Chapter 3

Coplanar Forces: Equilibrium

3.1 Introduction

We have so far studied the three systems of forces and to find the resultant of the system. If the resultant of the force system happens to be zero, the system is said to be in a state of equilibrium. Various practical examples can illustrate the state of equilibrium of a system of forces like:

i) *a lamp hanging from the ceiling is an example of Concurrent Force System in equilibrium.*

ii) *structures like buildings, dams etc. are examples of General Force System in equilibrium.*

iii) *students sitting on a bench is an example of Parallel Force System in equilibrium.*

In this chapter we shall study the *conditions of equilibrium* and applying these conditions with the help of free body diagrams, we shall learn to analyse a system in equilibrium.

3.2 Conditions of Equilibrium

A body is said to be in equilibrium if it is in a state of rest or uniform motion. This is precisely what Newton has stated in his first law of motion. For a body to be in equilibrium the resultant of the system should be zero. This implies that:

the sum of all forces should be zero i.e. $\sum \overline{F} = 0$

and the sum of all moments should also be zero i.e. $\sum \overline{M} = 0$

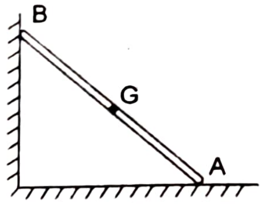
The above two equations are the conditions of equilibrium in vector form.
For a coplanar system of forces, the scalar equations of equilibrium are:

$\Sigma F_x = 0$ ------------------ sum of all forces in x direction is zero
$\Sigma F_y = 0$ ------------------ sum of all forces in y direction is zero
$\Sigma M = 0$ ------------------ sum of moments of all forces is zero

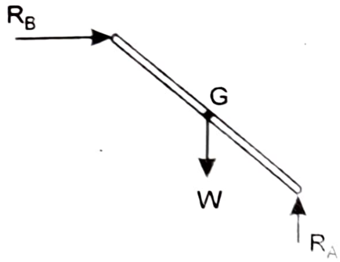
3.3 Free Body Diagram (FBD)

"A diagram formed by isolating the body from its surroundings and then showing all the forces acting on it is known as a Free Body Diagram". Such a diagram is required to be drawn for the body under analysis. For example consider a ladder AB of weight W resting against the smooth vertical wall and

smooth horizontal floor. If the ladder is under analysis then the FBD of the ladder shall show the weight W acting through its C.G. and the normal reactions R_A and R_B offered by the floor and the wall respectively.



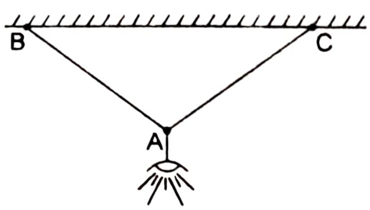
Ladder resting against wall and floor



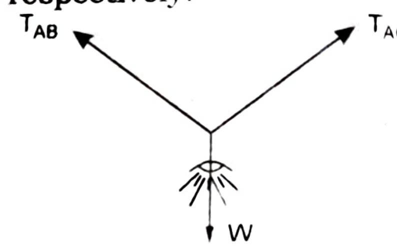
F.B.D. of Ladder

Fig. 3.1

Consider a lamp of weight W, suspended from two strings AB and AC tied to the ceiling. The FBD of the lamp will show the weight W of the lamp and the tensions T_{AB} and T_{AC} in the strings AB and AC respectively.



Lamp supported by two strings



F.B.D. of lamp

Fig. 3.2

3.4 Types of Supports and Reactions

Whenever a body is supported, the support offers resistance, known as reaction. For example you are sitting on a chair while reading this book. Your weight is being supported by the chair which offers a force of resistance (reaction) upwards. Likewise let us see the different types of supports and the reactions they offer.

1. Hinge Support

A hinge allows free rotation of the body but does not allow the body to have any linear motion. It therefore offers a force reaction which can be split into horizontal and vertical components. Figure shows a body having a hinge support at A. The horizontal component H_A and the vertical component V_A of the support reaction are shown.

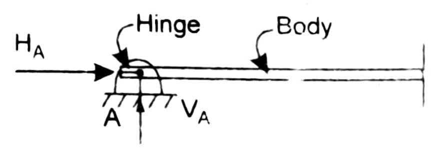


Fig. 3.3 (a)

When two bodies are connected such that the connection allows rotation between them and behaves as a hinge then such a connection is referred to as an *internal hinge* or *pin connection.* For example the two members of a scissor are connected by a pin which allows rotation but allows no linear movement.

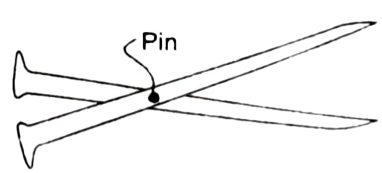
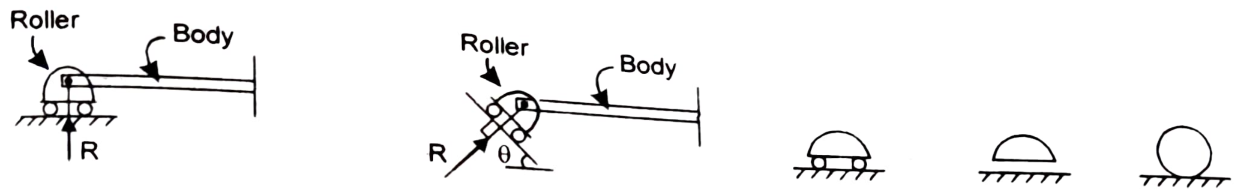


Fig. 3.3 (b)

2. **Roller Support**

A roller support is free to roll on a surface on which it rests. It offers a force reaction in a direction normal to the surface on which the roller is supported. A roller support may be shown in any of the three symbols as shown in figure 3.4 (c).



(a) Roller supported on horizontal surface

(b) Roller supported on inclined surface

(c) Representation of roller

Fig. 3.4

3. **Fixed Support**

A fixed support neither allows any linear motion nor allows any rotation. It therefore offers a force reaction which can be split into a horizontal and a vertical component and also a moment reaction. Figure shows a body having a fixed support at A. In addition to the horizontal component H_A and the vertical component V_A of the force reaction, there is a moment reaction M_A.

