1.3 CHARACTERISTIC EQUATION

If A is any square matrix of order n, we can form the matrix $A - \lambda I$, where I is the nth order unit matrix.

The determinant of this matrix equated to zero,

i.e.,

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

is called the characteristic equation of A.

On expanding the determinant, we get

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \ldots + k_n = 0$$

where k's are expressible in terms of the elements a_{ij}

The roots of this equation are called Characteristic roots or latent roots or eigen values of the matrix A.

$|A - \lambda I| = 0$

→ eigen value of A.

1.4 EIGEN VECTORS

Consider the linear transformation $Y = AX$ ...(1) which transforms the column vector X into the column vector Y. We often required to find those vectors X which transform into scalar multiples of themselves.

Let X be such a vector which transforms into λX by the transformation (1).

$(A - \lambda I) X = 0$

$\lambda =$

$X =$ eigen vector, corresponding eigen value λ

$|A - \lambda I| = 0$

$\lambda = 2,$

$X = \begin{pmatrix} \\ \end{pmatrix}$

$AX = 0$

$|A| = 0$

$C(A) \in$

$\gamma <$

Then $Y = \lambda X$ ... (2)

From (1) and (2), $AX = \lambda X \Rightarrow AX - \lambda I X = 0$

$$\Rightarrow (A - \lambda I)X = 0 \text{ ...(3)}$$

This matrix equation gives n homogeneous linear equations

$$\left.\begin{array}{llll}
(a_{11}-\lambda)x_1 & +a_{12}x_2 & +\ldots & +a_{1n}x_n = 0 \\
a_{21}x_1 & +(a_{22}-\lambda)x_2 & +\ldots & +a_{2n}x_n = 0 \\
\ldots. & \ldots. & \ldots. & \ldots. \\
a_{n1}x_1 & +a_{n2}x_2 & +\ldots & +(a_{nn}-\lambda)x_n = 0
\end{array}\right\} \text{ ... (4)}$$

These equations will have a non-trivial solution only if the co-efficient matrix $A - \lambda I$ is singular

i.e., if $|A - \lambda I| = 0$ ... (5)

Corresponding to each root of (5), the homogeneous system (3) has a non-zero solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_4 \end{bmatrix}$$ is called an eigen vector or latent vector

Properties of Eigen Values:-

1. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.
2. The product of the eigen values of a matrix A is equal to its determinant.
3. If λ is an eigen value of a matrix A, then $1/\lambda$ is the eigen value of A^{-1}.
4. If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen value.

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1 $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

eigen values $\lambda_1, \lambda_2, \lambda_3$

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$$

2 $$\underline{\lambda_1 \lambda_2 \lambda_3} = |A|$$

3 $\lambda_1, \lambda_2, \lambda_3$ eigen values of A, $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ eigen values of A^{-1}.

PROPERTY 1:- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then

i. $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the eigen values of the matrix kA, where k is a non – zero scalar.

ii. $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigen values of the inverse matrix A^{-1}.

iii. $\lambda_1^p, \lambda_2^p, \ldots, \lambda_n^p$ are the eigen values of A^p, where p is any positive integer.

$\lambda_1^2, \lambda_2^2, \lambda_3^2 - - - \lambda_n^2 \to A^2$

Proof:-

i. Let λ_r be an eigen value of A and X_r the corresponding eigen vector.

Then, by definition,

$$\boxed{AX_r = \lambda_r X_r}$$

$(A - \lambda I) X_r = 0$

$AX = \lambda X$

Multiplying both sides by k,

$$(kA)X_r = (k\lambda_r)X_r$$

Then $k\lambda_r$ is an eigen value of kA and the corresponding eigen vector is the same as that of λ_r, namely X_r.

ii. Pre multiplying both sides of $AX_r = \lambda_r X_r$ by A^{-1}

$$A^{-1}(A X_r) = A^{-1}(\lambda_r X_r)$$

$$X_r = \lambda_r (A^{-1} X_r)$$

$$\Rightarrow A^{-1} X_r = \frac{1}{\lambda_r} (X_r)$$

$Ax = \lambda x$

Hence $\frac{1}{\lambda_r}$ is an eigen value of A^{-1} and the corresponding eigen vector is the same as that of λ_r, namely X_r.

