25|3|2022

Propositional Logic

1. Mammals drink milk $\Rightarrow$ Mammal(Tom) $\rightarrow$ drink(Tom, milk)
2. Man is mortal $\Rightarrow$ man(Tom) $\rightarrow$ mortal(Tom)
3. Man is mammal $\Rightarrow$ man(Tom) $\rightarrow$ mammal(Tom)
4. Tom is man. $\Rightarrow$ man(Tom)

To prove :- Tom drinks milk. $\Rightarrow$ drink(Tom, milk)
Tom is mortal. $\Rightarrow$ mortal(Tom)

Proof :. Using Modus Ponens

4) man(Tom)
2) man(Tom) $\rightarrow$ mortal(Tom)

$\boxed{\text{mortal(Tom)}}$

$$\frac{P \qquad P \rightarrow Q}{Q}$$

4) man(Tom)
3) man(Tom) $\rightarrow$ mammal(Tom)

5) mammal(Tom)
1) mammal(Tom) $\rightarrow$ drink(Tom, milk)

$\boxed{\text{drink(Tom, milk)}}$

Modus Ponens
P
$P \rightarrow Q$

Limitations of Prop. Logic

All dogs are faithful. $\Rightarrow$ Dog(Tom) $\rightarrow$ Faithful(Tom)

PREDICATE LOGIC

- Also known as First Order Logic (FOL)

→ Terms :— A term denote some object other than True/False.

→ Predicates

→ Quantifier

Term
- Constant — name denotes a specific object
  e.g. Tom, 5
- Variable — name denotes any element in that set
  e.g. $x \in N$
- Function — Functional term of arity n takes n object as i/p and returns an object
  e.g. plus(3,4)
  plus(x, y), $x, y \in N$

plus(x, y) returns a natural no. (x+y)
where $x, y \in N$

plus(3, −1) ✗ plus(1.2, 3.5) ✗ plus(plus(5,2), plus(3,7)) ✓

$$\frac{\text{plus}(\underbrace{\text{plus}(5,2)}_{7}, \underbrace{\text{plus}(3,7)}_{10})}{17}$$

Predicate

Predicate are like functions except that their return type is True/False.

e.g. greater(x, y) is true iff $x > y$

$\Rightarrow$ greater(4, 3) is true
greater(1, 2) is false

- A predicate with no variable is called proposition.
- " " " one " " property.

e.g. $dog(x)$ is true iff x is dog.

- Let $P(x, y, --)$ and $Q(w, z, --)$ are two predicates

then
$P \vee Q$
$P \wedge Q$
$\neg P, \neg Q$
$P \rightarrow Q$
} valid Predicates

e.g. If x is man then x is mortal.

$man(x) \rightarrow mortal(x)$

If n is natural no. then n is either even or odd.

$natural(n) \rightarrow (even(n) \vee odd(n))$

Quantifiers

- Existential Quantifier "$\exists$" There exists
- Universal Quantifier $\forall$ "for all"

All dogs are faithful $\Rightarrow$ $\forall x\, (dog(x) \rightarrow faithful(x))$

All birds cannot fly $\Rightarrow$ $\exists x\, (bird(x) \wedge \neg Fly(x))$

↓

Not (all birds can fly) $\Rightarrow$ $\neg\left(\forall x(Bird(x) \rightarrow Fly(x))\right)$

$\neg\, (\forall x\, (\neg Bird(x) \vee Fly(x))$

Mammals drink milk $\Rightarrow$ $\forall x\, (Mammal(x) \rightarrow Drink(x, milk))$

Atleast one planet has life on it. $\Rightarrow$ $\exists x\, (Planet(x) \wedge haslife(x))$

Some dogs bark $\Rightarrow$ $\exists x\, (Dog(x) \wedge Bark(x))$

Duality of Quantifiers

$\neg(\exists x) = \forall x$

$\neg(\forall x) = \exists x$

All man are mortal $\quad \forall x(man(x) \rightarrow mortal(x))$

$\Updownarrow$

No man is immortal $\quad \neg \exists x(\,man(x) \wedge \neg\, mortal(x))$

Sentence

- A predicate is a sentence.
- If P, Q are sentence and x is a variable then

$(P), \quad \neg P, \quad \exists x P, \forall x P, \; P \wedge Q, \; P \vee Q, \; P \rightarrow Q$, are sentences.

