1.12 Types of Analysis

To analyze the given algorithm we need to know on what inputs the algorithm takes less time (performing well) and on what inputs the algorithm takes long time. We have already seen that an algorithm can be represented in the form of an expression. That means we represent the algorithm with multiple expressions: one for the case where it takes less time and other for the case where it takes the more time.

In general the first case is called the *best case* and second case is called the *worst case* of the algorithm. To analyze an algorithm we need some kind of syntax and that forms the base for asymptotic analysis/notation.

There are three types of analysis:

- **Worst case**
  - Defines the input for which the algorithm takes long time.
  - Input is the one for which the algorithm runs the slower.
- **Best case**
  - Defines the input for which the algorithm takes lowest time.
  - Input is the one for which the algorithm runs the fastest.
- **Average case**
  - Provides a prediction about the running time of the algorithm
  - Assumes that the input is random

Lower Bound <= Average Time <= Upper Bound

For a given algorithm, we can represent the best, worst and average cases in the form of expressions. As an example, let $f(n)$ be the function, which represents the given algorithm.

$f(n) = n^2 + 500$, for worst case
$f(n) = n + 100n + 500$, for best case

Similarly, for average case too. The expression defines the inputs with which the algorithm takes the average running time (or memory).

1.13 Asymptotic Notation

Having the expressions for the best, average case and worst cases, for all the three cases we need to identify the upper and lower bounds. To represent these upper and lower bounds we need some kind of syntax and that is the subject of the following discussion. Let us assume that the given algorithm is represented in the form of function $f(n)$.

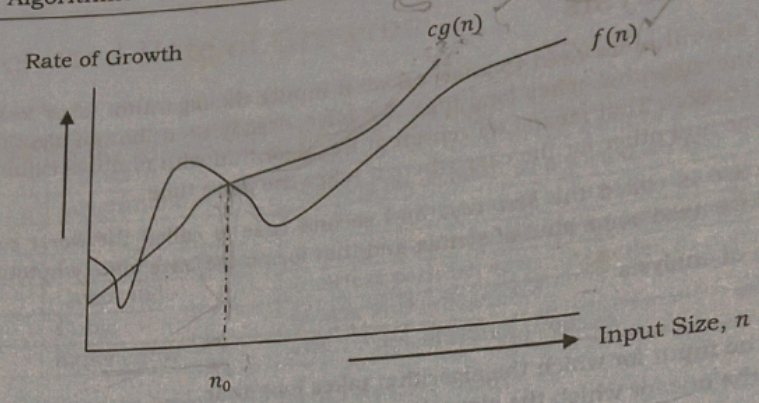
1.14 Big-O Notation

This notation gives the *tight* upper bound of the given function. Generally, it is represented as $f(n) = O(g(n))$. That means, at larger values of n, the upper bound of $f(n)$ is $g(n)$.

For example, if $f(n) = n^4 + 100n^2 + 10n + 50$ is the given algorithm, then n^4 is $g(n)$. That means, $g(n)$ gives the maximum rate of growth for $f(n)$ at larger values of n.

Let us see the O−notation with little more detail. O−notation defined as $O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$. $g(n)$ is an asymptotic tight upper bound for $f(n)$. Our objective is to give smallest rate of growth $g(n)$ which is greater than or equal to given algorithms rate of growth $f(n)$.

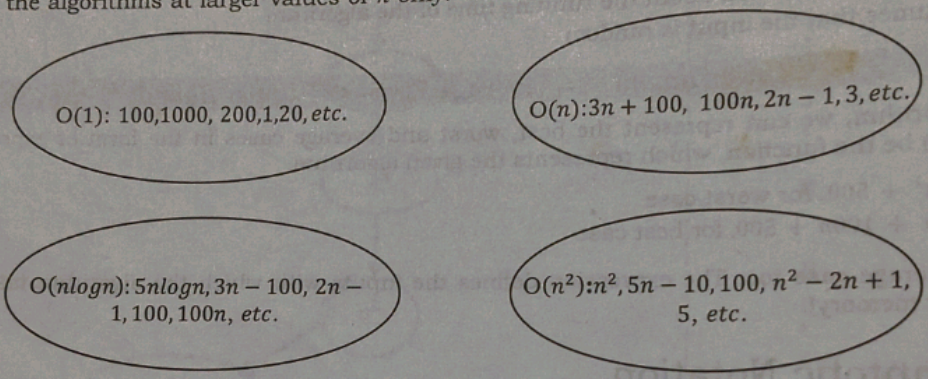
Generally we discard lower values of n. That means the rate of growth at lower values of n is not important. In the figure below, n_0 is the point from which we need to consider the rate of growths for a given algorithm. Below n_0 the rate of growths could be different.



