Lecture 12

Monday, 13 September 2021   3:02 PM

$x(n) = \alpha^n u(n)$, $|\alpha| < 1$

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$= \frac{1}{1-\alpha e^{-j\omega}} \times \frac{1-\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}}$$

$$= \frac{1-\alpha\cos\omega + j\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}$$

even $\leftarrow$ $$X_{Re}(e^{j\omega}) = \frac{1-\alpha\cos\omega}{1-2\alpha\cos\omega+\alpha^2}$$

odd $\leftarrow$ $$X_{im}(e^{j\omega}) = \frac{\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1-2\alpha\cos\omega+\alpha^2}}$$

even

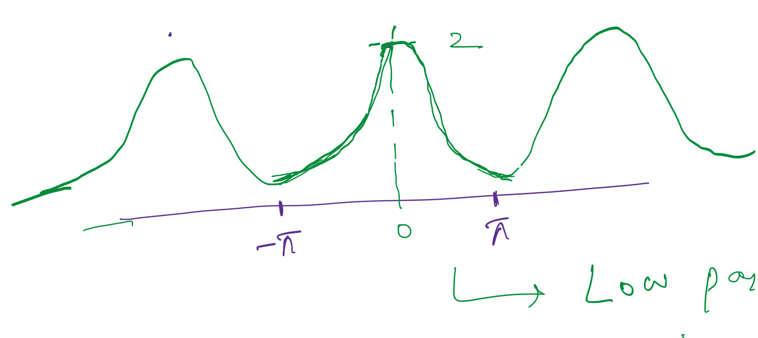
$$\theta(\omega) = -\tan^{-1}\left(\frac{\alpha\sin\omega}{1-\alpha\cos\omega}\right)$$

odd.

For $\alpha = 1/2$

$\max(|X(e^{j\omega})|) = 2$ at $\omega = 0$

$\min(|X(e^{j\omega})|) = \frac{2}{3}$ at $\omega = \pm\pi$



$\hookrightarrow$ Low pass signal.

Convergence of $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$\searrow$ This series may or mayn't converge.

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x(n)\, e^{-j\omega n}$$

Uniform convergence.

$$\lim_{K\to\infty} \underbrace{\left| X_K(e^{j\omega}) - X(e^{j\omega}) \right|}_{\text{error} = \varepsilon(\omega)} = 0 \quad \forall\omega.$$

$\Rightarrow$ The sequence is absolutely summable.

$\longrightarrow$ sufficient condition but not necessary.

If $x(n)$ is absolutely summable.

$\rightarrow$ Its DTFT exists.

$$|X(e^{j\omega})| = \left|\sum x(n)\, e^{-j\omega n}\right|$$

$$\le \sum_{n=-\infty}^{\infty} |x(n)| < \infty.$$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 \le \left(\sum |x(n)|\right)^2 < \infty.$$

$\rightarrow$ If $x(n)$ is absolutely summable then it will be definitly square summable.

$\rightarrow$ But the converse is not necessary true.

Sequences which are square summable but not absolutly summable :—

$$\lim_{K\to\infty} \underbrace{\int_{-\pi}^{\pi} \left|X(e^{j\omega}) - X_K(e^{j\omega})\right|^2 d\omega}_{\text{Total energy of the error.}} = 0.$$

$\downarrow$

Mean - square convergence of $X(e^{j\omega})$

$\rightarrow$ This convergence criteria is necessary for sq. summable sequences.

Ex

$$h(n) = \begin{cases} \dfrac{\sin\omega_c n}{\pi n}, & n \ne 0 \\ \dfrac{\omega_c}{\pi}, & n = 0. \end{cases}$$

$\searrow$ Sq. summable seq^n.

$h(n) \longleftrightarrow H(e^{j\omega})$

$$H(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le \omega_c \\ 0 & \omega_c < \omega \le \pi. \end{cases}$$

$\downarrow$

Low pass filter.

$\rightarrow$ The cond^{n}s absolutly summable, square summable are sufficient conditions but not necessary.

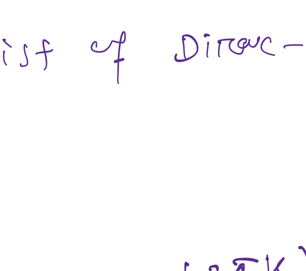
Neither absolutly summable nor square summable :—

$\rightarrow$ For such sequences F.T. is possible by using Dirac - delta function.

$\delta(\omega) \rightarrow$ Dirac - delta (Analog delta fun^n)

$$\int_{-\infty}^{\infty} \delta(\omega)\, d\omega = 1$$

$\delta(\omega) = \lim_{\Delta\to 0}$



$\delta(\omega) = 0, \quad \omega \ne 0.$

Example of such sequences :

$u(n)$, α^n, $e^{jn\omega_0}$, $\cos n\omega_0$, $\sin n\omega_0$ - -

$\rightarrow$ F.T. of such signals consist of Dirac - delta function $\delta(\omega)$.

Ex

$$\underset{\substack{\downarrow \\ x(n)}}{e^{jn\omega_0}} \longleftrightarrow \underbrace{\sum_{K=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0+2\pi K)}_{X(e^{j\omega})}$$

Proof

Since F.T. is one - to - one, if we can proove that IDFT$\{X(e^{j\omega})\} = x(n)$, it will be necessary & sufficient.

$\sum 2\pi\,\delta(\omega-\omega_0+2\pi K) \rightarrow$ periodic impulse train.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sum_{K=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0+2\pi K)}_{\text{Consider only } K=0.}\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\,\delta(\omega-\omega_0)\, e^{j\omega n}\, d\omega$$

$$= e^{j\omega_0 n}$$

DTFT of some Basic Sequences

$\delta(n) \longleftrightarrow 1$

$1, -\infty < n < \infty \longleftrightarrow \sum_{K=-\infty}^{\infty} 2\pi\,\delta(\omega+2\pi K)$

Neither Abs. summable nor sq. summable.

$$u(n) \longleftrightarrow \frac{1}{1-e^{-j\omega}} + \sum_{K=-\infty}^{\infty} \pi\,\delta(\omega+2\pi K)$$

$$\alpha^n u(n),\ |\alpha|<1 \longleftrightarrow \frac{1}{1-\alpha e^{-j\omega}}.$$

DTFT Theorems.

Linearity Theorem.

If $g(n) \longleftrightarrow G(e^{j\omega})$ &

$h(n) \longleftrightarrow H(e^{j\omega})$

then $\alpha\, g(n) + \beta\, h(n) \longleftrightarrow \alpha\, G(e^{j\omega}) + \beta\, H(e^{j\omega})$.

Time - reversal Theorem

If $g(n) \longleftrightarrow G(e^{j\omega})$

$g(-n) \longleftrightarrow G(e^{-j\omega})$

Time - shifting Theorem.

If $g(n) \longleftrightarrow G(e^{j\omega})$ then

$g(n-n_0) \longleftrightarrow e^{-j\omega n_0}\, G(e^{j\omega})$.

Frequency - shifting Theorem

$e^{j\omega_0 n}\, g(n) \longleftrightarrow G\left(e^{j(\omega-\omega_0)}\right)$

Differentiation - in - Frequency

If $g(n) \longleftrightarrow G(e^{j\omega})$

$$n\, g(n) \longleftrightarrow j\,\frac{d\,G(e^{j\omega})}{d\omega}.$$

Ex

$x(n) = (n+1)\alpha^n u(n)$, $|\alpha| < 1$.

$$(n+1)\alpha^n u(n) \longleftrightarrow \frac{1}{(1-\alpha e^{-j\omega})^2}.$$