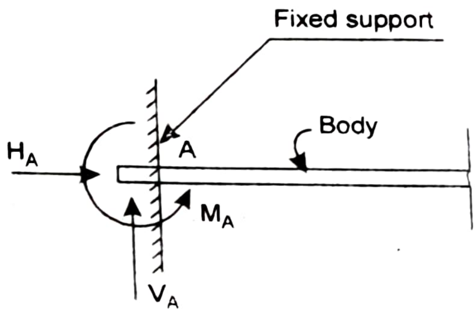


Fig. 3.5

4. **Smooth Surface Support**

A smooth surface offers a similar reaction as a roller support, i.e. a force reaction normal to the smooth surface. Fig. 3.6 shows a sphere supported between two smooth surfaces. Each surface offers one force reaction, normal to the surface at contact points.

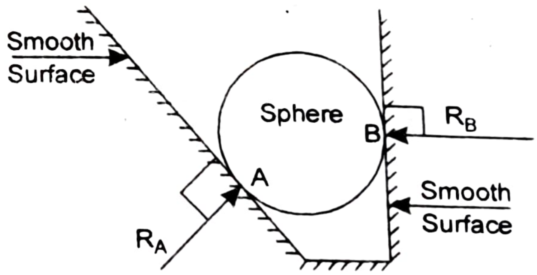


Fig. 3.6

5. **Rope/ String/ Cable Support**

It offers a pull force in a direction away from the body. This force is commonly referred to as the *tension* force. Fig. 3.7 shows a lamp suspended by two strings, each of them offers a tension force.

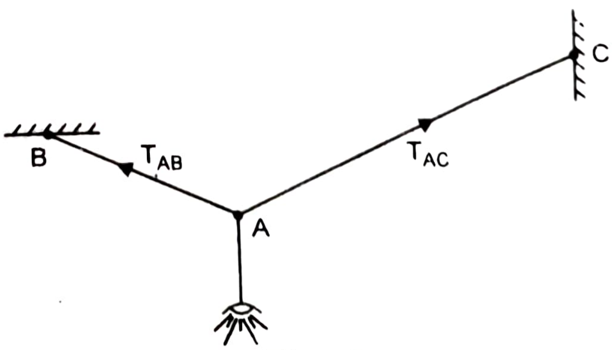


Fig. 3.7

6. Edge (Fulcrum) Support

It offers a force reaction normal to the surface of the body supported on the edge.

Fig. 3.8 shows a rod supported on a floor and rests against an edge. The edge offers a force reaction normal to the rod surface.

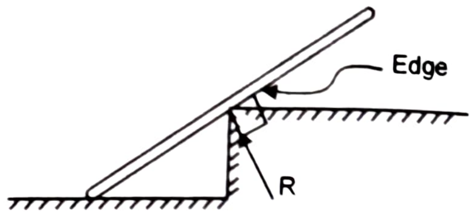


Fig. 3.8

3.5 Types of Loads

The following types of loads can act on bodies.

1. *Point load*

This load is concentrated at a point. Fig. 3.9 shows point loads F_1, F_2, F_3 acting on the beam.

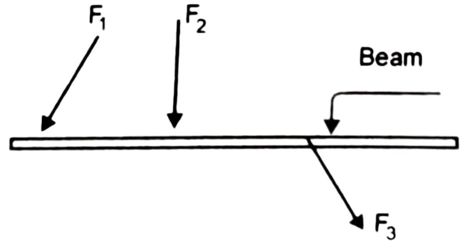
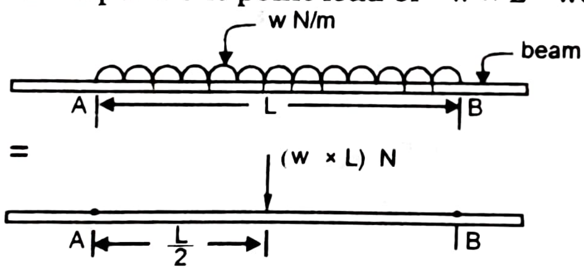


Fig. 3.9

2. *Uniformly distributed load (u.d.l)*

In this loading the load of uniform intensity is spread over a length. u.d.l can be converted into an equivalent point load by multiplying the load intensity with the length. This equivalent point load would act at center of the spread. Fig. 3.10 Shows a u.d.l of intensity w N/m spread over a length AB of L meters. The equivalent point load of $w \times L$ would therefore act at L/2 from A.

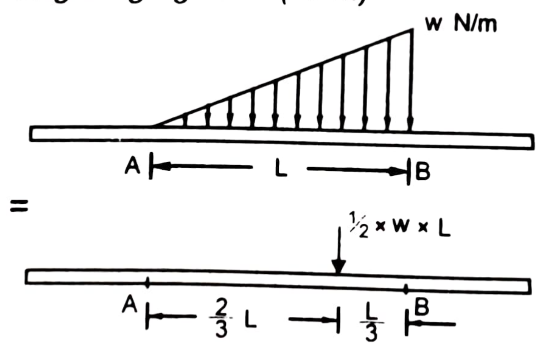


u.d.l spread over portion AB

Equivalent point Load

Fig. 3.10

3. *Uniformly varying load. (u.v.l.)*



u.v.l spread over portion AB

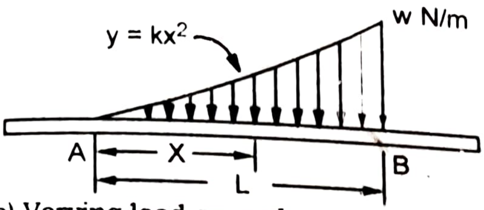
Equivalent point Load

Fig. 3.11

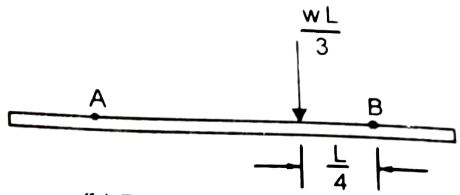
In this loading, the load of uniformly varying intensity is spread over a length. u.v.l. can be converted into an equivalent load, which is equal to the area under the load diagram. The equivalent point load would act at the centroid of the load diagram. Refer Fig. 3.11 which shows a u.v.l. of intensity varying from zero to w N/m over a spread length of L metres.