- Nothing else is sentence

Everyday, someone celebrates his/her birthday.

⇓

$\forall y \exists x\; Birthday(x, y)$ ✓

$Birthday(x, y)$

↓

x celebrates his/her birthday on y date.

Everyone Loves his/her mother.

$\forall x$ Loves (x, Mother(x)) ✓

Loves (x, y) : x loves y

Mother(x) : x's mother

Every number is successor of its predecessor.

$\forall x \forall y$ Succ.(x,y) $\rightarrow$ Pred(y, x)

Succ.(x,y) : x is succ of y
Pred.(x,y) : x is pre.of y

$\forall x$ equal(x, succ(pred(x)))

Succ(x) : Succ. of x

Pred.(x) : Pred. of x

equal(x, y) is true iff x = y

Inference Rules

1. Universal Elimination.

$\forall x$ Likes (x, flower)

$\downarrow$

Likes (y, flower)

Likes (Ram, flower)

2. Existential Elimination.

$\exists x$ Likes (x, flower)

$\downarrow$

Likes (Person, flower)

Any person/particular constant which satisfies

$\downarrow$

Skolemization

Existential Intro.

Likes (Ram, flower)

$\downarrow$

$\exists x$ Likes (x, flower) ✓

Example: ① If a perfect square is divisible by a prime p then it is also divisible by square of p.

② Every perfect square is divisible by some prime.

③ 36 is perfect square.

④ Does there exist a prime q such that square of q divides 36 ?

1: $\forall x \forall y$((PerfectSq (x) $\wedge$ Prime (y) $\wedge$ Divides (x,y)) $\rightarrow$ Divides (x, Square(y)))

Prime (x) : x is prime.

Divides(x,y) : x is divisible by y

2. $\forall x \exists y$ (PerfectSq.(x) $\wedge$ Prime(y) $\wedge$ Divides (x,y))

PerfectSq(x) :

3. PerfectSq.(36)

4. $\exists q$(Prime (q) $\wedge$ Divides (36, Square(q))) ← To prove

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② $\forall x \exists y$ (PerfectSq(x) $\wedge$ Prime(y) $\wedge$ Divides (x,y))

$\downarrow$ Remove $\forall$, Replace x/36

⑤ $\exists y$ (PerfectSq(36) $\wedge$ Prime (y) $\wedge$ Divides (36, y))

$\downarrow$ Remove $\exists$, Replace y by a constant p

⑥ PerfectSq.(36) $\wedge$ Prime (p) $\wedge$ Divides (36, p)

① $\forall x \forall y$ (PerfectSq (x) $\wedge$ Prime (y) $\wedge$ Divides(x,y)) $\rightarrow$ Divides (x, Square(y))

⑥ (Perfect Sq(36) $\wedge$ Prime(p) $\wedge$ Divides (36,p))

① $\forall x \forall y\,(\text{PerfectSq}(x) \land \text{Prime}(y) \land \text{Divides}(x,y)) \rightarrow \text{Divides}(x, \text{Square}(y))$

⑥ $\exists y\,(\text{Perfect Sq}(36) \land \text{Prime}(p) \land \text{Divides}(36, p))$

$\Downarrow$

⑦ $\text{Divides}(36, \text{square}(p))$

From ⑥ and ⑦,

$\text{Prime}(p) \land \text{Divides}(36, \text{Square}(p))$

$\downarrow$ Introduce $\exists$

$\exists y(\text{Prime}(y) \land \text{Divides}(36, \text{Square}(y)))$.

