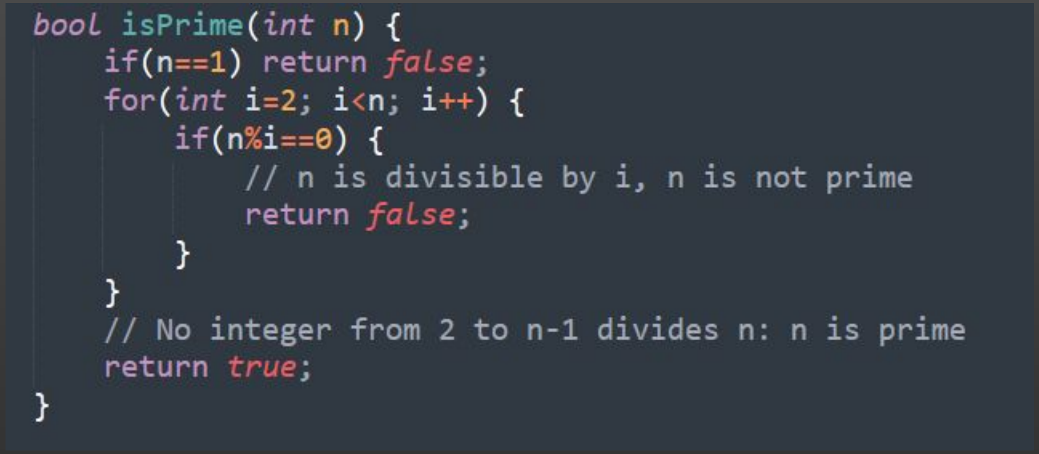
Number Theory 1

Contents:

Primality Test:
Check If a number n is prime or not.

O(n) approach – Brute force:

Check for every integer from 1 to n-1, if it divides n.

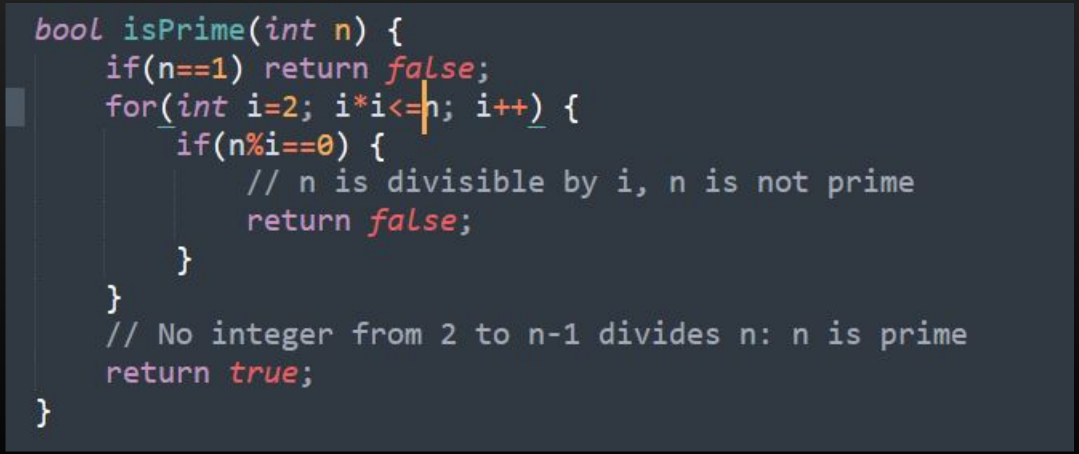


```
bool isPrime(int n) {
    if(n==1) return false;
    for(int i=2; i<n; i++) {
        if(n%i==0) {
            // n is divisible by i, n is not prime
            return false;
        }
    }
    // No integer from 2 to n-1 divides n: n is prime
    return true;
}
```

- Observation: Factors always occur in pairs.
- If (a, b) is a factor pair, a*b=n and a <= sqrt(n) <= b.
- For every composite number there will always be a factor from 2 to sqrt(n)
- It’s sufficient to check for all integers from 2 to sqrt(n) if it divides n

O(sqrt n) - Optimized Approach:

Check for every integer from 1 to sqrt(n), if it divides n.



```
bool isPrime(int n) {
    if(n==1) return false;
    for(int i=2; i*i<=n; i++) {
        if(n%i==0) {
            // n is divisible by i, n is not prime
            return false;
        }
    }
    // No integer from 2 to n-1 divides n: n is prime
    return true;
}
```

Find all Prime number from 1 to n (1 <= n <= 1e6)

Traditional Approach will make n*sqrt(n) operation ~1e9 operations in worst case. Can we do better?

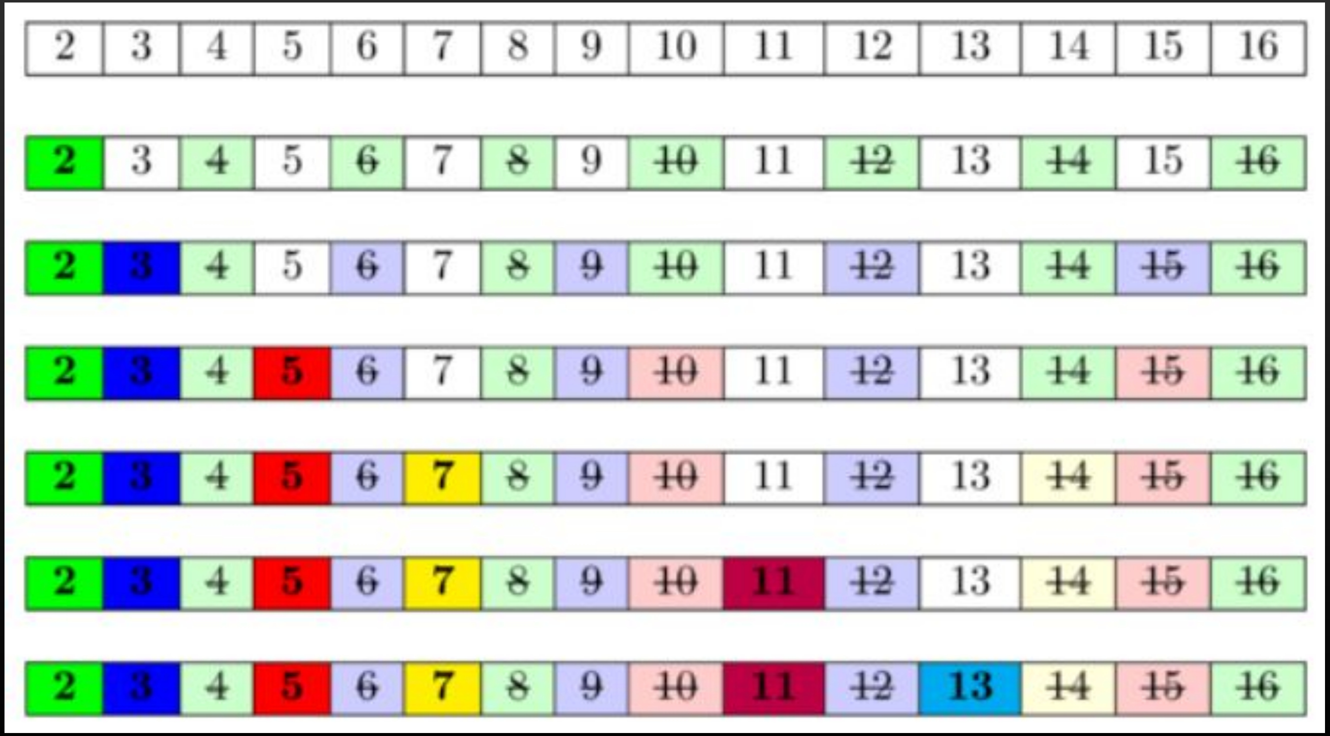
Sieve Of Eratosthenes:

(Sieve of Eratosthenes is an algorithm for finding all the prime numbers in a segment [1;n] using O(n*loglogn) operations)

Idea: Don’t check for all numbers, multiples of prime numbers are always composite.

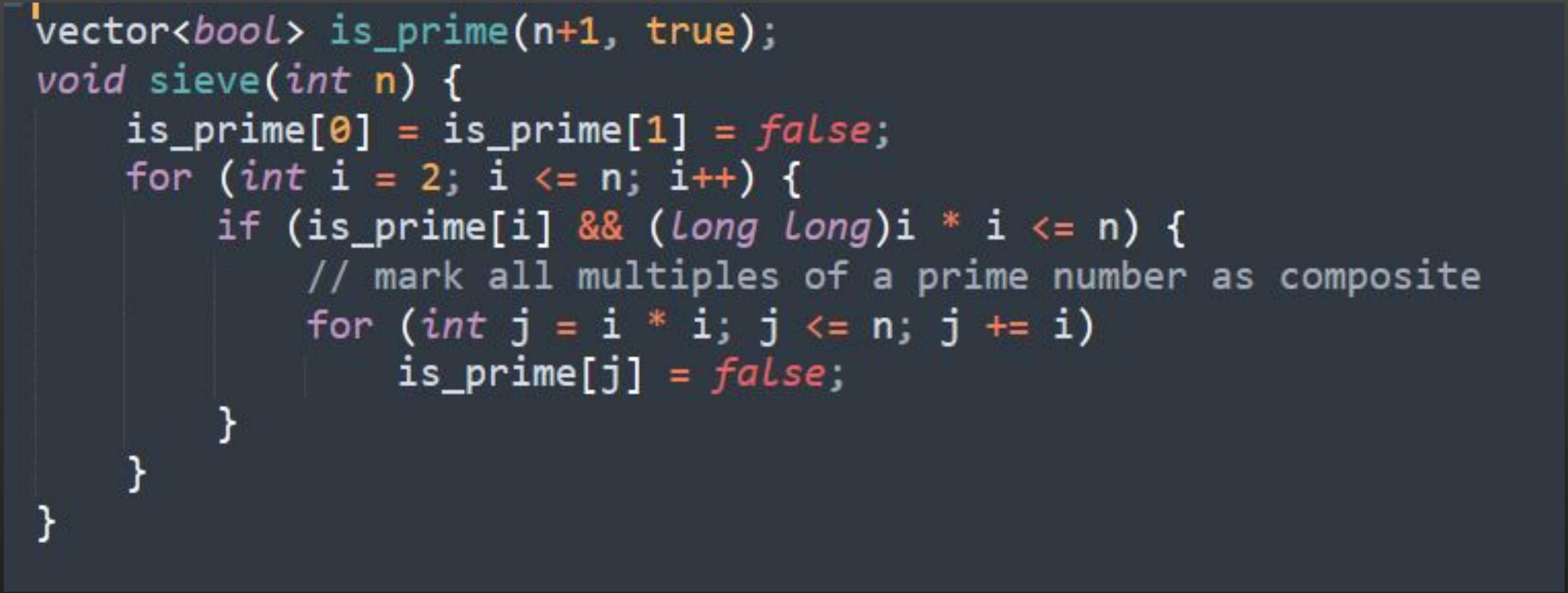
- Mark multiples of prime numbers as composite
- Complexity – n*log(log(n))

N = 16 Dry Run



2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~	11	~~12~~	13	~~14~~	15	~~16~~
2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~
2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~
2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~
2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~
2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~	~~16~~

Implementation:



```
vector<bool> is_prime(n+1, true);
void sieve(int n) {
    is_prime[0] = is_prime[1] = false;
    for (int i = 2; i <= n; i++) {
        if (is_prime[i] && (long long)i * i <= n) {
            // mark all multiples of a prime number as composite
            for (int j = i * i; j <= n; j += i)
                is_prime[j] = false;
        }
    }
}
```

Addiitonal Algorithms:

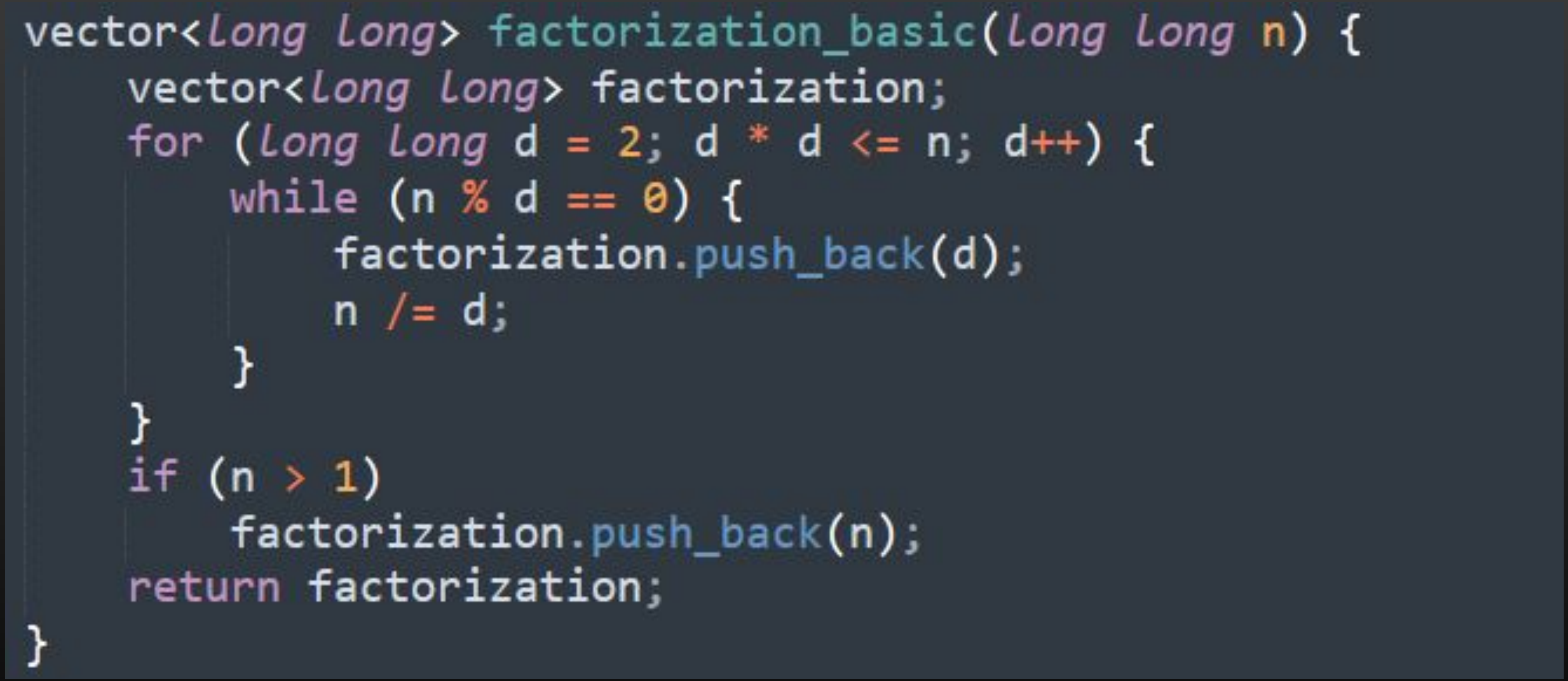
- Segmented Sieve – for finding primes between large numbers.
- Linear Sieve – finding primes less than 1e7 in O(n), also used for computing smallest prime factor of a number

Prime Factorization of a number

Trial Division Method O(sqrt(n)):

- If n is composite, it will have factors in pair, and one factor from each pair will be less than sqrt(n) and the other can be found using the first factor.
- If n is prime, it will have just 1 prime factor that is n.

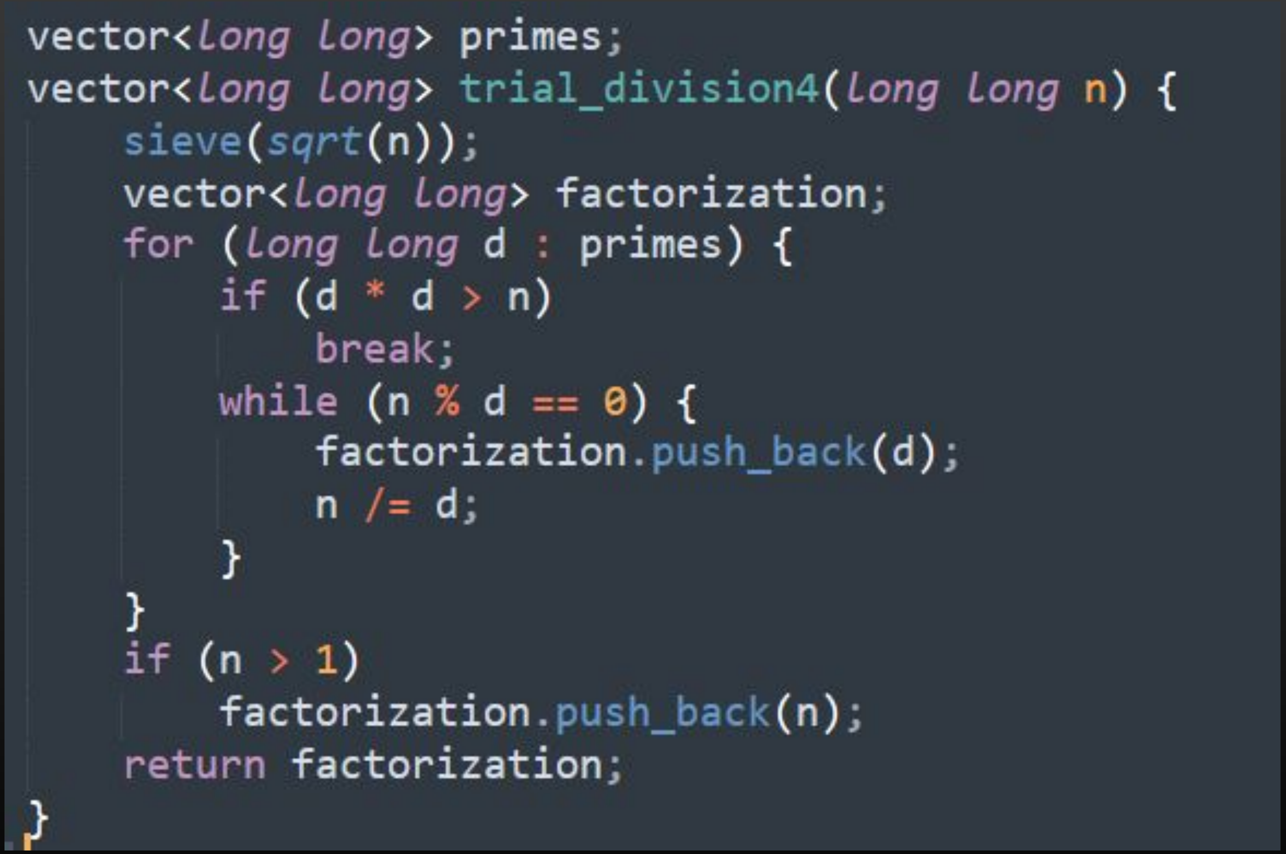
Implementation:



```cpp
vector<long long> factorization_basic(long long n) {
    vector<long long> factorization;
    for (long long d = 2; d * d <= n; d++) {
        while (n % d == 0) {
            factorization.push_back(d);
            n /= d;
        }
    }
    if (n > 1)
        factorization.push_back(n);
    return factorization;
}
```

Using Precomputed primes with SOE.

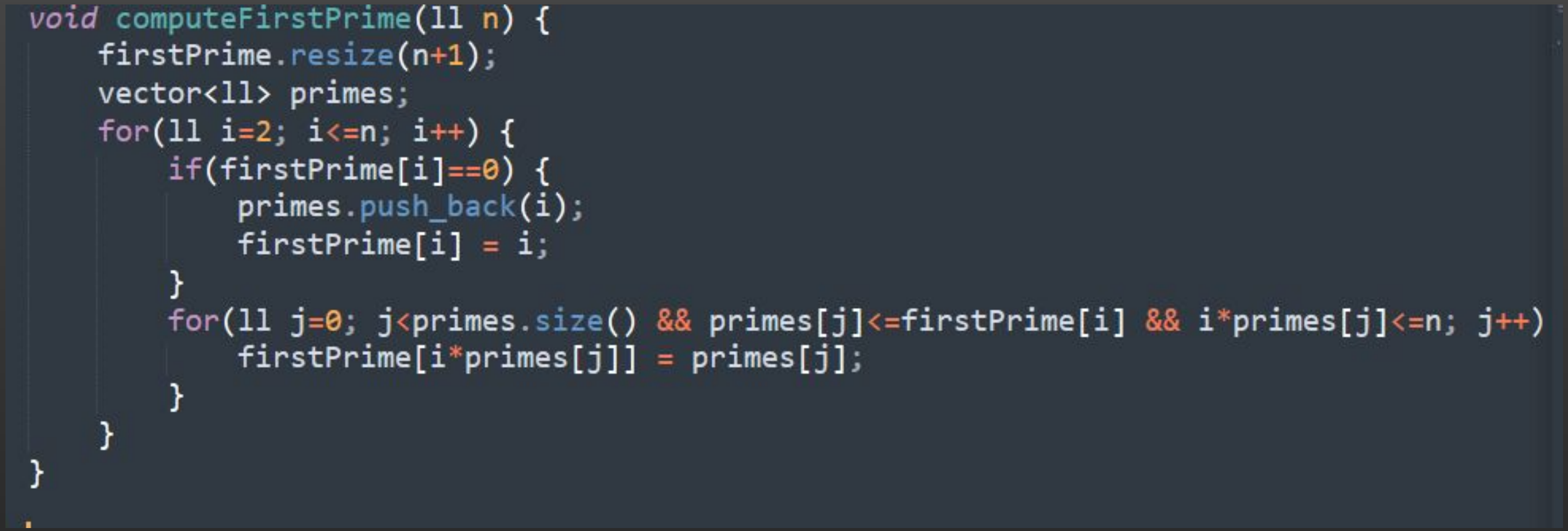
- For finding prime factors, we need not check with composite numbers
- Using sieve find primes less than sqrt(n) and check with only those.



```cpp
vector<long long> primes;
vector<long long> trial_division4(long long n) {
    sieve(sqrt(n));
    vector<long long> factorization;
    for (long long d : primes) {
        if (d * d > n)
            break;
        while (n % d == 0) {
            factorization.push_back(d);
            n /= d;
        }
    }
    if (n > 1)
        factorization.push_back(n);
    return factorization;
}
```

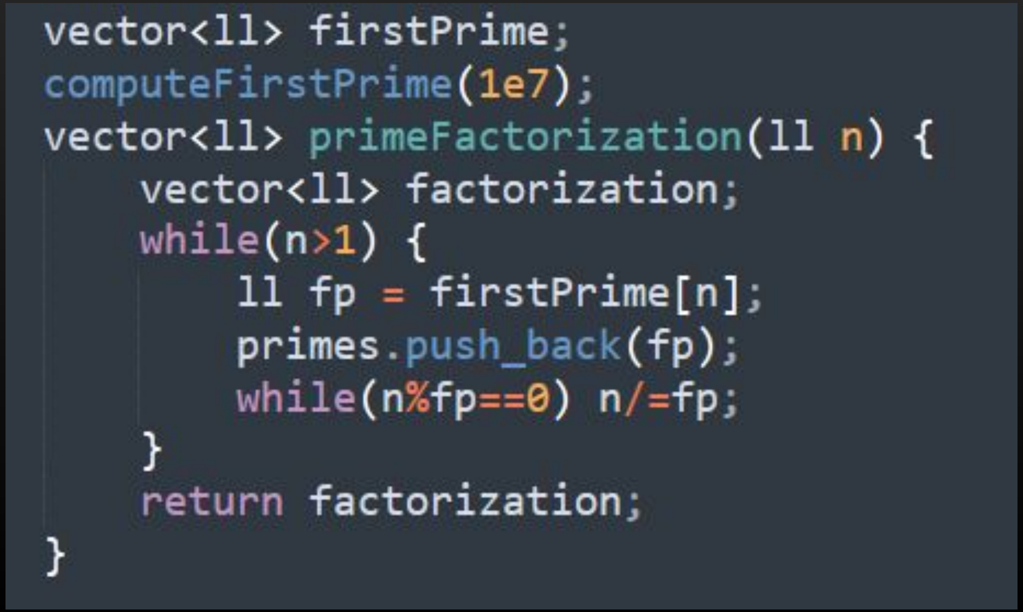
Using Sieve in O(logn):

The first prime function using linear sieve –



```
void computeFirstPrime(ll n) {
    firstPrime.resize(n+1);
    vector<ll> primes;
    for(ll i=2; i<=n; i++) {
        if(firstPrime[i]==0) {
            primes.push_back(i);
            firstPrime[i] = i;
        }
        for(ll j=0; j<primes.size() && primes[j]<=firstPrime[i] && i*primes[j]<=n; j++)
            firstPrime[i*primes[j]] = primes[j];
        }
    }
}
```

Prime factorization function:



```
vector<ll> firstPrime;
computeFirstPrime(1e7);
vector<ll> primeFactorization(ll n) {
    vector<ll> factorization;
    while(n>1) {
        ll fp = firstPrime[n];
        primes.push_back(fp);
        while(n%fp==0) n/=fp;
    }
    return factorization;
}
```

Modular Arithmetic:

What does the expression a ≡ b (mod m) signify?

Modular Congruences:
Number a and b which leaves the same reminder when divided by some integer m.
Example –

19 = 40 mod 5
23 = 3 mod 4

Important Properties of modular arithmetic: (mod is distributive over addition, subtraction and multiplication)

- (a mod m)+(b mod m) mod m=a+b mod m

- (a mod m)−(b mod m) mod m=a+m−b mod m

- (a mod m)*(b mod m) mod m=a*b mod m

Remember: (a mod m)/(b mod m) mod m != a/b mod m
Mod is <u>not</u> distributive over division.

For division we use something called a modular multiplicative inverse.
(a mod m)/(b mod m) mod m = (a mod m)*mod_inv(b) mod m = a*mod_inv(b) mod m

Modular multiplicative inverse (mod_inv) :

There are 2 faster ways of calculating mod_inv of a number.

- Extended GCD algorithm.
- Fermat’s little theorm.

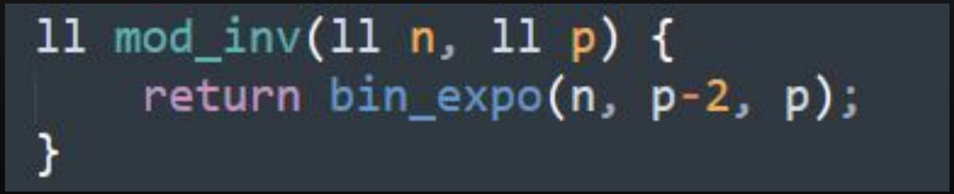
Though the extended GCD algorithm is more versatile and sometimes slightly faster, the Fermat's little theorem method is more popular and simpler, but it works only when m is prime.

Fermat’s little theorm: Let mod_inv(a)=b We want to find a number b such that a*b mod m = 1
We know,

a^m mod m = a mod m
a^(m-1) mod m = 1
a*a^(m-2) mod m = 1

Hence b = a^(m-2) mod m.

