Lecture 13

Wednesday, 15 September 2021 3:52 PM

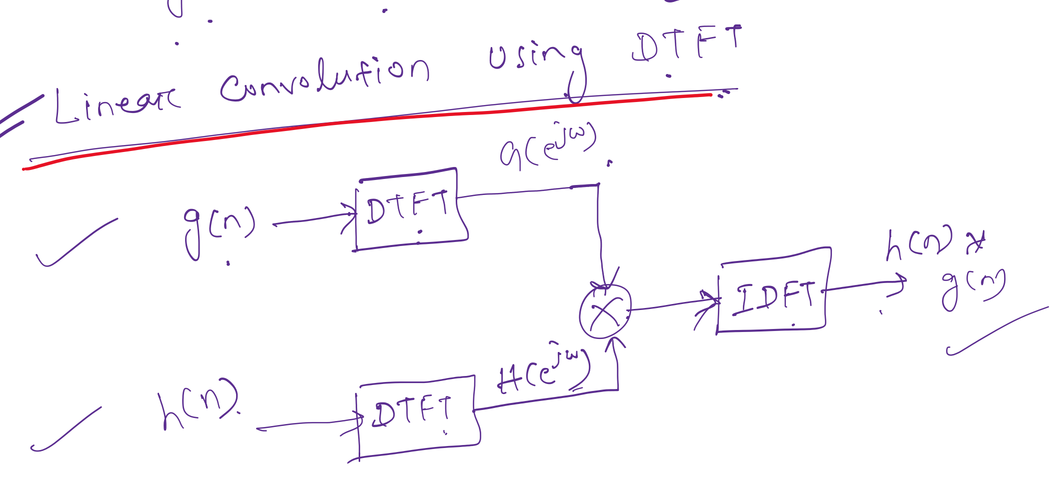
Convolution Theorem

If $g(n) \longleftrightarrow G(e^{j\omega})$

$h(n) \longleftrightarrow H(e^{j\omega})$

$g(n) * h(n) \longleftrightarrow G(e^{j\omega})\, H(e^{j\omega})$

Linear Convolution using DTFT



Modulation Theorem

$y(n) = g(n)\, h(n)$

$$g(n)\, h(n) \longleftrightarrow \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi} G(e^{j\theta})\, H(e^{j(\omega-\theta)})\, d\theta}_{\text{Convolution integral.}}$$

Proof → Home Assignment.

Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} g(n)\, h^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})\, H^*(e^{j\omega})\, d\omega.$$

complex sequ.

$g(n) \longleftrightarrow G(e^{j\omega})$

$h(n) \longleftrightarrow H(e^{j\omega})$

Proof

$$\sum_{n=-\infty}^{\infty} g(n)\, h^*(n) = \sum_{n=-\infty}^{\infty} g(n)\Big(\underbrace{\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega}_{h(n)}\Big)^*$$

$$= \sum_{n=-\infty}^{\infty} g(n) \int_{-\pi}^{\pi} H^*(e^{j\omega})\, e^{-j\omega n}\, d\omega$$

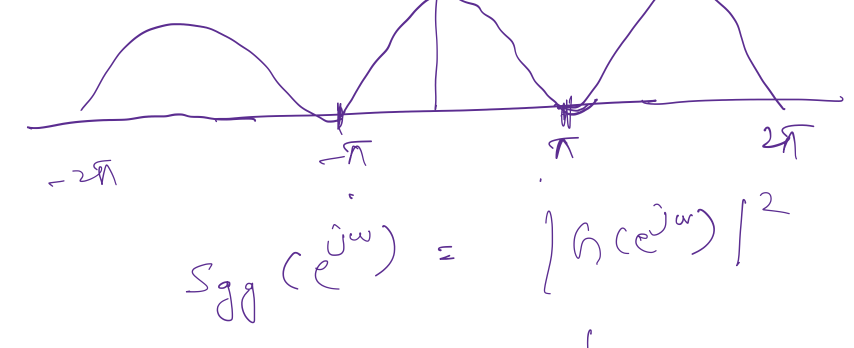
$$= \int_{-\pi}^{\pi} H^*(e^{j\omega}) \underbrace{\sum_{n=-\infty}^{\infty} g(n)\, e^{-j\omega n}}_{G(e^{j\omega})}\, d\omega$$

$$= \int_{-\pi}^{\pi} G(e^{j\omega})\, H^*(e^{j\omega})\, d\omega.$$

Energy of $g(n) = \mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g(n)|^2$

$$= \sum_{n=-\infty}^{\infty} g(n)\, g^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})\, G^*(e^{j\omega})\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2\, d\omega$$



$S_{gg}(e^{j\omega}) = |G(e^{j\omega})|^2$

↓

Energy density spectrum.

$\mathcal{E}_g$ = Energy of $g(n)$ = $\dfrac{\text{Area under the Energy density spectrum curve in } \omega \in [-\pi, \pi]}{2\pi}$

Discrete Fourier Transform (DFT)

→ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$ → Infinite seq.

continuous.

