Lecture 22

Friday, 8 October 2021 3:02 PM

Inverse z-transform

$g(n) \overset{z}{\longleftrightarrow} G(z)$ $\quad \boxed{z = re^{j\omega}}$

$$G(z) = \sum_{n=-\infty}^{\infty} g(n)\, r^{-n} e^{-j\omega n} \rightarrow DTFT\left[g(n)\, r^{-n}\right]$$

$G(re^{j\omega})$

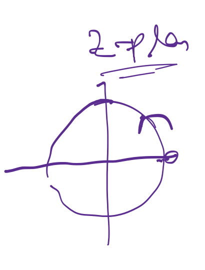
$$\Rightarrow g(n)\, r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega})\, e^{jn\omega}\, d\omega.$$

$$\Rightarrow g(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(z)\, r^{n} e^{jn\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(z)\, z^{n}\, d\omega.$$

$z = re^{j\omega}$

$\Rightarrow dz = \underbrace{re^{j\omega}}_{z}\, j\, d\omega$



$\Rightarrow d\omega = \dfrac{dz}{jz}$

$$\Rightarrow g(n) = \frac{1}{2\pi}\oint_C G(z)\, z^{n}\, \frac{dz}{jz}$$

$$= \frac{1}{2\pi j}\oint_C G(z)\, z^{n-1}\, dz.$$

Methods to compute inverse ZT

1) Partial Fraction Expansion
2) Long division Method.

Partial fraction Expansion :-

$G(z) = \dfrac{P(z)}{Q(z)}$ → polynomials in z^{-1}.

$g(n) \longleftrightarrow G(z)$.

→ ROC information is necessary to compute inverse ZT.

Degree of $P(z) = M$.
Degree of $Q(z) = N$.

If $M \geq N$

$G(z)$ is an improper fraction.

If $M < N$

$G(z)$ is a proper fraction

If $G(z)$ is a proper fraction

$G(z) = \dfrac{P(z)}{Q(z)}$ → N poles

$= \displaystyle\sum_{k=1}^{N} \frac{A_k}{(1-\alpha_k z^{-1})}$ → poles are at $z = \alpha_k$.

$A_k = \left[(1-\alpha_k z^{-1})\, G(z)\right]\Big|_{\alpha_k z^{-1}=1 \Rightarrow z=\alpha_k}$

residues

Ex $\quad H(z) = \dfrac{z(z+2)}{(z-0.2)(z+0.6)}, \quad |z| > 0.6$

Sol^n

$$H(z) = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

$$= \frac{A}{1-0.2z^{-1}} + \frac{B}{1+0.6z^{-1}}$$

$A = (1-0.2z^{-1})\, H(z)\Big|_{0.2z^{-1}=1}$

$= \dfrac{1+2z^{-1}}{1+0.6z^{-1}}\Big|_{0.2z^{-1}=1}$

$= \dfrac{1+10}{1+3} = \dfrac{11}{4} = 2.75.$

$B = (1+0.6z^{-1})\, H(z)\Big|_{0.6z^{-1}=-1}$

$= -1.75.$

$$H(z) = \frac{2.75}{1-0.2z^{-1}} - \frac{1.75}{1+0.6z^{-1}}$$

$\downarrow$ $2.75\,(0.2)^n u(n)$ $\qquad \downarrow$ $-1.75\,(-0.6)^n u(n)$

$(a)^n u(n) \rightleftharpoons \dfrac{1}{1-az^{-1}}, \quad |z| > a.$

$$h(n) = 2.75\,(0.2)^n u(n) - 1.75\,(-0.6)^n u(n)$$

Ex $\quad X(z) = \dfrac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}, \quad |z| > \frac{1}{2}$

Ans → $x(n) = 2\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$

$$\boxed{\begin{array}{l} (a)^n u(n) \longleftrightarrow \dfrac{1}{1-az^{-1}}, \quad |z| > 1 \\ -(a)^n u(-n-1) \longleftrightarrow \dfrac{1}{1-az^{-1}}, \quad |z| < 1 \end{array}}$$

If $G(z)$ is an improper fraction

$G(z) = \dfrac{P(z)}{Q(z)} \begin{array}{l}\rightarrow M \\ \rightarrow N\end{array} \quad M \geq N.$

$= \displaystyle\sum_{l=0}^{M-N} \eta_l z^{-l} + \underbrace{\frac{P_1(z)}{Q(z)}}_{\text{Proper fraction.}}$

$\boxed{z^{-1} \overset{IZT}{\longleftrightarrow} \delta(n-1)}$

Ex $\quad X(z) = \dfrac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$

$|z| > \frac{1}{2}.$

Sol^n

$$X(z) = \frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$= C + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{4}z^{-1}}$$

$$= C + \frac{P_1(z)}{Q(z)}.$$

$$\begin{array}{r|l} & 2 \\ \hline \frac{1}{8}z^{-2} - \frac{3}{4}z^{-1} + 1 & \frac{1}{4}z^{-2} - \frac{7}{4}z^{-1} + 4 \\ & \frac{1}{4}z^{-2} - \frac{3}{2}z^{-1} + 2 \\ & - \quad + \quad - \\ \hline & -\frac{1}{4}z^{-1} + 2 \end{array}$$

$$X(z) = 2 + \frac{2 - \frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$= 2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{4}z^{-1}}$$

$A_1 = 3, \quad A_2 = -1.$

$$\Rightarrow X(z) = 2 + \frac{3}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}$$

$$x(n) = 2\delta(n) + 3\left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{4}\right)^n u(n).$$

Ex $\quad X(z) = \dfrac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}},$ Causal.

Sol^n

$$X(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

Find $x(n)$.

Multiple poles.

→ If $G(z)$ is a proper fraction with multiple poles.

$$G(z) = \frac{N(z)}{(1-pz^{-1})^L\, Q_1(z)}$$

$(1-pz^{-1})^L$: L no. of poles at $z = p$ $\qquad Q_1(z)$: $N-L$ distinct poles.

$(N(z))^{\circ} < \underbrace{L + (Q_1(z))^{\circ}}_{= N}$ → proper fraction.

$$G(z) = \sum_{i=1}^{L} \frac{A_i}{(1-pz^{-1})^i} + \frac{N_1(z)}{Q_1(z)}$$

$$= \sum_{i=1}^{L} \frac{A_i}{(1-pz^{-1})^i} + \sum_{l=1}^{N-L} \frac{B_l}{1-\alpha_l z^{-1}}$$

↓ L repeated poles at $z = p$ $\qquad$ ↓ simple poles at $z = \alpha_l$.