Big-O Visualization

$O(g(n))$ is the set of functions with smaller or same order of growth as $g(n)$. For example, $O(n^2)$ includes $O(1), O(n), O(nlogn)$ etc..

Note: Analyze the algorithms at larger values of n only. What this means is, below n_0 we do not care for rate of growth.



Big-O Examples

Example-1 Find upper bound for $f(n) = 3n + 8$
Solution: $3n + 8 \leq 4n$, for all $n \geq 8$
$\therefore 3n + 8 = O(n)$ with c = 4 and $n_0 = 8$

Example-2 Find upper bound for $f(n) = n^2 + 1$
Solution: $n^2 + 1 \leq 2n^2$, for all $n \geq 1$
$\therefore n^2 + 1 = O(n^2)$ with $c = 2$ and $n_0 = 1$

Example-3 Find upper bound for $f(n) = n^4 + 100n^2 + 50$
Solution: $n^4 + 100n^2 + 50 \leq 2n^4$, for all $n \geq 11$
$\therefore n^4 + 100n^2 + 50 = O(n^4)$ with $c = 2$ and $n_0 = 11$

Example-4 Find upper bound for $f(n) = 2n^3 - 2n^2$
Solution: $2n^3 - 2n^2 \leq 2n^3$, for all $n \geq 1$
$\therefore 2n^3 - 2n^2 = O(2n^3)$ with $c = 2$ and $n_0 = 1$

Example-5 Find upper bound for $f(n) = n$
Solution: $n \leq n$, for all $n \geq 1$
$\therefore n = O(n)$ with $c = 1$ and $n_0 = 1$

Example-6 Find upper bound for $f(n) = 410$
Solution: $410 \leq 410$, for all $n \geq 1$
$\therefore 410 = O(1)$ with $c = 1$ and $n_0 = 1$

No Uniqueness?

There are no unique set of values for n_0 and c in proving the asymptotic bounds. Let us consider, $100n + 5 = O(n)$. For this function there are multiple n_0 and c values possible.

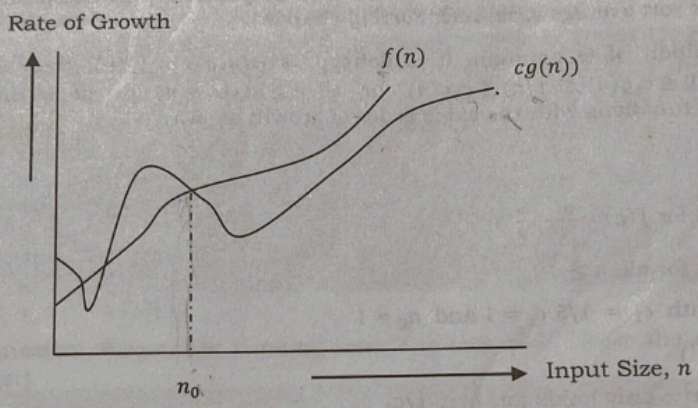
Solution1: $100n + 5 \leq 100n + n = 101n \leq 101n$, for all $n \geq 5$, $n_0 = 5$ and $c = 101$ is a solution.

Solution2: $100n + 5 \leq 100n + 5n = 105n \leq 105n$, *for all* $n \geq 1$, $n_0 = 1$ *and* $c = 105$ *is also a solution.*

1.15 Omega-Ω Notation

Similar to O discussion, this notation gives the tighter lower bound of the given algorithm and we represent it as $f(n) = \Omega(g(n))$. That means, at larger values of n, the tighter lower bound of $f(n)$ is $g(n)$. For example, if $f(n) = 100n^2 + 10n + 50$, $g(n)$ is $\Omega(n^2)$.

The Ω notation can be defined as $\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq cg(n) \leq f(n)$ for all n $\geq n_0\}$. $g(n)$ is an asymptotic tight lower bound for $f(n)$. Our objective is to give largest rate of growth $g(n)$ which is less than or equal to given algorithms rate of growth $f(n)$.