4. *Varying Load.*

In this loading, the loading intensity varies as some relation. Fig. 3.12 shows a varying distributed load of parabolic nature. The equivalent point load is the area under the curve acting at the C.G. of the area.



(a) Varying load spread over portion AB



(b) Equivalent point load

Fig. 3.12

5. *Couple*

We have studied about couple earlier in article 2.12. A couple acting on a body tends to cause rotation of the body. Couples are free vectors and hence they can be located anywhere on the body. Figure 3.13 shows three couples M_1, M_2 and M_3 acting on the beam. Couples are represented by curved arrows.

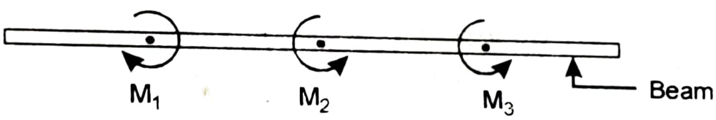
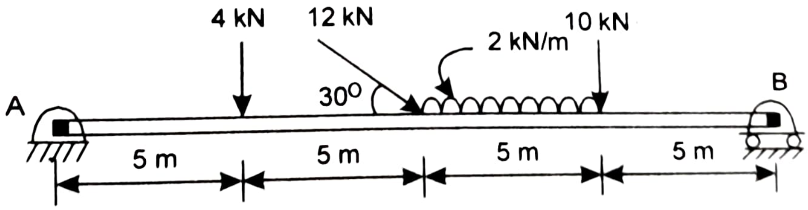


Fig. 3.13

3.6 Basic Problems in Equilibrium.

Ex. 3.1 A beam AB is hinged at end A and roller supported at end B. It is acted upon by loads as shown. Find the support reactions.



Solution:

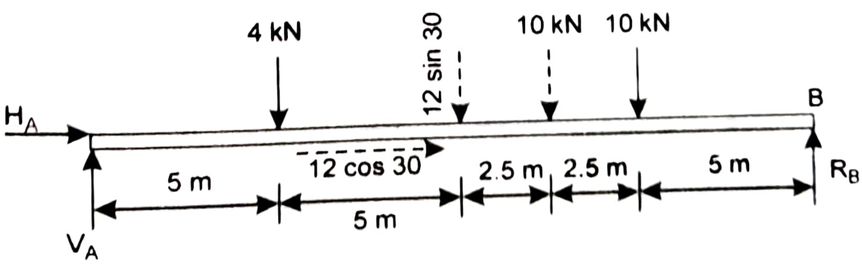


Figure shows the FBD of the beam AB. Hinge at A gives reaction R_A having components H_A and V_A. Roller at B gives a vertical reaction R_B.

The u.d.l has been converted into a point load of 2 kN/m × 5 m = 10 kN acting at the center of u.d.l. The 12 kN inclined load has been resolved into components.

Applying Conditions of Equilibrium (COE) to the beam AB

$\Sigma M_A = 0$ + ve

$-4 \times 5 - 12 \sin 30 \times 10 - 10 \times 12.5 - 10 \times 15 + R_B \times 20 = 0$

$R_B = 17.75$ kN

$R_B = 17.75$ kN ↑ .... **Ans.**

$\Sigma F_X = 0$ → + ve

$H_A + 12 \cos 30 = 0$

$H_A = -10.39$ kN

$H_A = 10.39$ kN ←

$\Sigma F_Y = 0$ ↑ + ve

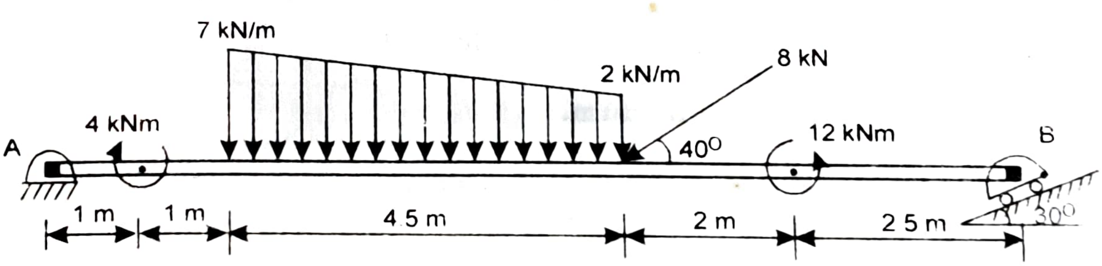
$V_A - 4 - 12 \sin 30 - 10 - 10 + 17.75 = 0$

$V_A = 12.25$ kN ↑

Adding vectorially the components H_A and V_A, the reaction

$R_A = 16.06$ kN $\theta = 49.69°$ .... **Ans.**

Ex. 3.2 The beam AB is loaded by forces and couples as shown. Find the reaction force offered by the supports to keep the system in equilibrium.



Solution:

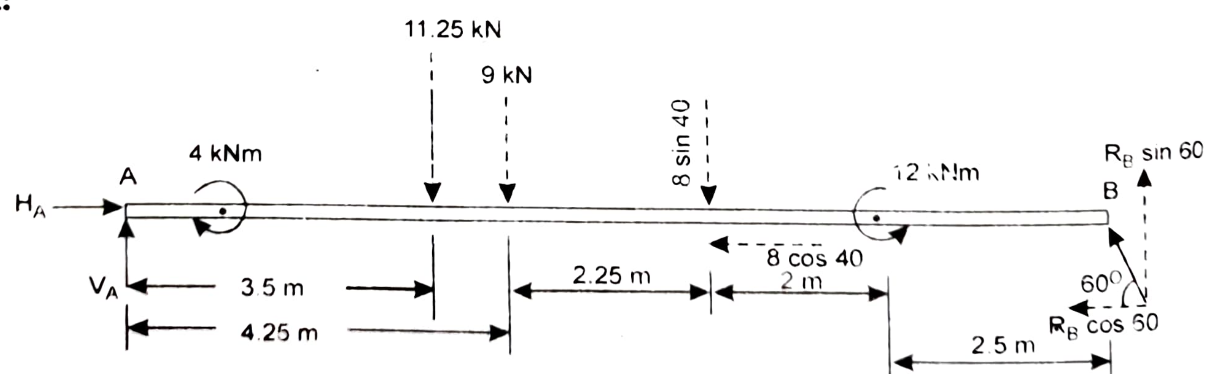


Figure shows the FBD of the beam AB. The hinge at A offers reaction R_A having components H_A and V_A.