Implementation:



```
ll mod_inv(ll n, ll p) {
    return bin_expo(n, p-2, p);
}
```

Here exponentiation is done using a faster algorithm known as binary exponentiation, which will be discussed later.

Modular Exponentiation

Modular Binary Exponentiation in O(logn):

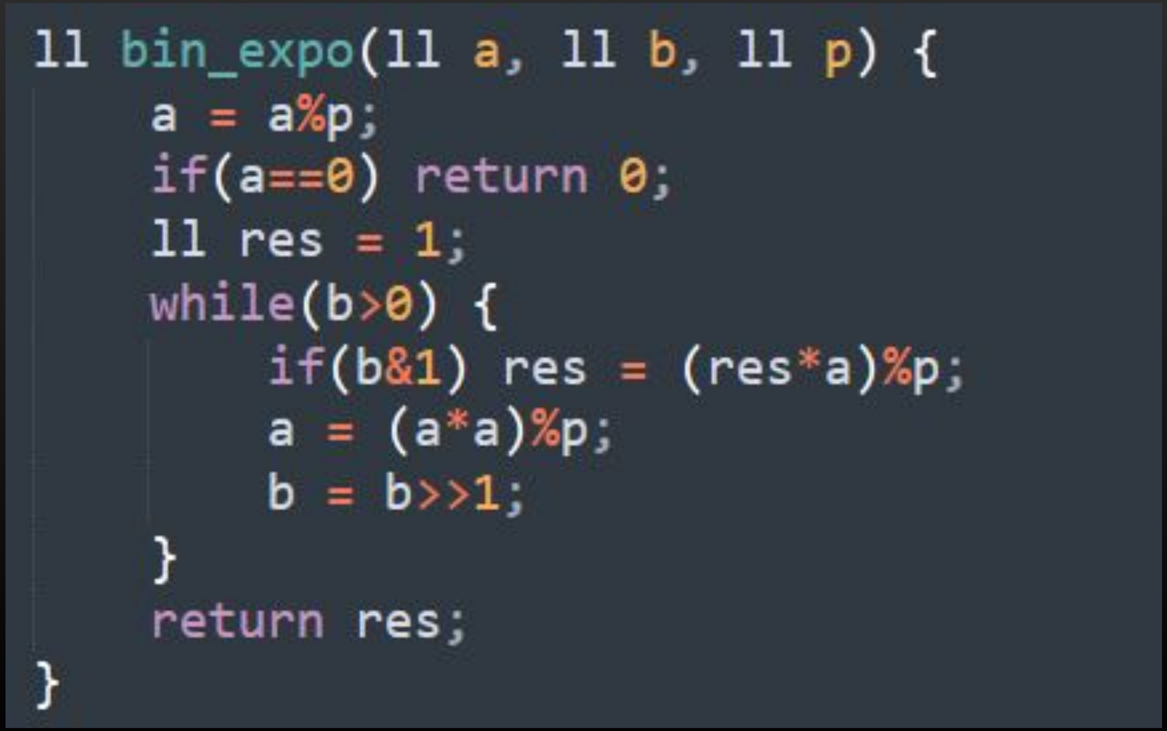
Instead of multiplying linearly, multiply by squaring.

Example: find 3^13

13 can be written as 1101 or 8+4+1.

3^13 can be written as 3^(8+4+1) = 3^8 * 3^4 * 3^1.

Implementation:



```
ll bin_expo(ll a, ll b, ll p) {
    a = a%p;
    if(a==0) return 0;
    ll res = 1;
    while(b>0) {
        if(b&1) res = (res*a)%p;
        a = (a*a)%p;
        b = b>>1;
    }
    return res;
}
```

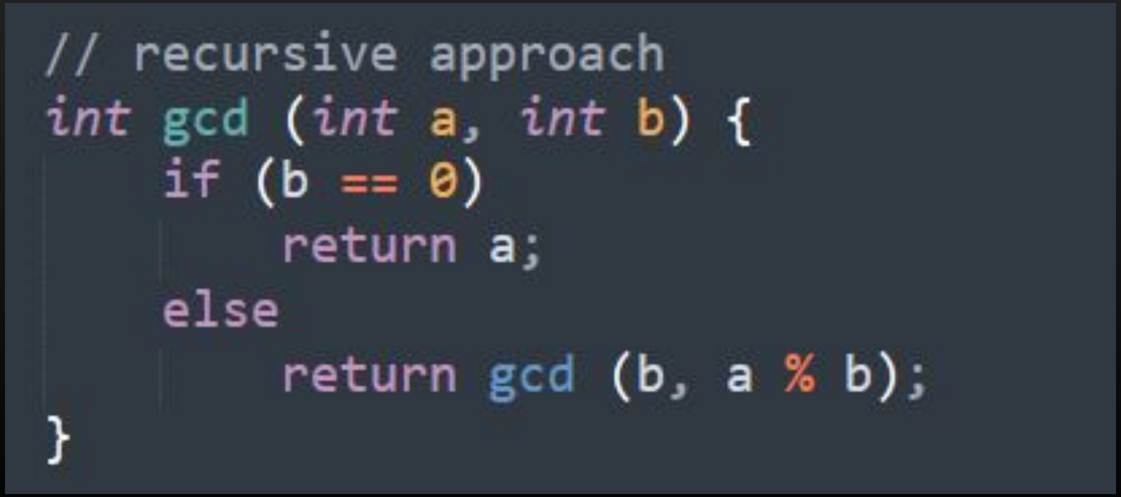
Euclidean GCD

Originally, the Euclidean algorithm was formulated as follows: subtract the smaller number from the larger one until one of the numbers is zero.

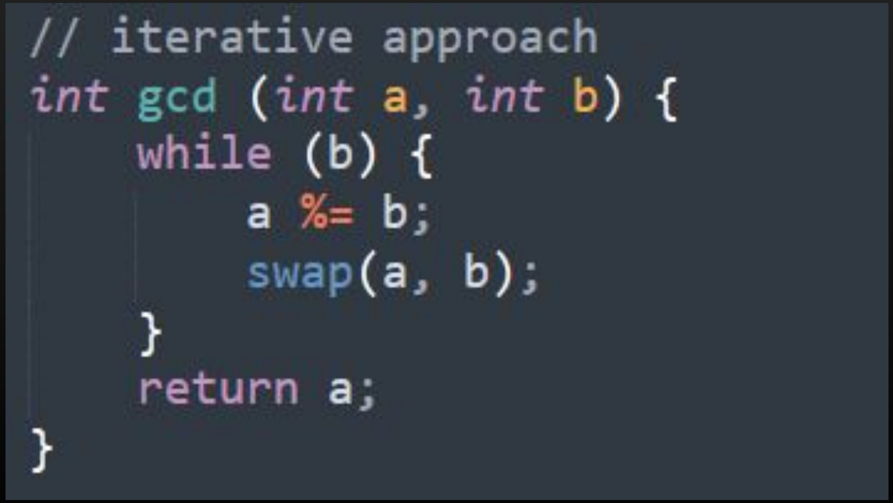
Instead of subtracting multiple times to get to the remainder, we can use mod. The relation then becomes:

$$\gcd(a,b) = \begin{cases} a, & \text{if } b = 0 \\ \gcd(b, a \bmod b), & \text{otherwise.} \end{cases}$$

Implementation:



```
// recursive approach
int gcd (int a, int b) {
    if (b == 0)
        return a;
    else
        return gcd (b, a % b);
}
```



```
// iterative approach
int gcd (int a, int b) {
    while (b) {
        a %= b;
        swap(a, b);
    }
    return a;
}
```

Binomial Coefficient nCr.

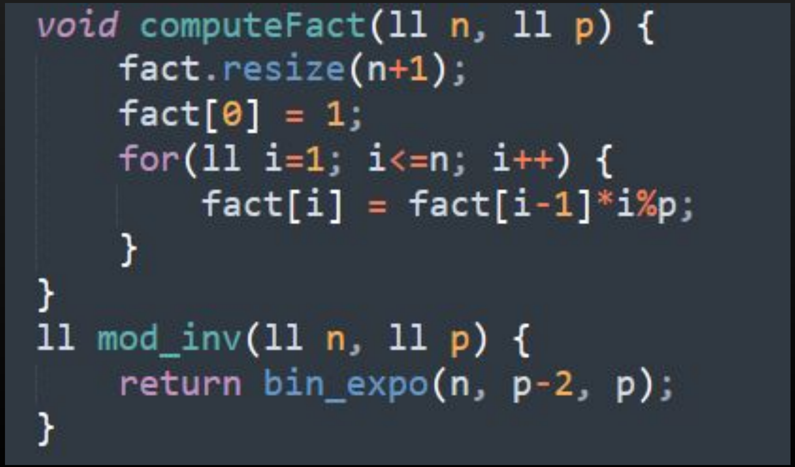
Find nCr(n, r) in log n:

Dp approach for finding nrc will be discussed later.

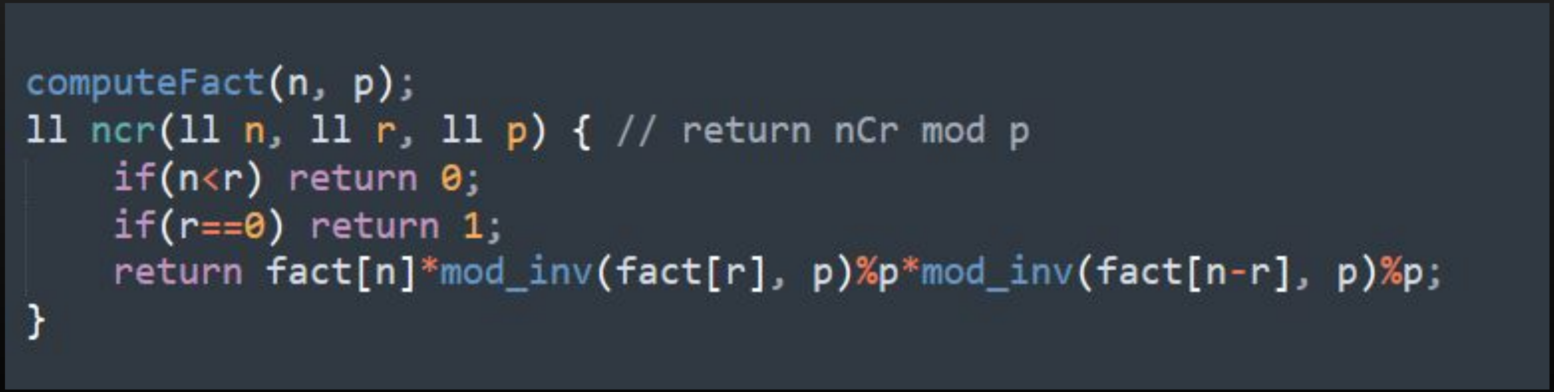
Using Fermat's little theorm:

- We know that nCr = n! / (r! * (n-r)!)
- We can precompute n! for all integers from [1, n].
- nCr can run out of bound very quickly so most of the time the answer is returned modulo some prime number.
- nCr = n! / (r! * (n-r)!) mod m
- We can use fermat little theorm for finding mod_inv.
- nCr = n! mod m * mod_inv(r!) mod m * mod_inv((n-r)!) mod m.

Implementation:



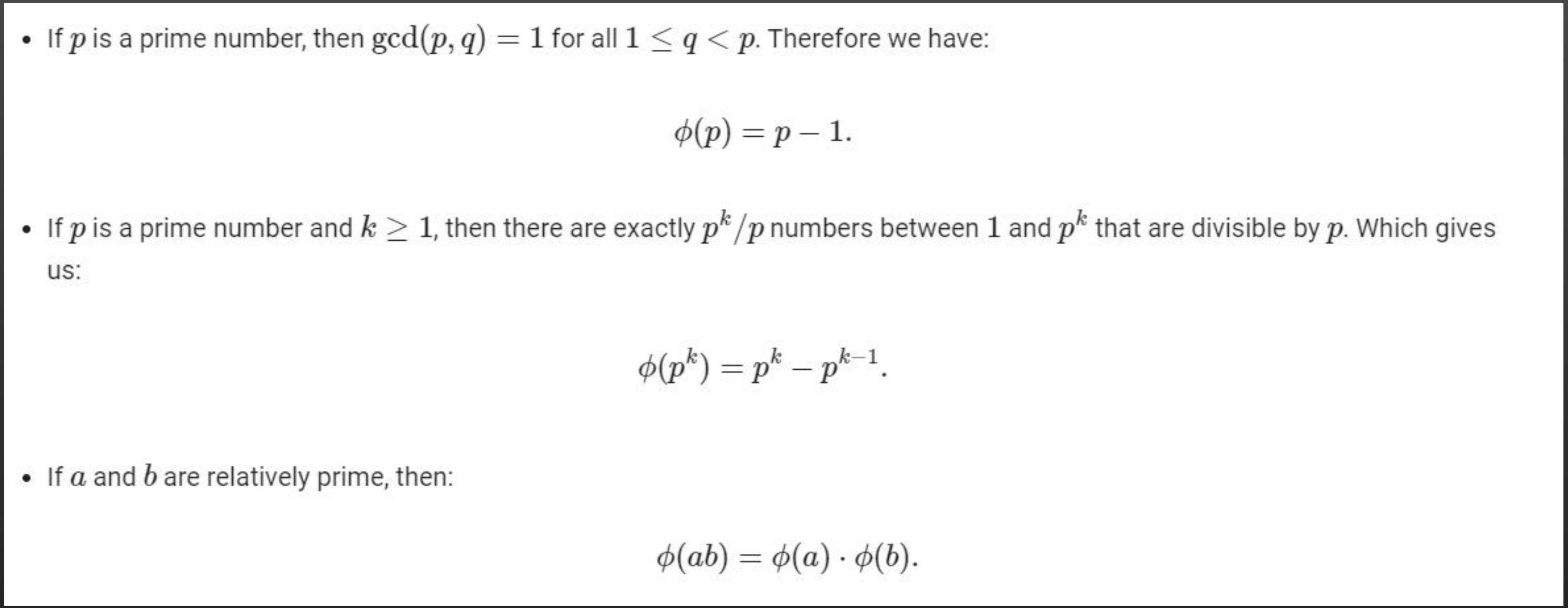
```
void computeFact(ll n, ll p) {
    fact.resize(n+1);
    fact[0] = 1;
    for(ll i=1; i<=n; i++) {
        fact[i] = fact[i-1]*i%p;
    }
}
ll mod_inv(ll n, ll p) {
    return bin_expo(n, p-2, p);
}
```



```
computeFact(n, p);
ll ncr(ll n, ll r, ll p) { // return nCr mod p
    if(n<r) return 0;
    if(r==0) return 1;
    return fact[n]*mod_inv(fact[r], p)%p*mod_inv(fact[n-r], p)%p;
}
```

Count number of integers from [1, n] which are coprime to n.

Euler Totient Function: phi(n) – returns count of integers in [1, n] which are coprime to n.



- If p is a prime number, then $\gcd(p, q) = 1$ for all $1 \le q < p$. Therefore we have:

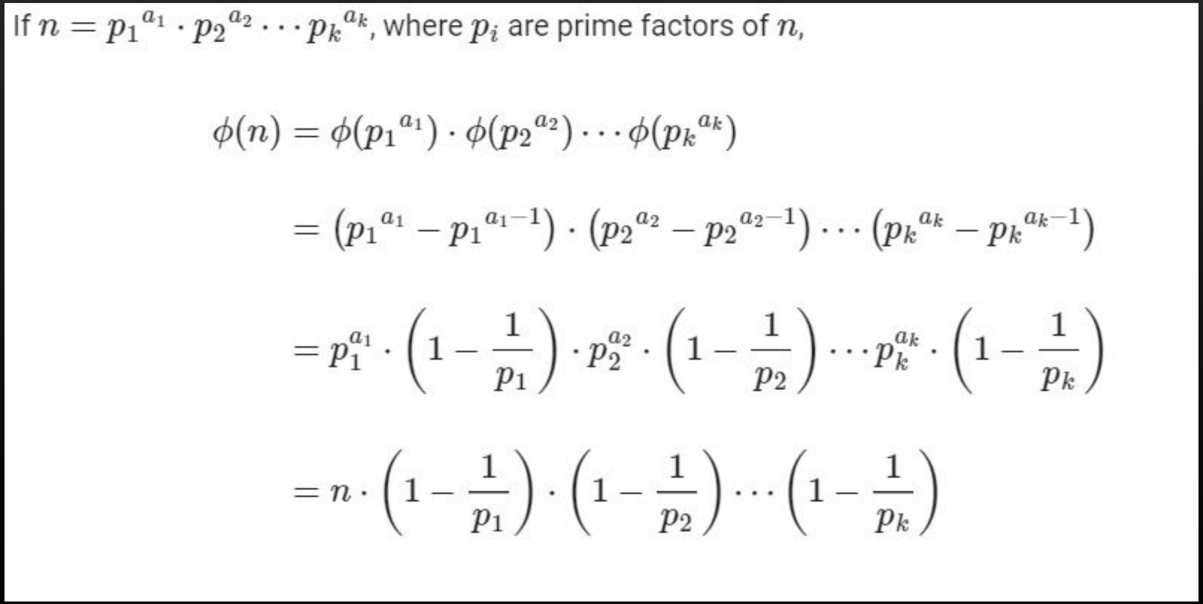
$$\phi(p) = p - 1.$$

- If p is a prime number and $k \ge 1$, then there are exactly p^k / p numbers between 1 and p^k that are divisible by p. Which gives us:

$$\phi(p^k) = p^k - p^{k-1}.$$

- If a and b are relatively prime, then:

$$\phi(ab) = \phi(a) \cdot \phi(b).$$



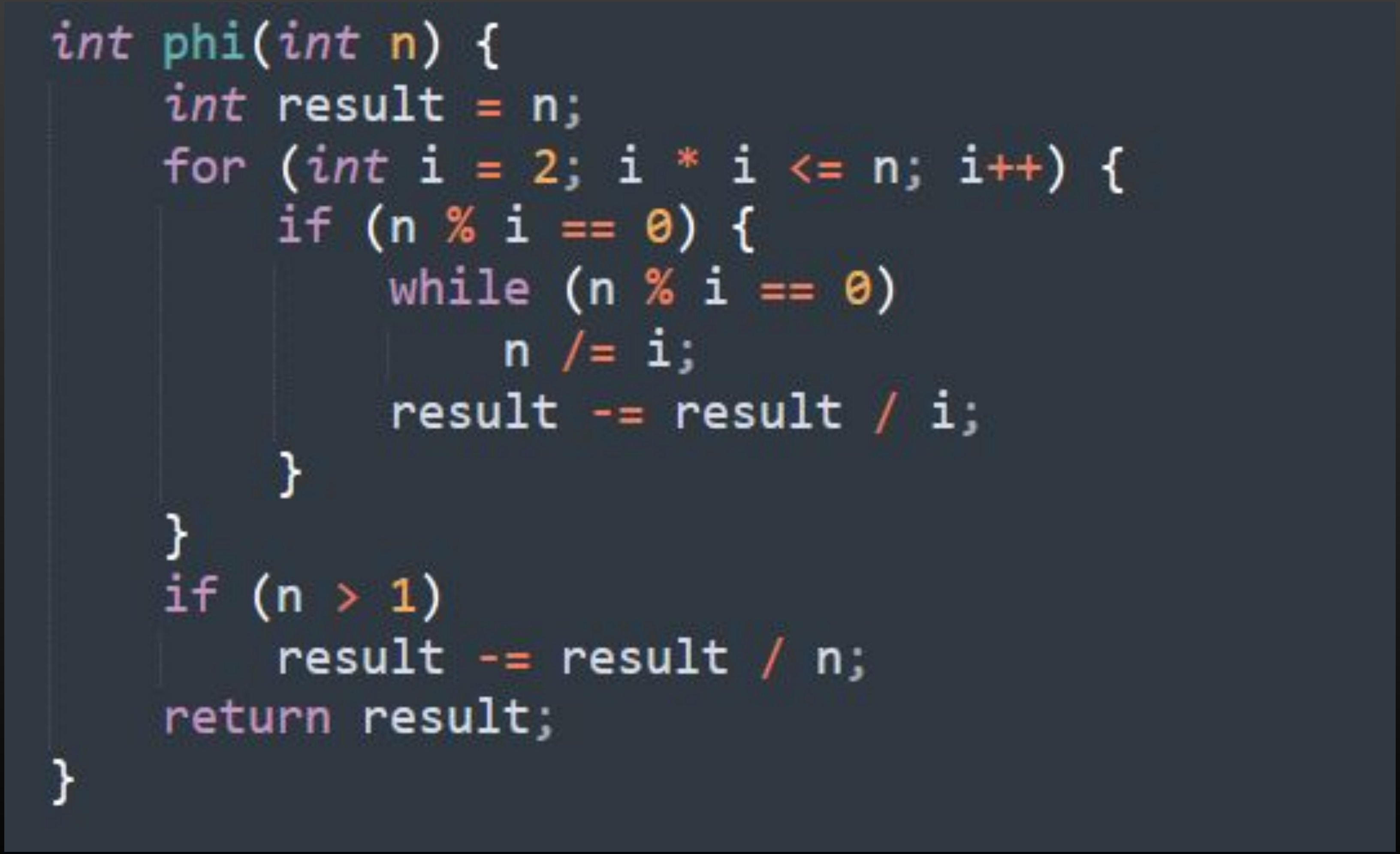
If $n = {p_1}^{a_1} \cdot {p_2}^{a_2} \cdots {p_k}^{a_k}$, where p_i are prime factors of n,