Ω Examples

Example-1 Find lower bound for $f(n) = 5n^2$

Solution: $\exists\, c,\ n_0$ Such that: $0 \leq cn^2 \leq 5n^2 \Rightarrow cn^2 \leq 5n^2 \Rightarrow c = 1$ and $n_0 = 1$

$\therefore 5n^2 = \Omega(n^2)$ with $c = 1$ and $n_0 = 1$

Example-2 Prove $f(n) = 100n + 5 \neq \Omega(n^2)$

Solution: $\exists\, c,\ n_0$ Such that: $0 \leq cn^2 \leq 100n + 5$

$100n + 5 \leq 100n + 5n\ (\forall n \geq 1) = 105n$

$cn^2 \leq 105n \Rightarrow n(cn - 105) \leq 0$

Since n is positive $\Rightarrow cn - 105 \leq 0 \Rightarrow n \leq 105/c$

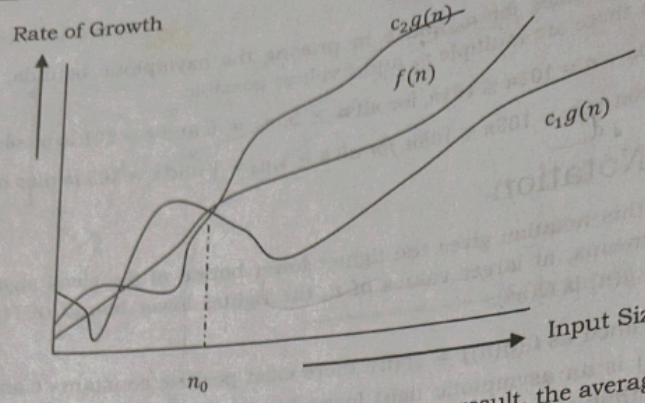
$\Rightarrow$ Contradiction: n cannot be smaller than a constant

Example-3 $2n = \Omega(n)$, $n^3 = \Omega(n^3)$, $logn = \Omega(logn)$

1.16 Theta-Θ Notation

This notation decides whether the upper and lower bounds of a given function (algorithm) are same. The average running time of algorithm is always between lower bound and upper bound. If the upper bound (O) and lower bound (Ω) give the same result then Θ notation will also have the same rate of growth. As an example, let us assume that $f(n) = 10n + n$ is the expression.

Then, its tight upper bound $g(n)$ is $O(n)$. The rate of growth in best case is $g(n) = O(n)$.



In this case, rate of growths in the best case and worst are same. As a result, the average case will also be same. For a given function (algorithm), if the rate of growths (bounds) for O and Ω are not same then the rate of growth Θ case may not be same. In this case, we need to consider all possible time complexities and take average of those (for example, quick sort average case, refer *Sorting* chapter).

Now consider the definition of Θ notation. It is defined as $\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0\}$. $g(n)$ is an asymptotic tight bound for $f(n)$. $\Theta(g(n))$ is the set of functions with the same order of growth as $g(n)$.

Θ Examples

Example-1 Find Θ bound for $f(n) = \frac{n^2}{2} - \frac{n}{2}$

Solution: $\frac{n^2}{5} \leq \frac{n^2}{2} - \frac{n}{2} \leq n^2$, for all, $n \geq 1$

$\therefore \frac{n^2}{2} - \frac{n}{2} = \Theta(n^2)$ with $c_1 = 1/5, c_1 = 1$ and $n_0 = 1$

Example-2 Prove $n \neq \Theta(n^2)$

Solution: $c_1 n^2 \leq n \leq c_2 n^2 \Rightarrow$ only holds for: $n \leq 1/c_1$

$\therefore n \neq \Theta(n^2)$

Example-3 Prove $6n^3 \neq \Theta(n^2)$

Solution: $c_1 n^2 \leq 6n^3 \leq c_2 n^2 \Rightarrow$ only holds for: $n \leq c_2/6$

$\therefore 6n^3 \neq \Theta(n^2)$

Example-4 Prove $n \neq \Theta(logn)$

Solution: $c_1 logn \leq n \leq c_2 \log n \Rightarrow c_2 \geq \frac{n}{\log n}, \forall n \geq n_0$ – Impossible

1.17 Important Notes

For analysis (best case, worst case and average) we try to give upper bound (O) and lower bound (Ω) and average running time (Θ). From the above examples, it should also be clear that, for a given function (algorithm) getting upper bound (O) and lower bound (Ω) and average running time (Θ) may not be possible always. For example, if we are discussing the best case of an algorithm, then we try to give upper bound (O) and lower bound (Ω) and average running time (Θ).

In the remaining chapters we generally focus on upper bound (O) because knowing lower bound (Ω) of an algorithm is of no practical importance and we use θ notation if upper bound (O) and lower bound (Ω) are same.

1.18 Why is it called Asymptotic Analysis?

From the discussion above (for all the three notations: worst case, best case and average case), we can easily understand that, in every case for a given function $f(n)$ we are trying to find other function $g(n)$ which approximates $f(n)$ at higher values of n. That means, $g(n)$ is also a curve which approximates $f(n)$ at higher values of n. In mathematics we call such curve as *asymptotic curve*. In other terms, $g(n)$ is the asymptotic curve for $f(n)$. For this reason, we call algorithm analysis as *asymptotic analysis*.