The roller at B offers reaction R_B normal to the surface on which the roller is supported.

The trapezoidal load has been replaced by two equivalent point loads.
The inclined 8 kN force has been resolved into components.
The beam is loaded with two couples viz. 4 kNm anti-clockwise and 12 kNm clockwise couples.

Applying COE to the beam AB.

$\Sigma M_A = 0$ ↺ + ve

$-4 - 11.25 \times 3.5 - 9 \times 4.25 - 8 \sin 40 \times 6.5 + 12 + R_B \sin 60 \times 11 = 0$

$R_B = 10.82$ kN

$\therefore$ $R_B = 10.82$ kN $\theta = 60°$ ……… **Ans.**

$\Sigma F_X = 0$ → + ve

$H_A - 8 \cos 40 - 10.82 \cos 60 = 0$

$H_A = 11.54$ kN

$H_A = 11.54$ kN →

$\Sigma F_Y = 0$ ↑ + ve

$V_A - 11.25 - 9 - 8 \sin 40 + 10.82 \sin 60 = 0$

$V_A = 16.02$ kN

$V_A = 16.02$ kN ↑

Adding vectorially the components H_A and V_A the reaction

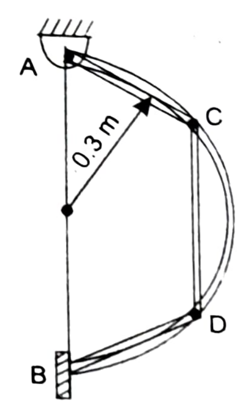
$R_A = 19.74$ kN, $\theta = 54.2°$

Ex. 3.3 A uniform rod weighing 16 N per meter length is used to form a structure as shown. This structure is hung in a vertical plane from a hinge A. The other end B is resting against a smooth wall.

Knowing length AC = CD = BD = 0.3 m

(a) Determine the reactions at A and B.

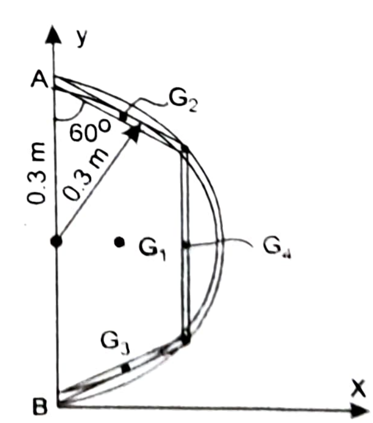
(b) If the wall at B was removed determine the angle which the line AB will make with the vertical when the structure aquires equilibrium position.

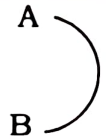
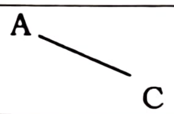


Solution:

(a) To find suport reactions.

Let us first find out the center of gravity of the structure. The calculations for the same are tabulated.



Part	Weight W_i (N) W_i = 16 N/m × length (m)	x_i (m)	$W_i x_i$ (Nm)
	$16 \times (\pi \times 0.3) = 15.08$	$\frac{2r}{\pi} = \frac{2 \times 0.3}{\pi}$ $= 0.191$	2.88
	$16 \times 0.3 = 4.8$	0.13	0.624
	$16 \times 0.3 = 4.8$	0.26	1.248
	$16 \times 0.3 = 4.8$	0.13	0.624
	$\Sigma W_i = 29.48$ N		$\Sigma W_i x_i$ = 5.376 Nm

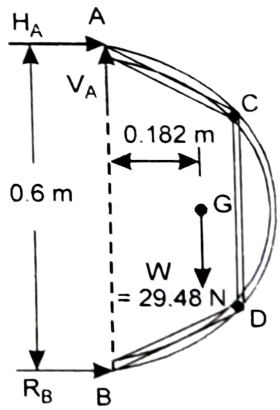
$$\overline{X} = \frac{\Sigma W_i x_i}{\Sigma W_i} = \frac{5.376}{29.48} = 0.182 \text{ m}$$

∴ The centroid G lies at $\overline{X}$ = 0.182 m on the axis of symmetry.

Figure shows the FBD of the structure. The support reactions are,
1) hinge reaction R_A at A having two components H_A and V_A.
2) Horizontal reaction R_B at B given by the smooth vertical wall.

The structure is in equilibrium under the action of three forces viz, R_A, R_B and W.

Applying COE



$\Sigma M_A = 0$ ↺ + ve

$-29.48 \times 0.182 + R_B \times 0.6 = 0$

$R_B = 8.942$ N

$R_B = 8.942$ N →......**Ans.**

$\Sigma F_X = 0$ → + ve

$H_A + 8.942 = 0$

$H_A = -8.942$ N

$H_A = 8.942$ N ←

$\Sigma F_Y = 0$ ↑ + ve

$V_A - 29.48 = 0$

$V_A = 29.48$ N

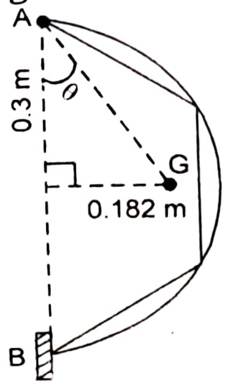
$V_A = 29.48$ N ↑

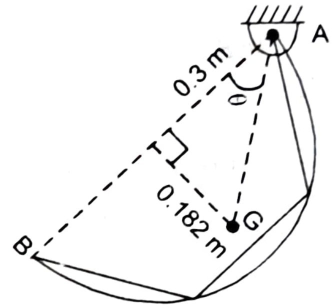
Adding vectorially the components H_A and V_A the total reaction at A is

$R_A = 30.81$ kN $\theta = 73.13°$ .... **Ans.**

(b) To find angle made by line AB when wall is removed.

When the wall is removed the structure tilts and aquires an equilibrium position such that the center of gravity G lies vertically below A. The line AG which was initially inclined at an angle θ to the line AB now aquires a vertical position, while the line AB is now inclined at an angle θ with vertical.

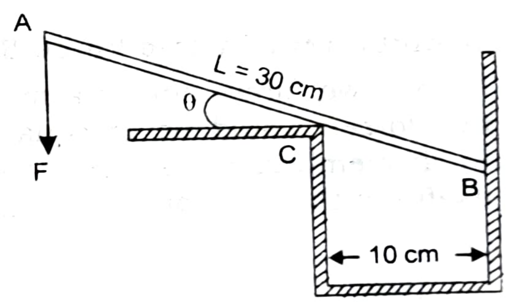




From Figure -------- $\tan\theta = \dfrac{0.182}{0.3}$

$\therefore \quad \theta = 31.24°$ .... **Ans.**

Ex. 3.4 A uniform rod AB of length 30 cm and weight 25 N is in equilibrium in a channel of width 10 cm. If a force F = 50 N acts at A, determine the angle θ the rod makes in the equilibrium position and the reaction at supports. Neglect friction.



Solution:

Figure shows the FBD. The system contains a single body AB whose support reactions are,

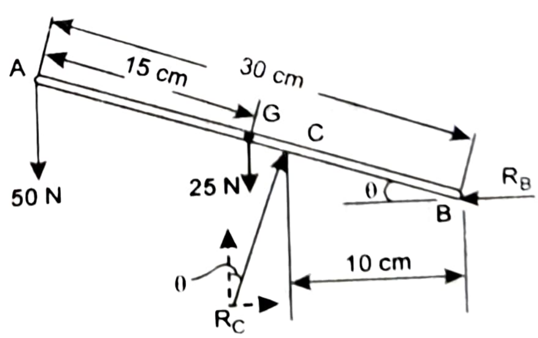
1. Smooth surface reaction R_B being $\perp$ to the vertical surface
2. Edge reaction R_C which is $\perp$ to the body AB.

Applying COE

$\Sigma F_Y = 0 \uparrow +$ ve

$-50 - 25 + R_C \cos\theta = 0$

$\therefore \quad R_C \cos\theta = 75$ -------(1)



$\Sigma M_B = 0$ ↺ + ve

$$50 \times 30 \cos\theta + 25 \times 15 \cos\theta - \frac{R_C \times 10}{\cos\theta} = 0$$

$$1875 \cos^2\theta - 10\, R_C = 0$$

$$\therefore \quad R_C = 187.5 \cos^2\theta \qquad \text{-----(2)}$$

Solving equations (1) and (2)

$$187.5 \cos^3\theta = 75$$
$$\cos\theta = 0.7368$$
$$\therefore \quad \theta = 42.54° \qquad \text{......} \textbf{ Ans.}$$

Substituting value of θ in equation (1)

$R_C = 101.79$ N $\theta = 47.46°$ ...... **Ans.**

Applying COE

$\Sigma F_X = 0$ → + ve

$$R_C \sin\theta - R_B = 0$$
$$101.79 \sin 42.54 - R_B = 0$$
$$\therefore \quad R_B = 68.82 \text{ N} \leftarrow \text{......} \textbf{ Ans.}$$

3.7 Equilibrium of a Two Force Body

If only two forces act on a member and the member is in equilibrium then th two forces would be of equal magnitude, opposite in direction and collinear Such members are referred to as *two force members* and their identification i useful in the solution of equilibrium problems.

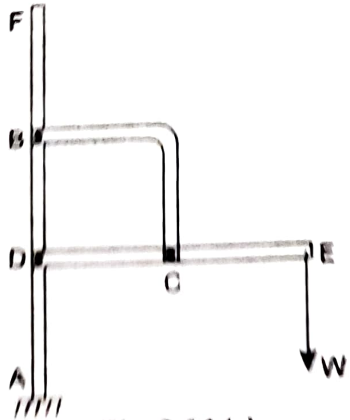


Fig. 3.14 (a)

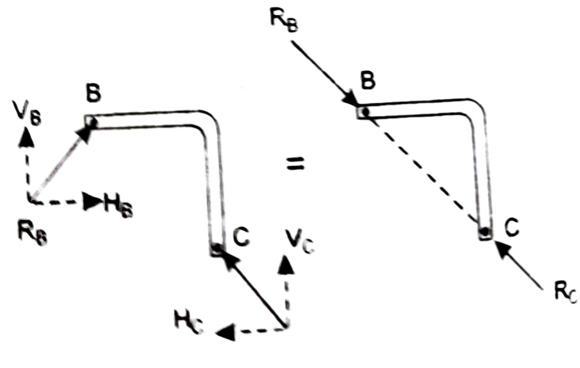


Fig. 3.14 (b)

Fig. 3.14 (a) shows a frame consisting of three members AF, BC and D Member BC is isolated and shown in Fig. 3.14 (b). Let R_B and R_C be the reactions at B and C respectively. Since only two forces act on member BC, i a two force member. Therefore $R_B = R_C$ in magnitude, opposite in direction a are collinear (i.e. both are directed along line BC)

3.6 Equilibrium of a Three Force Body.

If three coplanar forces act on a member and the member is in equilibrium, then the forces would be either Concurrent or Parallel.
Fig. 3.15 (a) shows a uniform rod AB of weight W, one end of which is resting against a smooth vertical wall, while the other end is supported by a string. The member AB is acted upon by three forces viz., a horizontal reaction R_A at A, the self weight W acts through the C.G. of the rod, while the tension T in the string acts at B. These three forces keep the rod in equilibrium and should therefore be concurrent. Refer Fig. 3.15 (b).