iii. Pre multiplying both sides of $AX_r = \lambda_r X_r$ by A

$$A(A X_r) = A(\lambda_r X_r)$$
$$A^2 X_r = \lambda_r (A X_r)$$
$$= \lambda_r (\lambda_r x_r)$$
$$= \lambda_r^2 X_r.$$

Similarly, we can prove that $A^3 X_r = \lambda_r^3 X_r, \ldots,$

$A^p X_r = \lambda_r^p X_r$, where p is any positive integer. Hence λ_r^p is any eigen value of A^p and the corresponding eigen vector is the same as that of λ_r, namely X_r.

THEOREM :-

A matrix A is singular if and only if 0 is an eigen value of A.

$|A| = 0$

1.5 PROBLEMS

1. Find the sum and product of the eigen values of the matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

without finding the eigen values.

$\lambda_1, \lambda_2, \lambda_3$

$\lambda_1 + \lambda_2 + \lambda_3 = -2 + 1 = -1.$

$\lambda_1 \lambda_2 \lambda_3 = |A|.$

Solution:-

Sum of the eigen values of A = sum of its diagonal elements.

$$= -2 + 1 + 0$$

$$= -1.$$

Product of the eigen values of A = | A |

$$= \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$$= 45.$$

2. Two eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

are equal to 1 each. Find the third eigen value.

Solution:- Let a be the third eigen value of A.

Since sum of the eigen values = sum of the diagonal elements,

$$1 + 1 + a = 2 + 3 + 2$$

$$a = 5$$

Therefore, the third eigen value of A is 5.

3. The product of two eigen values of the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$ is 16. Find the third eigen value.

Solution:-

Let a be the third eigen value of A.

Since product of the eigen values = | A |

$$16a = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

Therefore, $a = 2.$

4. Find the sum of the eigen values of the inverse of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 0 & 5 & 2 \end{pmatrix}$$

Solution:-

The eigen values of the lower triangular matrix A is **1, -3, 2.** Then the eigen values of A^{-1} are

$$1, -\frac{1}{3}, \frac{1}{2}.$$

Sum of the eigen values of A^{-1} = $1+\frac{-1}{3}+\frac{1}{2}.$

$$= \frac{7}{6}$$

5. If $A = \begin{pmatrix} 2 & 7 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$, find the eigen values of 3A, A^{-1} and – $2A^{-1}$.

Solution:-

The eigen values of A are 2, -1, 4.

The eigen values of 3A are 3×2, 3×(-1), 3×4

i.e., 6, -3, 12

The eigen values of A^{-1} are

$$\frac{1}{2}, \frac{1}{-1}, \frac{1}{4}$$

i.e., $$\frac{1}{2}, -1, \frac{1}{4}$$

The eigen values of $-2A^{-1}$ are

$$-2\left(\frac{1}{2}\right), (-2)(-1), -2\left(\frac{1}{4}\right)$$

i.e., $$-1, 2, -\frac{1}{2}$$

6. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$$

2×2

Solution:- The characteristic equation of the given matrix is

$$|A - \lambda I| = 0$$

or $\begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) - 10 = 0$

$4 - 5\lambda + \lambda^2 - 10 = 0$

$=> \lambda^2 - 5\lambda - 6 = 0$ ✓

$=> \lambda = 6, -1$ ✓

Thus, the eigen values of A are 6, -1.