$$\begin{aligned}\phi(n) &= \phi({p_1}^{a_1}) \cdot \phi({p_2}^{a_2}) \cdots \phi({p_k}^{a_k}) \\ &= \left({p_1}^{a_1} - {p_1}^{a_1 - 1}\right) \cdot \left({p_2}^{a_2} - {p_2}^{a_2 - 1}\right) \cdots \left({p_k}^{a_k} - {p_k}^{a_k - 1}\right) \\ &= p_1^{a_1} \cdot \left(1 - \frac{1}{p_1}\right) \cdot p_2^{a_2} \cdot \left(1 - \frac{1}{p_2}\right) \cdots p_k^{a_k} \cdot \left(1 - \frac{1}{p_k}\right) \\ &= n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)\end{aligned}$$

Source: Cp Algorithms

Phi(n) $= n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$

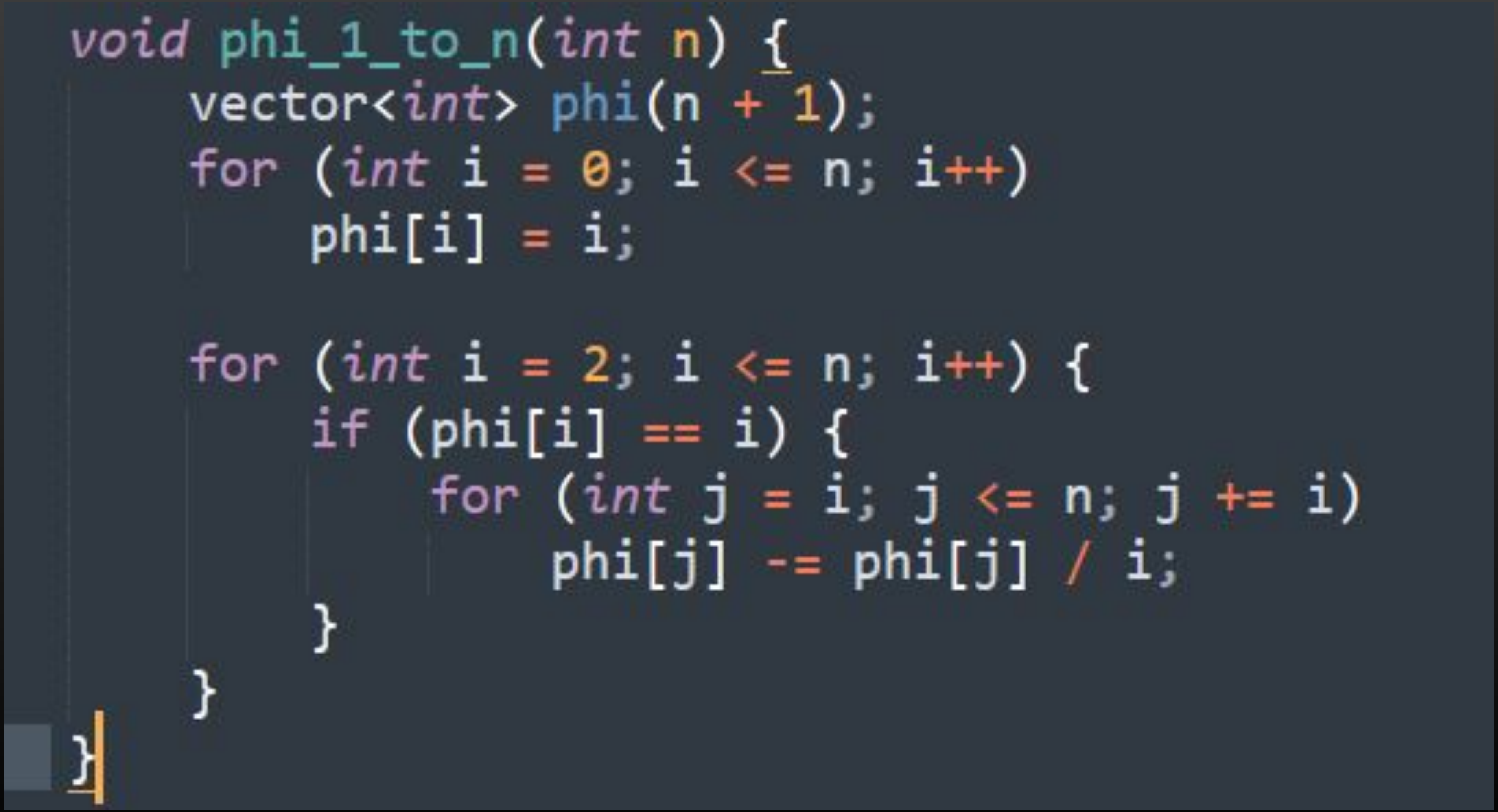
Implementation Using factorization in O(sqrt(n)):



```
int phi(int n) {
    int result = n;
    for (int i = 2; i * i <= n; i++) {
        if (n % i == 0) {
            while (n % i == 0)
                n /= i;
            result -= result / i;
        }
    }
    if (n > 1)
        result -= result / n;
    return result;
}
```

Phi(n) $$= n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

Calculating phi(n) for all numbers from 1 to n using sieve in O(n*loglogn)



```cpp
void phi_1_to_n(int n) {
    vector<int> phi(n + 1);
    for (int i = 0; i <= n; i++)
        phi[i] = i;

    for (int i = 2; i <= n; i++) {
        if (phi[i] == i) {
            for (int j = i; j <= n; j += i)
                phi[j] -= phi[j] / i;
        }
    }
}
```