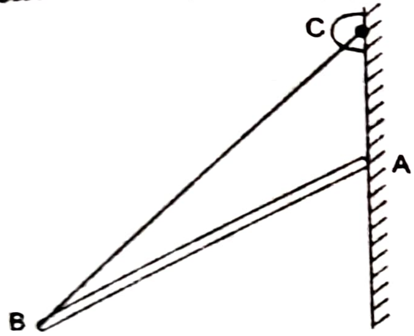


Fig. 3.15 (a)

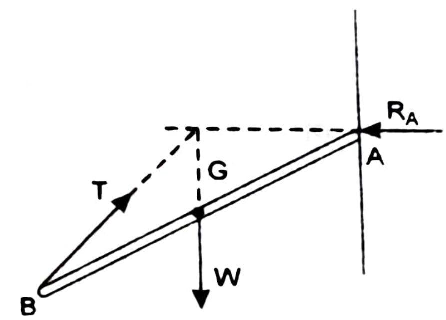


Fig. 3.15 (b)

Fig. 3.16 (a) shows a beam AB which is hinge supported at A and roller supported at B. Let a vertical load P be applied on the beam at C. We know that the reaction R_B would be vertical. Since the beam is in equilibrium, hinge reaction R_A would also be a vertical force. This is therefore a case of three parallel forces in equilibrium. Refer Fig. 3.16 (b).

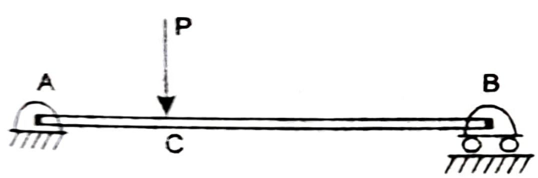


Fig. 3.16 (a)

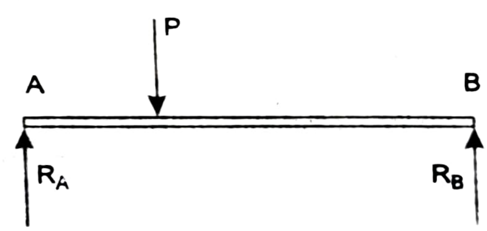
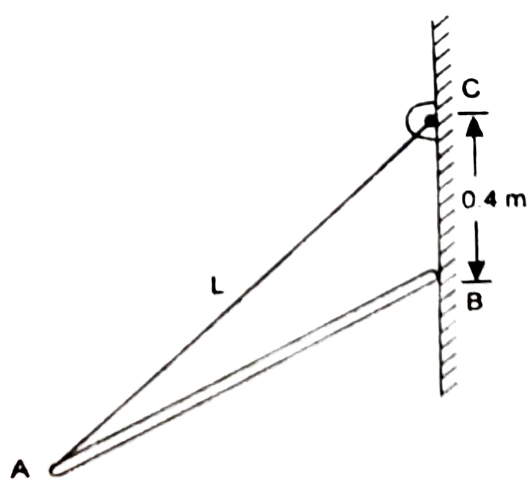
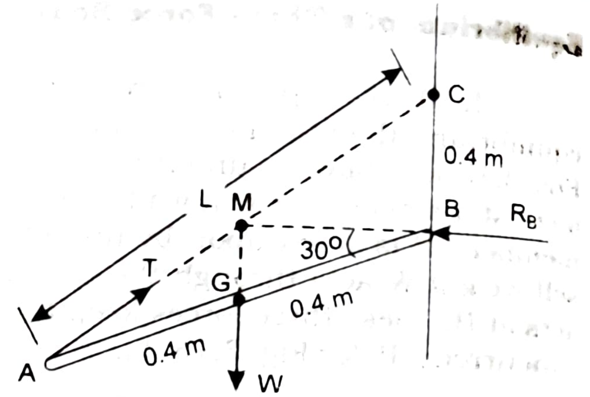


Fig. 3.16 (b)

Ex. 3.8 A uniform rod of weight W and length 0.8 m is held in equilibrium with one end resting against a smooth vertical wall, while the other end is supported by a rope. Determine the length L of the rope.



Solution: Figure shows the FBD of the rod AB.



The external supports for the rod are

1) a smooth surface at B, giving reaction R_B $\perp$ to the smooth surface.
2) a rope support at A, giving tension reaction force T.

The weight W acts through the rod's C.G. Since the system has three forces in equilibrium, the forces should be concurrent. Let M be the point of concurrence of R_B, T and W.

Δ CAB being similar to Δ MAG $\therefore$ MG = 0.2 m

$\therefore$ In Δ BMG $\angle B = 30°$

In Δ ABC $\angle B = 120°$. Now using cosine rule

$L^2 = (0.8)^2 + (0.4)^2 - 2 \times 0.8 \times 0.4 \cos 120$

$\therefore$ L = 1.058 m .... **Ans.**

3.9 Lami's Theorem

Lami's theorem deals with a particular case of equilibrium involving three forces only. It states "*If three concurrent forces act on a body keeping it in equilibrium, then each force is proportional to the sine of the angle between the other two forces*".

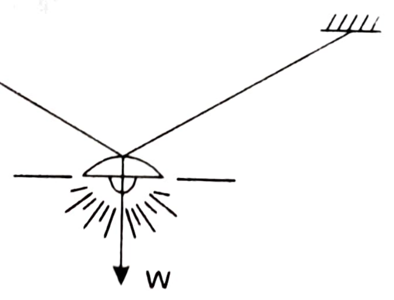


Fig. 3.17 (a)

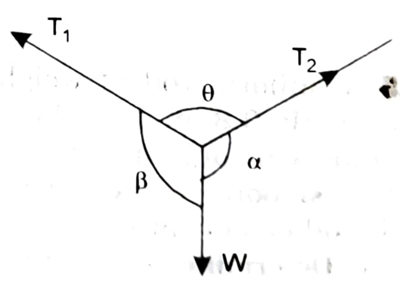


Fig. 3.17 (b)

Fig. 3.17 (a) shows a lamp held by two cables. The two tensile forces T_1 and T_2 in the string and the weight W of the lamp form a system of three forces in equilibrium. The forces would form a concurrent system. If α is the angle between T_2 and W, β is the angle between T_1 and W and θ is the angle between T_1 and T_2 then according to Lami's theorem

$$\frac{T_1}{\sin\alpha} = \frac{T_2}{\sin\beta} = \frac{W}{\sin\theta}$$

or in general for a system of three forces P, Q and R as shown in Fig. 3.17 (c) we write Lami's equation as

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\theta}$$

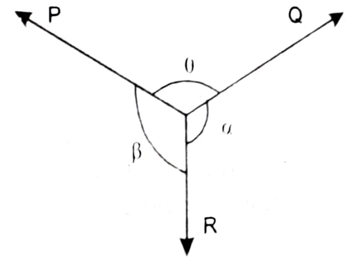


Fig. 3.17 (c)

Note that while using Lami's theorem, the three concurrent forces should either act towards the point of concurrence or act away from it. If this is not the case then using the principle of transmissibility they can be made in the required form. Fig. 3.18 (a) shows such a case for a sphere resting against smooth surfaces. The reactions R_A and R_B act ⊥ to smooth surfaces. To apply Lami's equation the forces have been arranged acting away from the point of concurrence as shown in Fig. 3.18 (b).

