Lecture 23

Saturday, 9 October 2021 2:00 PM

$$G(z) = \sum_{i=1}^{L} \frac{A_i}{(1-pz^{-1})^i} + \sum_{l=1}^{N-L} \frac{B_l}{1-\alpha_l z^{-1}}$$

$$A_i = \frac{1}{(L-i)!\,(-p)^{L-i}} \frac{d^{L-i}}{(dz^{-1})^{L-i}} (1-pz^{-1})^L G(z) \Big|_{z=p,\ pz^{-1}=1}$$

$$z^{-1}\left\{ \frac{A_i}{(1-pz^{-1})^i} \right\} \rightarrow \text{will discuss later.}$$

uses this property

$$\boxed{n\,g(n) \longleftrightarrow -z \frac{dG(z)}{dz}}$$

Ex

$$G(z) = \frac{N(z)}{(1-pz^{-1})^3(1-\alpha z^{-1})}$$

$N^{\circ}(z) < 4.$

$$= \frac{A_1}{1-pz^{-1}} + \frac{A_2}{(1-pz^{-1})^2} + \frac{A_3}{(1-pz^{-1})^3} + \frac{B}{1-\alpha z^{-1}}$$

$$B = (1-\alpha z^{-1})\, G(z) \Big|_{\alpha z^{-1}=1}$$

$$A_3 = (1-pz^{-1})^3\, G(z) \Big|_{pz^{-1}=1}$$

$$G(\infty) = A_1 + A_2 + \underbrace{A_3 + B}_{\text{Known}} \quad \text{— (1)}$$

$$G(1) = \frac{A_1}{1-p} + \frac{A_2}{(1-p)^2} + \frac{A_3}{(1-p)^3} + \frac{B}{1-\alpha} \quad \text{— (2)}$$

Inverse z-transform using Long Division

1) $G(z)$ causal.

$$G(z) = \frac{P(z)}{Q(z)} = g(0) + g(1)z^{-1} + g(2)z^{-2} + \cdots$$

Causal seq^n. Quotient of the division

→ May n't be able to find the general formula for $g(n)$. Only, we can find the coefficients.

If $G(z)$ is anticausal.

$$G(z) = g(-1)z + g(-2)z^2 + \cdots$$

$g(-1)\delta(n+1)$ $\quad$ $g(-2)\delta(n+2)$

Ex

$$G(z) = \frac{1}{1-z^{-1}}, \quad \text{ROC is specified.}$$

1) Assume causal ($|z| > 1$)

$$G(z) = \frac{z}{z-1}$$

$$z-1 \,\overline{)\, z \,}\, (1 + z^{-1} + z^{-2} + \cdots$$

$z - 1$

1

$1 - z^{-1}$

z^{-1}

$z^{-1} - z^{-2}$

$\vdots$

$$\Rightarrow G(z) = 1 + z^{-1} + z^{-2} + \cdots$$

$$g(n) = \delta(n) + \delta(n-1) + \cdots$$

$$= \{\underset{\uparrow}{1}, 1, 1, \cdots\}$$

$$= u(n)$$

2) Assume Anti causal ($|z| < 1$)

$$G(z) = \frac{1}{1-z^{-1}}.$$

$$-z^{-1}+1 \,\overline{)\, 1 \,}\, (-z - z^2 - z^3 - \cdots$$

$1 - z$

z

$z - z^2$

z^2

$\vdots$

$$g(n) = -\delta(n+1) - \delta(n+2) - \cdots$$

$$= \{-1, -1, \cdots \underset{\overset{\uparrow}{n=-1}}{-1}\}$$

$$= -u(-n-1).$$

Ex

Causal $H(z)$

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

Ans

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} \cdots$$

$$h(n) = \{\underset{\underset{n=0}{\uparrow}}{1}, 1.6, -0.52, 0.4, \cdots\}$$

Ex

$$X(z) = \frac{2}{(z+0.5)(z+1)} \quad \text{ROC is not specified.}$$

$z = -0.5, \quad z = -1.$

$$X(z) = \frac{2z^{-2}}{(1+0.5z^{-1})(1+z^{-1})} \rightarrow \text{improper fraction}$$

$$= 4 + \frac{4}{1+z^{-1}} - \frac{8}{1+0.5z^{-1}}$$

Possible ROCs:

1) $0.5 < |z| < 1$
2) $|z| < 0.5$
3) $|z| > 1.$

$0.5 < |z| < 1$

$$x(n) = 4\delta(n) - 4(-1)^n u(-n-1) - 8(-0.5)^n u(n).$$

$|z| < 0.5$

$$x(n) = 4\delta(n) - 4(-1)^n u(-n-1) + 8(-0.5)^n u(-n-1)$$

$|z| > 1$

$$x(n) = 4\delta(n) + 4(-1)^n u(n) - 8(-0.5)^n u(n)$$

Properties of ZT.

$$g(n) \overset{z}{\longleftrightarrow} G(z), \quad \text{ROC: } R_g.$$

$$h(n) \overset{z}{\longleftrightarrow} H(z), \quad \text{ROC: } R_h.$$

R_g: $R_{g_1} < |z| < R_{g_2}$.

R_h: $R_{h_1} < |z| < R_{h_2}$.

1) $g^*(n) \longleftrightarrow G^*(z^*)$, ROC: R_g.

2) $g(-n) \longleftrightarrow G(1/z)$, ROC: $1/R_g$.

$\frac{1}{R_g}$: $\frac{1}{R_{g_2}} < |z| < \frac{1}{R_{g_1}}$

3) Linearity:

$$\alpha g(n) + \beta h(n) \longleftrightarrow \alpha G(z) + \beta H(z)$$

ROC includes $R_g \cap R_h$.

(It can be wider than $R_g \cap R_h$.)

4) Time-shifting.

$$g(n-n_0) \longleftrightarrow z^{-n_0} G(z)$$

ROC: R_g excluding possibly $z=0$ or $z=\infty$

5) $\alpha^n g(n) \longleftrightarrow G(z/\alpha)$

ROC: $|\alpha| R_g$.

$|\alpha| R_g$: $|\alpha| R_{g_1} < |z| < |\alpha| R_{g_2}$.

6) $n\,g(n) \longleftrightarrow -z \frac{dG(z)}{dz}$,

ROC: R_g, excluding possibly $z=0$ or $z=\infty$.

7) $g(n) * h(n) \longleftrightarrow G(z)H(z)$

ROC includes $R_g \cap R_h$.

8) $$g(n)\,h^*(n) \longleftrightarrow \frac{1}{2\pi j} \oint_C G(v)\, H^*(z^*/v^*)\, v^{-1}\, dv.$$

ROC includes $R_g R_h$.

$R_g R_h$: $R_{g_1} R_{h_1} < |z| < R_{g_2} R_{h_2}$

$$g(n)\,h(n) \longleftrightarrow \frac{1}{2\pi j} \oint_C G(v)\, H(z/v)\, v^{-1}\, dv.$$

Same ROC.

Prove it → Assignment.

9) Parseval's Theorem.

$$\sum_{n=-\infty}^{\infty} g(n)\,h^*(n) = \frac{1}{2\pi j} \oint G(v)\, H^*\!\left(\frac{1}{v^*}\right) v^{-1}\, dv.$$

Proof: