svd

step ① find $A^T A$

step ② find eigen values & eigen vectors of $A^T A$.

step ③ arrange them in descending order and then label them by
$\lambda_1, \lambda_2, \lambda_3 \ldots$

step ④ write

$$D = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix}$$

step ⑤ ~~find~~ if eigen vectors are labeled as x_1, x_2, x_3 & so on ~~way~~ name them unit vectors by dividing vectors by its norm

step ⑥ write V as

$$V = \begin{bmatrix} \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \end{bmatrix}$$

step ⑦ write $U = [u_1 \; u_2 \; u_3]$

where $u_1 = \frac{1}{\sqrt{\lambda_1}} AV_1$

$u_2 = \frac{1}{\sqrt{\lambda_2}} AV_2$

$u_3 = \frac{1}{\sqrt{\lambda_3}} AV_3$

Ex find the singular value

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

$A = U \; D \; V'$

$$A' = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}$$

$$AA' = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

$$(B - \lambda I = \begin{bmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{bmatrix}$$

$= (25-\lambda)^2 - 49$

$= \cancel{\ell}\,\lambda^2 - 50\lambda + 576 = 0$

$\lambda = 18, 32$

$$D = \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}$$

for eigen vectors of $A'A$

$\lambda = 32$

$(B \cancel{\ell} - \lambda I)x = 0$

$$\begin{bmatrix} 25-32 & 7 \\ 7 & 25-32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-7x_1 + 7x_2 = 0$

$x_1 = x_2$

$$x_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = 18$

$(B - \lambda I)x = 0$

$$\begin{bmatrix} 25-18 & 7 \\ 7 & 25-18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$v_1 = \frac{x_1}{\|x_1\|} = \frac{[1 \ 1]'}{\sqrt{1+1}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$v_2 = \frac{x_2}{\|x_2\|} \quad \frac{[1 \ -1]'}{\sqrt{1+1}} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$v_f = [v_1 \ v_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$U_1 = \frac{1}{\sqrt{\lambda_1}} A V_1 = \frac{1}{\sqrt{32}} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U_2 = \frac{1}{\sqrt{\lambda_2}} A V_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$U = [u_1 \ \ u_2]$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\therefore$ SVD of A is,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{32} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}'$$