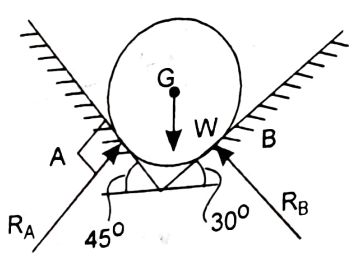


Fig. 3.18 (a)

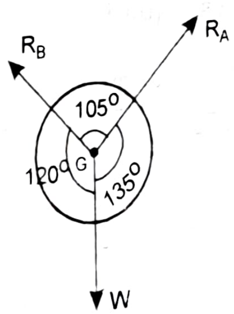


Fig. 3.18 (b)

Proof: Let P, Q and R be the three concurrent forces in equilibrium as shown in Fig. 3.19 (a).

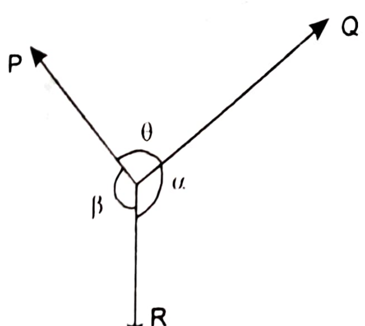


Fig. 3.19 (a)

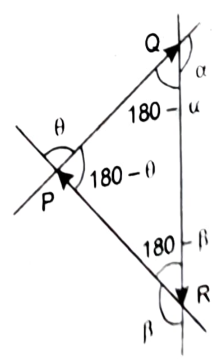


Fig. 3.19 (b)

Since the forces are vectors they are added vectorially by head and tail connections. We get a closed triangular polygon as shown in Fig. 3.19 (b).

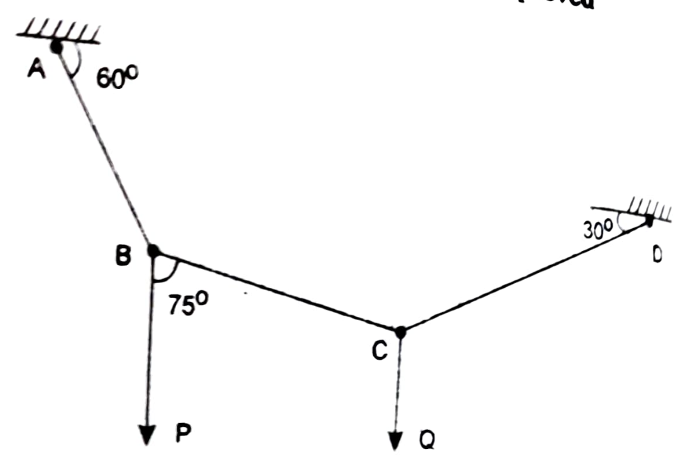
...we get

$$\frac{P}{\sin(180-\alpha)} = \frac{Q}{\sin(180-\beta)} = \frac{R}{\sin(180-\theta)}$$

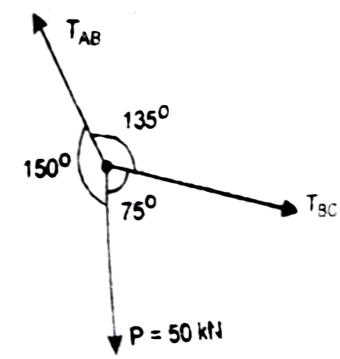
$\therefore$

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\theta}$$ .... proved

Ex. 3.6 **A string ABCD carries two loads P and Q. If P = 50 kN, find force Q and tensions in different portions of the string.**



Solution: Isolating joint B of the string. Let T_{AB} and T_{BC} be the tensions in the string portions AB and BC respectively.

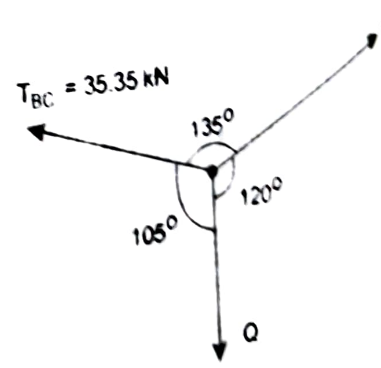


Using Lami's equation

$$\frac{T_{AB}}{\sin 75} = \frac{T_{BC}}{\sin 150} = \frac{50}{\sin 135}$$

$\therefore$ $T_{AB} = 68.3$ kN .... **Ans.**

$T_{BC} = 35.35$ kN ....**Ans.**



Now isolating joint C.

Let T_{CD} be the tension in portion CD.

Using Lami's equation

$$\frac{35.35}{\sin 120} = \frac{T_{CD}}{\sin 105} = \frac{Q}{\sin 135}$$

$\therefore$ $T_{CD} = 39.43$ kN .... **Ans.**

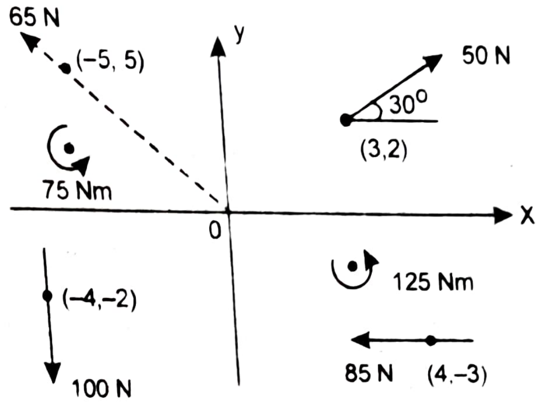
$Q = 28.86$ kN .... **Ans.**

10 Equilibrant Force

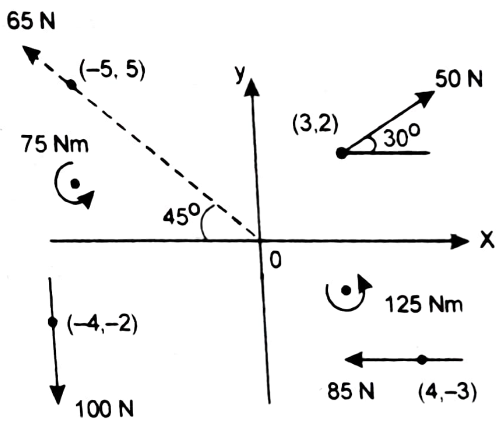
An unbalanced force system can be brought to equilibrium by adding an quilibrant force in the system. The equilibrant force has magnitude, direction and oint of application as of the resultant of the system but has a sense opposite to that of he Resultant.

To find the Equilibrant of a force system we first find magnitude, direction and location of the Resultant of the force system. The Equilibrant of the force system shall therefore be a force of the same magnitude, direction and location as of the resultant but having an opposite sense to that of the Resultant.

Ex. 3.7 For the system shown, determine the Equilibrant.



Solution: The force system is a general system of three forces and two couples. Let us first find the resultant of the system.



$\Sigma F_X \quad \rightarrow +\text{ve}$

$= 50 \cos 30 - 65 \cos 45 - 85$

$= -87.66$

$= 87.66 \text{ N} \leftarrow$

$\Sigma F_Y \quad \uparrow +\text{ve}$

$= 50 \sin 30 - 65 \sin 45 - 100$

$= -29.04$

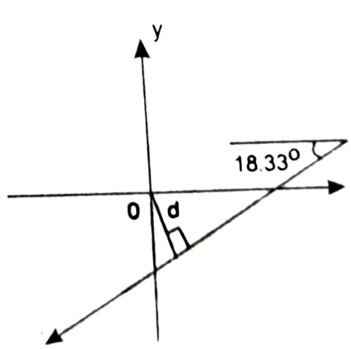
$= 29.04 \text{ N} \downarrow$

$$R = \sqrt{\Sigma F_x^{\,2} + \Sigma F_y^{\,2}}$$

$$= \sqrt{87.66^2 + 29.04^2}$$

$$= 92.34 \text{ N}$$

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x} = \frac{29.04}{87.66} \quad \therefore \quad \theta = 18.33° \;\swarrow$$



To locate the position of the resultant
Let us assume that the resultant is located at a ⊥ distance d to the right of O.

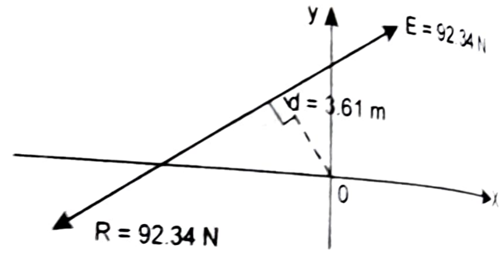
Using Varignon's Theorem

$\sum M_o^F = M_o^R$ ↺ + ve

$-50 \cos 30 \times 2 + 50 \sin 30 \times 3 - 85 \times 3 + 100 \times 4 + 75 + 125 = -92.34 \times d$

$\therefore \quad d = -3.61$ m

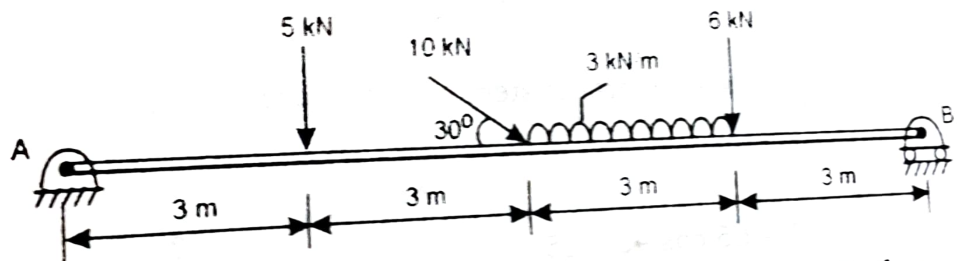
Since d is –ve the assumption is incorrect. The resultant is therefore to the left of O. The Equilibrant force being equal in magnitude and direction but having an opposite sense therefore acts as shown.



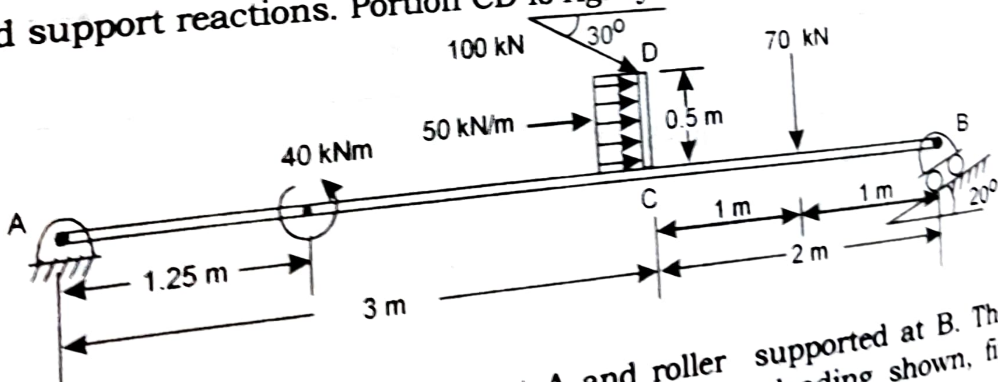
$\therefore$ E = 92.34 N, θ = 18.33° ∠ is located at a ⊥ distance d = 3.61 m to the left of O... A

Excercise 3.1

P1. A beam AB is loaded as shown. Find support reactions.



P2. Find support reactions. Portion CD is rigidly connected to the beam.



P3. Figure shows beam AB hinged at A and roller supported at B. The L shaped portion DEF is welded at D to the beam AB. For the loading shown, find support reactions.