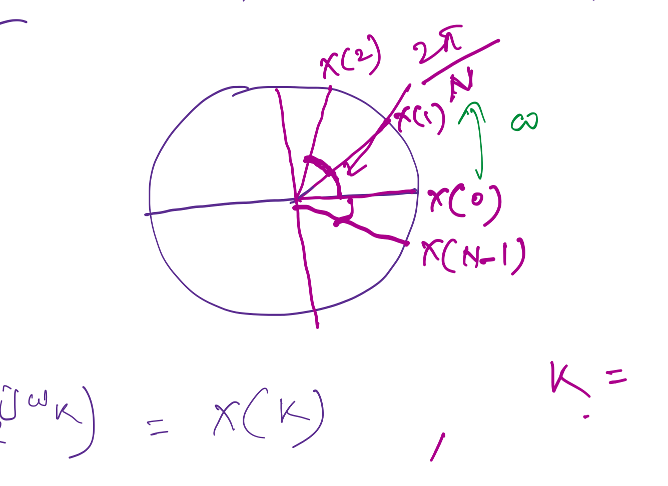
→ This a finite transformation.

If $x(n)$ → length 0 to $N-1$.

then $X(e^{j\omega}) = x(0) + x(1)e^{-j\omega} + \dots + x(N-1)e^{-j(N-1)\omega}$.

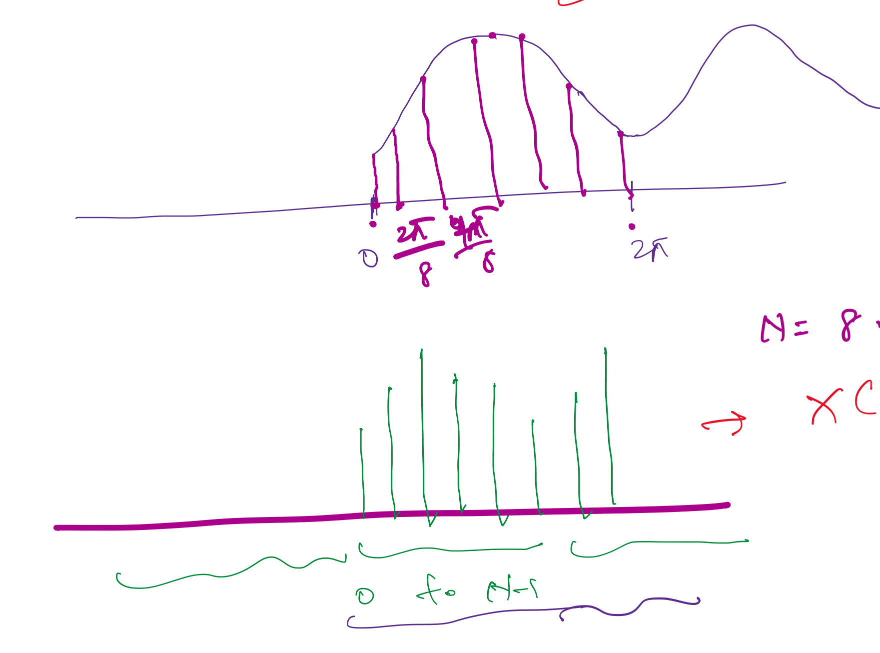
→ Sample $X(e^{j\omega})$ uniformly at N distinct points.

$X(e^{j\omega})\big|_{\omega=\omega_K}$, $\boxed{\omega_K = \dfrac{2\pi K}{N}}$



$X(e^{j\omega_K}) = X(K)$, $K = 0$ to $N-1$

$\frac{2\pi}{N}$ → sampling interval in freq. domain.



$X(K) = \text{DFT}[x(n)]$. → $e^{-j\omega_K n}$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi K n}{N}}$$

K → 0 to $N-1$.

→ $X(K)$ → in general it is complex signal even if $x(n)$ is real.

$\{x(n)\} \longleftrightarrow \{X(K)\}$ → finite length.

Finite length; one-to-one

IDFT

$x(n) = \text{IDFT}[X(K)]$.

$$= \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi n K}{N}}$$

n → 0 to $N-1$

$e^{-j2\pi/N} = W_N$.

DFT: $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$, $K = 0$ to $N-1$

IDFT: $x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$, $n = 0$ to $N-1$

→ W_N^{Kn} → periodic with period N.

$W_N^{Kn} = W_N^{n(K+N)} = W_N^{Kn}$.

⇒ $X(K)$ is periodic fun of K with period N.

→ $W_N^{-Kn} = W_N^{-K(n+N)} = W_N^{-Kn}$.

⇒ $x(n)$ is forced to be periodic.

Proof of IDFT

$$\sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$$

$$= \sum_{K=0}^{N-1} \underbrace{\sum_{l=0}^{N-1} x(l)\, W_N^{Kl}}_{X(K)}\, W_N^{-Kn}.$$

$$= \sum_{l=0}^{N-1} x(l) \underbrace{\sum_{K=0}^{N-1} W_N^{K(l-n)}}_{= \begin{cases} N & \text{if } l = n \\ 0 & \text{if } l \neq n. \end{cases}}$$

$= x(n)\, N.$

$$\Rightarrow \boxed{x(n) = \frac{1}{N}\sum X(K)\, W_N^{-Kn}}$$