$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U \Sigma V^T$ be svd of A.

STEP 1: Compute V:

Consider the matrix $A^T \cdot A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

$$A^T \cdot A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

We have to find eigen values & eigen vectors of $A^T \cdot A$.

Let $\lambda^2 - S_1 \lambda + S_2 = 0$ be the C.E. of $A^T \cdot A$.

$S_1 = Tr(A^T \cdot A) = 2 + 3 = 5$

$S_2 = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$

$\lambda^2 - 5\lambda + 6 = 0$

$\boxed{\lambda = 3, 2}$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U \Sigma V^T$ be svd of A.

STEP 1 : Compute V : $A^T \cdot A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

For $\lambda = 3$: Let $x_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ be e.v. of $A^T \cdot A$

Then, $(A^T A) x_1 = \lambda x_1$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 3 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$-a + 0b = 0 \quad \text{—— (1)}$$

$$0a + 0b = 0 \quad \text{—— (2)}$$

From eqn (1), $-a + 0b = 0$,

Put $a = 0$ & $b = 1$ $\therefore x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow N(x_1) = \sqrt{0^2 + 1^2} = 1$ $\quad 1/1$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U \Sigma V^T$ be svd of A.

STEP 1: Compute V: $A^T \cdot A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

For $\lambda = 2$: Let $x_2 = \begin{pmatrix} a \\ b \end{pmatrix}$ be e.v. of $A^T \cdot A$

Then $(A^T \cdot A)\, x_2 = \lambda x_2$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$0a + 0b = 0 \quad \text{— } (1)$$

$$0a + b = 0 \quad \text{— } (2)$$

From eqn. (2), $0a + b = 0$, put $a = 1$ & $b = 0$

$$\therefore x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow N(x_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\sqrt{1^2 + 0^2} = 1$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U \Sigma V^T$ be svd of A.

STEP 1 : Compute V : $A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Now, $N(X_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ & $N(X_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\therefore \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let $A = U\Sigma V^T$ be svd of A.

<u>STEP 2</u>: Compute Σ.

Order of Σ = Order of A.

Now, $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$

No. of non-zero eigen values = Rank.

$\therefore$ Rank = 2

$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$, i.e., Σ_1 is a diagonal matrix whose diagonal entries are σ_1, σ_2

$\Sigma_1 = R \times R = 2 \times 2$

$\therefore \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U\Sigma V^T$ be svd of A.

<u>STEP 3</u>: Compute U.

Consider the matrix $A \cdot A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U \Sigma V^T$ be svd of A.

STEP 3: Compute U.

Consider the matrix $A \cdot A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$A \cdot A^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Let $\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0$ be the C.E. of $A \cdot A^T$.

$S_1 = 2 + 1 + 2 = 5$.

$S_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 + 4 + 1 = 6$

$S_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2(1) - 1(2) = 0$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U\Sigma V^T$ be svd of A.

<u>STEP 3</u>: Compute U

$$\lambda^3 - 5\lambda^2 + 6\lambda - 0 = 0$$

$$\begin{array}{c|cccc} 2 & 1 & -5 & 6 & 0 \\ & 0 & 2 & -6 & 0 \\ \hline & 1 & -3 & 0 & \boxed{0} \end{array}$$

$$(\lambda - 2)(\lambda^2 - 3\lambda) = 0$$

$$\lambda - 2 = 0 \Rightarrow \lambda = 2$$

$$\lambda^2 - 3\lambda = 0 \Rightarrow \lambda(\lambda - 3) = 0 \Rightarrow \lambda = 0, 3$$

$$\therefore \boxed{\lambda = 3, 2, 0}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U\Sigma V^T$ be svd of A.

STEP 3: Compute U

For $\lambda = 3$: Let $x_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be e.v. of $A \cdot A^T$.

Then $(AA^T)x_1 = \lambda x_1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$-a + b + 0c = 0$ — (1)

$a - 2b + c = 0$ — (2)

$0a + b - c = 0$ — (3)

$$\begin{matrix} 1 & 0 & -1 & 1 \\ -2 & 1 & 1 & -2 \end{matrix}$$

$A = U\Sigma V$ be SVD of A

Compute U

Let $x_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be e.v. of $A \cdot A^T$.

Then $(AA^T)x_1 = \lambda x_1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$b + 0c = 0$ — 1

$2b + c = 0$ — 2

$b - c = 0$ — 3

$$\frac{a}{1} = \frac{b}{1} = \frac{c}{1} = k = 1$$

$\therefore x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\Rightarrow$ $N(x_1) = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$\begin{bmatrix} \ldots & \ldots \\ -1 & 1 \end{bmatrix}$$

Let $A = U\Sigma V^T$ be svd of A.

STEP 3: Compute U

For $\lambda = 2$: Let $x_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be e.v. of $A \cdot A^T$

Then $(A \cdot A^T)x_2 = \lambda x_2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$0a + b + 0c = 0$ — (1)

$a - b + c = 0$ — (2)

$0a + b + 0c = 0$ — (3)

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{a}{1} = \frac{b}{0} = \quad = k$$

$$\therefore x_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow N(x_2) = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

Let $A = U\Sigma V^T$ be svd of A.

STEP3: Compute U

For $\lambda = 0$: Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be e.v. of $A \cdot A^T$.

Then $(A \cdot A^T) X_3 = \lambda X_3$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$2a + b + 0c = 0$ — (1)

$a + b + c = 0$ — (2)

$0a + b + 2c = 0$ — (3)

$$\begin{matrix} 1 & & 0 & & 2 & & 1 \\ & \times & & \times & & \times & \\ 1 & & 1 & & 1 & & \end{matrix}$$

$\frac{a}{1} = \frac{b}{-2}$

$$\therefore X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \implies \therefore N(X_3) = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Let $A = U\Sigma V^T$ be svd of A.

STEP 3: Compute U

$$N(X_1) = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} ; \quad N(X_2) = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} ; \quad N(X_3) = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$$\therefore \quad U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

Sub : Eng Statistics and

Singular Value deComposition : [SVD]

For rectangular matrix : $A = U \Sigma V^T$

$A \rightarrow$ Given input matrix : $\underset{r}{m} \times \underset{c}{n}$

For V : $n \times n$ Columns are EV of $A^T A$
For U : $m \times m$ Columns are EV of $A A^T$
Σ : diagonal matrix : $m \times n$

Find Singular Value decomposition of matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2}$$

To find $A = U \Sigma V^T$

Sol^n:

$\boxed{\text{Step 1}}$: Compute $A^T A$ for V : EV : $n \times n = 2$
$A A^T$ for U : EV : $m \times m = 3$

- $A^T A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 1+0-1 \\ 1+0-1 & 1+1+1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 0+1 & -1+1 \\ 0+1 & 0+1 & 0+1 \\ -1+1 & 0+1 & 1+1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{\substack{3 \times 3 \\ m \;\; m}}$$ for U ✓

$\boxed{\text{Step 2}}$: Find Eigen Valu… for $A^TA \Rightarrow V$.
So we can …lculate V^T

$A^TA = A$

char eqn : $|A - \lambda I| = 0$

$$\therefore \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) = 0$$

$$6 - 2\lambda - 3\lambda - \lambda^2 = \ldots$$

4th Sem | ECE | VTU

$$AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 0+1 & -1+1 \\ 0+1 & 0+1 & 0+1 \\ -1+1 & 0+1 & 0+1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}_{\substack{3 \times 3 \\ m \times m}}$$ for U ✓

$\boxed{\text{Step 2}}$: Find Eigen values for $A^T A \Rightarrow V$.
So we can Calculate V^T

char eqn : $|A - \lambda I| = 0$ Here $A^T A = A$

$$\therefore \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda) = 0$$

$$6 - 2\lambda - 3\lambda - \lambda^2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$\boxed{\lambda = 3, 2}$ write in descending order

↳ eigen values.

ES & LA | 4th Sem | ECE | VTU

EV for $\lambda = 3$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

put $\lambda = 3$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

P F

$\Rightarrow$ Let $x_2 = x_2$ $x_1 = 0$

$\therefore$ eigen Vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$V_1 = \frac{X_1}{\|X_1\|} = \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{1}$$

EV for $\lambda = 2$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Big[$$

put $\lambda = 2$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \Big[$$

F P

$\Rightarrow$ Let $x_1 = x_1$ x_2

$x_1 / x_1 = 1$

EV $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$$V_2 = \frac{X_2}{\|X_2\|} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1}$$

$V_2 = \Big[1$

EV for $\lambda = 3$

$(A - \lambda I) X = 0$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

put $\lambda = 3$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

P F

$\Rightarrow$ Let $x_2 = x_2$ $x_1 = 0$

$\therefore$ eigen Vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$V_1 = \frac{X_1}{\|X_1\|} = \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{1}$$

$$V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \checkmark$$

EV for $\lambda = 2$

$(A - \lambda I) X = 0$

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

put $\lambda = 2$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

F P

$\Rightarrow$ Let $x_1 = x_1$ $x_2 = 0$

$x_1 / x_1 = 1$

EV $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$V_2 = \frac{X_2}{\|X_2\|} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1}$$

$$V_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \checkmark$$

NOTE : $\|X_1\| = \sqrt{0^2 + 1^2} = 1$ $\|X_2\| = \sqrt{1^2 + 0^2} = 1$

$\boxed{\text{Step 3}}$: Find matrix U : m×m : 3×3 : AA^T

we have

$$U = \left\{ \frac{AV_1}{\sigma_1}, \frac{AV_2}{\sigma_2} \right\}$$

u_1 u_2

$$u_1 = \frac{AV_1}{\sigma_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 0+1 \\ 0+1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\therefore \quad u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \checkmark$$

$$u_2 = \frac{AV_2}{\sigma_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+0 \\ 0+0 \\ -1+0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$U_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \qquad U_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$U_1^T U_3 = 0 \qquad\qquad U_2^T U_3 = 0$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \qquad\qquad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$\frac{\alpha}{\sqrt{3}} + \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{3}} = 0 \qquad\qquad \frac{\alpha}{\sqrt{2}} - \frac{\gamma}{\sqrt{2}} = 0$$

$$\frac{\alpha + \beta + \gamma}{\sqrt{3}} = 0 \qquad\qquad \frac{\alpha - \gamma}{\sqrt{2}} = 0$$

$$\beta + 2\alpha = 0 \qquad\qquad \alpha - \gamma = 0$$

$$\boxed{\beta = -2\alpha} \qquad\qquad \boxed{\alpha = \gamma}$$

$\therefore$ vector : $\begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$U_3 = \frac{U_3}{\|U_3\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \Bigg| \quad \begin{aligned} \|U_3\| &= \sqrt{1^2 + 4 + 1} \\ \|U_3\| &= \sqrt{6} \end{aligned}$$

$\therefore$ $U = \{ U_1, U_2, U_3 \}$

$u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ $\quad u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

$u_1^T u_3 = 0$

$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

$\frac{\alpha}{\sqrt{3}} + \frac{\beta}{\sqrt{3}} + \frac{\gamma}{\sqrt{3}} = 0$

$\frac{\alpha + \beta + \gamma}{\sqrt{3}} = 0$

$\beta + 2\alpha = 0$

$\boxed{\beta = -2\alpha}$

$u_2^T u_3 = 0$

$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$

$\frac{\alpha}{\sqrt{2}} - \frac{\gamma}{\sqrt{2}} = 0$

$\frac{\alpha - \gamma}{\sqrt{2}} = 0$

$\alpha - \gamma = 0$

$\boxed{\alpha = \gamma}$

$\therefore$ Vector : $\begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$u_3 = \frac{u_3}{\| u_3 \|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\| u_3 \| = \sqrt{1^2 + 4 + 1}$

$\| u_3 \| = \sqrt{6}$

$\therefore \quad U = \{ u_1, u_2, u_3 \}$

$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ \ldots \end{bmatrix}$

$$\therefore \quad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}_{\substack{3 \times 2 \\ m \; n}}$$

$\therefore$ SVD is method of decomposing a rectangular matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3 \times 2}$ into 3 matrices.

First matrix : $U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}_{\substack{3 \times 3 \\ m \times m}}$

Second matrix : $\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}_{\substack{3 \times 2 \\ m \times n}}$

$$= \begin{bmatrix} 1+0+0 & 0+1+0 \\ 1+0+0 & 0+0+0 \\ 1+0+0 & 0-1+0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & 1+0 \\ 0+0 & 1+0 \\ 0-1 & \text{[illegible]} \end{bmatrix}$$

$$U \Sigma V^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$