1.19 Guidelines for Asymptotic Analysis

There are some general rules to help us determine the running time of an algorithm.

1) **Loops:** The running time of a loop is, at most, the running time of the statements inside the loop (including tests) multiplied by the number of iterations.

```
// executes n times
for (i=1; i<=n; i++)
    m = m + 2; // constant time, c
```

Total time = a constant $c \times n = cn = O(n)$.

2) **Nested loops:** Analyze from inside out. Total running time is the product of the sizes of all the loops.

```
//outer loop executed n times
for (i=1; i<=n; i++) {
      // inner loop executed n times
      for (j=1; j<=n; j++)
            k = k+1; //constant time
}
```

Total time = $c \times n \times n = cn^2 = O(n^2)$.

3) **Consecutive statements:** Add the time complexities of each statement.

```
x = x +1; //constant time
// executed n times
for (i=1; i<=n; i++)
     m = m + 2; //constant time
//outer loop executed n times
for (i=1; i<=n; i++) {
     //inner loop executed n times
     for (j=1; j<=n; j++)
          k = k+1; //constant time
}
```

Total time = $c_0 + c_1 n + c_2 n^2 = O(n^2)$.

4) **If-then-else statements:** Worst-case running time: the test, plus *either* the *then* part *or* the *else* part (whichever is the larger).

```
//test: constant
if(length( ) == 0 ) {
     return false; //then part: constant
}
else {  // else part: (constant + constant) * n
      for (int n = 0; n < length( ); n++) {
            // another if : constant + constant (no else part)
            if(!list[n].equals(otherList.list[n]))
                  //constant
                  return false;
      }
}
```

Total time = $c_0 + c_1 + (c_2 + c_3) * n = O(n)$.

5) **Logarithmic complexity:** An algorithm is O(*logn*) if it takes a constant time to cut the problem size by a fraction (usually by ½). As an example let us consider the following program:

```
for (i=1; i<=n;)
     i = i*2;
```

If we observe carefully, the value of i is doubling every time. Initially $i = 1$, in next step $i = 2$, and in subsequent steps $i = 4, 8$ and so on. Let us assume that the loop is executing some k times. At k^{th} step $2^k = n$ and we come out of loop. Taking logarithm on both sides, gives

$log(2^k) = logn$
$klog2 = logn$
$k = logn$ //if we assume base-2

Total time = O(*logn*).

Note: Similarly, for the case below also, worst case rate of growth is O(*logn*). The same discussion holds good for decreasing sequence as well.

```
for (i=n; i>=1;)
    i = i/2;
```

Another example: binary search (finding a word in a dictionary of n pages)

- Look at the center point in the dictionary
- Is word towards left or right of center?
- Repeat process with left or right part of dictionary until the word is found

1.20 Properties of Notations

- Transitivity: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$.
- Reflexivity: $f(n) = \Theta(f(n))$. Valid for O and Ω also.
- Symmetry: $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.
- Transpose symmetry: $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$.

1.21 Commonly used Logarithms and Summations

Logarithms

$$\log x^y = y \log x \qquad \log n = \log_{10}^n$$

$$\log xy = \log x + \log y \qquad \log^k n = (\log n)^k$$

$$\log \log n = \log(\log n) \qquad \log \frac{x}{y} = \log x - \log y$$

$$a^{\log_b^x} = x^{\log_b^a} \qquad \log_b^x = \frac{\log_a^x}{\log_a^b}$$

Arithmetic series

$$\sum_{K=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometric series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 \ldots + x^n = \frac{x^{n+1} - 1}{x - 1} (x \neq 1)$$

Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \approx \log n$$

Other important formulae

$$\sum_{k=1}^{n} \log k \approx n \log n$$

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + \cdots + n^p \approx \frac{1}{p+1} n^{p+1}$$

1.22 Master Theorem for Divide and Conquer

All divide and conquer algorithms (Also discussed in detail in the *Divide and Conquer* chapter) divide the problem into sub-problems, each of which is part of the original problem, and then perform some additional work to compute the final answer. As an example, merge sort algorithm [for details, refer *Sorting* chapter] operates on two sub-problems, each of which is half the size of the original and then performs $O(n)$ additional work for merging. This gives the running time equation:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n)$$

The following theorem can be used to determine the running time of divide and conquer algorithms. For a given program (algorithm), first we try to find the recurrence relation for the problem. If the recurrence is of the below form then we can directly give the answer without fully solving it. If the recurrence is of the form $T(n) = aT(\frac{n}{b}) + \Theta(n^k \log^p n)$, where $a \geq 1, b > 1, k \geq 0$ and p is a real number, then: