = RHS

Hence Proved

Q6) Find the value of $\log[\sin(x+iy)]$

Sol^n We know that,

$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$\therefore \log(\sin x \cosh y + i \cos x \sinh y)$

We know,

$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$\therefore \log(\sin x \cosh y + i \cos x \sinh y)$

$= \frac{1}{2}\log[\cosh^2 y \sin^2 x + \cos^2 x \sinh^2 y] + i\tan^{-1}\left(\frac{\sin x \cosh y}{\cos x \sinh y}\right)$

$= \frac{1}{2}\log(\cosh^2 y - \cos^2 x \cdot \cosh^2 x + \cos^2 x \cosh^2 y - \cos^2 x) + i\tan^{-1}(\cot x \cdot \tanh y)$

$= \frac{1}{2}\log\left[\left(\frac{1+\cosh 2y}{2}\right) - \left(\frac{1+\cos 2x}{2}\right)\right] + i\tan^{-1}(\cot x \cdot \tanh y)$

$= \frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right] + i\tan^{-1}(\cot x \cdot \tanh y)$

Hence,

$\log(\sin(x+iy)) =$

$\frac{1}{2}\log\left[\frac{1}{2}\cosh 2y - \cos 2x\right] + i\tan^{-1}(\cot x \cdot \tan\cancel{x} hy)$