Corresponding to $\lambda=6$, the eigen vectors are given by

$(A - 6I)\, X_1 = 0$

$$\text{or} \begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \text{or} \begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$A\, X_1 = 0$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} -5 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

e

We get only one independent equation $-5x_1 - 2x_2 = 0$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-5} = k_1 \text{ (say)}$$

$$x_1 = 2k_1$$

$$x_2 = -5k_1$$

$\therefore$ The eigen vectors are $X_1 = k_1 \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

$n - r = 2 - 1 = 1$

$x_1 = k$

$2x_1 = -5k$

$\boxed{x_2 = \frac{-5k}{2}}$

$X_1 = \begin{bmatrix} k \\ \frac{-5k}{2} \end{bmatrix} = k \begin{bmatrix} 1 \\ -5/2 \end{bmatrix}$

$= k_1 \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Corresponding to λ = -1, the eigen vectos are given by $(A + I)\,X_2 = 0$

$$\Rightarrow \quad \begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 - x_2 = 0$$

$$x_1 = x_2 = k_2$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{1} = k_2 (\text{say})$$

$$X_2 = \begin{bmatrix} k_2 \\ k_2 \end{bmatrix}$$

$$x_1 = k_2, \; x_2 = k_2$$

$$= k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \text{ The eigen vectors are } X_2 = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

7. Find the eigen values and eigen vectors of the matrix A = $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution:- The characteristic equation of the given matrix is $|A-\lambda I|=0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda)[-\lambda(1-\lambda)-12]-2[-2\lambda-6]-3[-4+1(1-\lambda)]=0$$

$$\Rightarrow \quad \lambda^3+\lambda^2-21\lambda-45=0$$

$\lambda = 0, 1, -1, 2, -2$

$3, -3, 4, -4$

By trial, $\lambda=-3$ satisfies it.

$\lambda=-3$				
	1	1	-21	-45
	0	-3	6	45
	1	-2	-15	0

$$\therefore (\lambda+3)(\lambda^2-2\lambda-15)=0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0$$

$$\Rightarrow \lambda=-3,-3,5$$

$(\lambda+3)(\lambda^2-2\lambda-15)$

Thus, the eigen values of A are -3, -3, 5.

Corresponding to $\lambda = -3$, the eigen vectors are given by

$R_2 \to \frac{R_2}{2}$

$$(A + 3I)X_1 = 0 \qquad \text{or} \qquad \begin{bmatrix} -1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

We get only one independent equation $x_1 + 2x_2 - 3x_3 = 0$

Let $x_3 = k_1$, $x_2 = k_2$ then $x_1 = 3k_1 - 2k_2$

$\therefore$ The eigen vectors are given by

$$X_1 = \begin{bmatrix} 3k_1 - 2k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 \\ 1 & 2 & -3 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} -1 & 2 & -3 \\ 0 & 4 & -6 \\ 0 & -4 & 6 \end{bmatrix}$$

$R_3 \to R_3 + R_2.$

$$\begin{bmatrix} -1 & 2 & -3 \\ 0 & 4 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rho(A) = 2 = r$ no. of unknowns $= 3$

$r < n$ infinite. $n - r = 3 - 2 = 1$

$-x_1 + 2x_2 - 3x_3 = 0$

$4x_2 - 6x_3 = 0$

$x_3 = k,$ $4x_2 = 6k$

$$\boxed{x_2 = \frac{6k}{4} = \frac{3k}{2}}$$

$$-a_1 + \cancel{2} \times \frac{3k}{\cancel{2}} - 3k = 0$$

$$\boxed{a_1 = 0}$$

$$\therefore \quad X_1 = \begin{pmatrix} 0 \\ 3k/2 \\ k \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 0 \\ 3k \\ 2k \end{pmatrix} = k \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

Corresponding to $\lambda = 5$, the eigen vectors are given by $(A - 5I)X_2 = 0$.

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -7x_1 + 2x_2 - 3x_3 = 0$$

$$x_1 - 2x_2 - 3x_3 = 0$$

$$-x_1 - 2x_2 - 5x_3 = 0$$

From first two equations. ,

$$\frac{x_1}{10-6}=\frac{x_2}{3+5}=\frac{x_3}{-2-2}$$

$$\Rightarrow \quad \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{-1}=k_3\,(say)$$

$$\therefore \quad x_1=k_3,\ x_2=2k_{3,}\ x_3=-k_3$$

Hence the eigen vectors are given by

$$X_2=k_3\begin{bmatrix}1\\2\\-1\end{bmatrix}$$