Horn Sentence

↳ Implication with conjunction of atomic sentence on left and a single atomic sentence on right

e.g. $\forall x \forall y\ \text{PerfectSq}(x) \land \text{Prime}(y) \land \text{Divides}(x,y) \rightarrow \text{Divides}(x, \text{Square}(y))$

- Horn sentence does not contain "$\exists$".

Substitution

Replace variable with constant

$\text{Perfect Sq.}(x)$

$\downarrow x/49$

$\text{Perfect Sq.}(49)$

Unification

Process of finding a substitution that makes two atomic sentences identical.

$\text{Unify}(\text{Prime}(7), \text{Prime}(x)) = \{x/7\}$

$\text{Unify}(\text{Divides}(49, x), \text{Divides}(y, 7)) = \{x/7, y/49\}$

$\text{Unify}(\text{Divides}(49, x), \text{Divides}(x, 7)) = \times$

$\text{Unify}(\text{prime}(7), \text{prime}(17)) = \times$

RESOLUTION

$(x \lor S_1) \quad (\neg x \lor S_2)$

$S1 \lor S2$

$S1 \lor S2$: Resolvant

x: Resolved upon.

Example :- 1. All people who are graduating are happy.

$\forall x(\text{grad}(x) \rightarrow \text{happy}(x))$

2. All happy people smile.

$\forall x(\text{happly}(x) \rightarrow \text{smile}(x))$

3. Someone is graduating

$\exists x\, \text{grad}(x)$

4. Is someone smiling? (Conclusion)

$\exists x\, \text{Smile}(x)$

Proof by refutation

Step 1 :- Select predicates.

Step 2 :- Writing in predicate logic

(Negate the conclusion) 4. $\neg \exists x\, \text{Smile}(x)$

Step 3:- Convert into canonical form.

(i) Remove →

1) $\forall x\ (\neg grad(x) \vee happy(x))$

2) $\forall x\ (\neg happy(x) \vee smile(x))$

3) $\exists x\ (grad(x))$

4) $\neg \exists x\ smile(x)$

(ii) Reduce scope of negation.

4) $\forall x\ \neg smile(x)$

(iii) Standardize the variables apart

1) $\forall x\ (\neg grad(x) \vee happy(x))$

2) $\forall y\ (\neg happy(y) \vee smile(y))$

3) $\exists z\ (grad(z))$

4) $\forall w\ \neg smile(w)$

(iv) Move all quantifiers to left

✓

v) Eliminate $\exists$ (Skolemization)

1) $\forall x\ (\neg grad(x) \vee happy(x))$

2) $\forall y\ (\neg happy(y) \vee smile(y))$

3) $(grad(\underline{name1}))$

4) $\forall w\ \neg smile(w)$

vi) Drop all $\forall$

1) $(\neg grad(x) \vee happy(x))$

2) $(\neg happy(y) \vee smile(y))$

3) $grad(name1)$

4) $\neg smile(w)$

vii) Convert to CNF

viii) Make all conjuncts seperate clause

ix) Standardize variable

$\overbrace{\neg smile(w)}^{x} \vee$ $\quad \neg happy(y) \vee \overbrace{smile(y)}^{x}$

$\{y/w\}$

$\neg happy(w)$ $\quad \neg grad(x) \vee happy(x)$

$\{x/w\}$

$\neg grad(w)$ $\quad grad(name1)$

$\{w/name1\}$

Null clause

$S1 \vee x \quad \neg x \vee S2$

$S1 \vee S2$

$x \quad \neg x$

ϕ

Conclusion:- Someone is smiling. $smile(w)$ ✓

Classwork Assignment 2 (Deadline Monday 3:59 PM)

Example :- (1) If a perfect - square is divisible by a prime p then it is also divisible by square of p.

(2) Every perfect square is divisible by some prime.

(3) 36 is perfect square.

(4) Does there exist a prime q such that square of q divides 36 ?

By Resolution