Lecture 14

Friday, 17 September 2021 3:01 PM

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}, \quad K \to 0 \text{ to } N-1$$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}, \quad n \to 0 \text{ to } N-1.$$

Examples

1) $$x(n) = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le N-1 \end{cases}$$

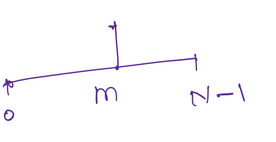
$$x(n) = \delta(n)$$

$$X(K) = \sum_{n=0}^{N-1} \delta(n) W_N^{Kn} = 1.$$

$$X(K) = 1, \quad \text{for } K = 0 \text{ to } N-1$$

2) $$x(n) = \begin{cases} 1, & n = m, \quad 0 \le m \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

$$\Rightarrow x(n) = \delta(n-m)$$



$$X(K) = \sum_{n=0}^{N-1} \delta(n-m) W_N^{Kn}$$

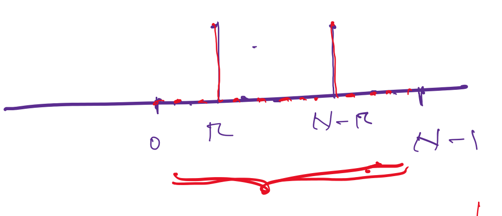
$$= W_N^{Km}, \quad 0 \le K \le N-1$$

$$X(K) \to \{ \underset{\uparrow}{1}, W_N^{m}, W_N^{2m} \cdots W_N^{(N-1)m} \}.$$

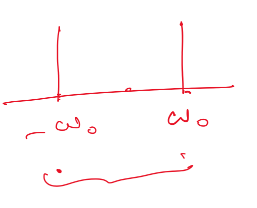
3) $$x(n) = \cos\left(\frac{2\pi r n}{N}\right)$$

$$X(K) = \begin{cases} N/2, & \text{for } K = r \\ N/2, & K = N-r \\ 0, & \text{otherwise.} \end{cases}$$

$\hookrightarrow$ Home assignment.



$\cos\omega_0 t \to$



Computational complexity

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}, \quad 0 \le K \le N-1$$

For each $K \to N$ complex multiplications

$\Rightarrow N^2$ complex multiplications

For each $K \to (N-1)$ complex additions

$\Rightarrow N(N-1)$ complex additions

$$\boxed{\begin{array}{l} N^2 \text{ complex multiplications} \\ N(N-1) \text{ complex additions} \end{array}} \to \text{Total computational complexity of DFT.}$$

$\to$ Fast Fourier Transform (FFT) is an algorithm used to compute DFT efficiently.

Matix Relations

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nK} \to N\text{-Point DFT.}$$

$0 \le n \le N-1$

$0 \le K \le N-1$

$$\bar{x} = [x(0) \; x(1) \; x(2) \cdots x(N-1)]^T$$

$$\bar{X} = [X(0) \; X(1) \; X(2) \cdots X(N-1)]^T.$$

$N \times 1$

$$\underbrace{\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}}_{\bar{X}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}}_{D_N \text{ (DFT matrix)}, \; N \times N} \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}}_{\bar{x}}$$

$$\Rightarrow \boxed{\bar{X} = D_N \bar{x}.}$$

$$\Rightarrow \boxed{\bar{x} = D_N^{-1} \bar{X}} \to \text{IDFT.} \quad -(1)$$

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-nK}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & & & & \\ 1 & W_N^{-(N-1)} & \cdots & & W_N^{-(N-1)^2} \end{bmatrix}}_{= D_N^*} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\Rightarrow \boxed{\bar{x} = \frac{1}{N} D_N^* \bar{X}.} \quad -(2)$$

$$\boxed{D_N^{-1} = \frac{1}{N} D_N^*}$$

$$D_N D_N^{-1} = I_N$$

$$\Rightarrow D_N \left(\frac{1}{N} D_N^*\right) = I_N.$$

$$\Rightarrow \boxed{D_N D_N^* = N I_N}$$

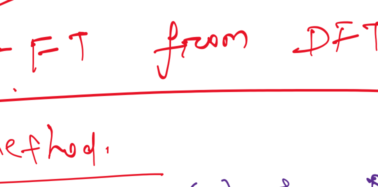
$$\boxed{D_N^* = D_N^H} \to (D_N^*)^H.$$

$D_N \to$ DFT matrix

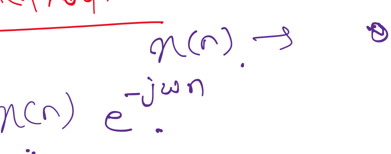
$D_N^* \to$ IDFT matrix

$$X(K) = X(e^{j\omega_K}) \qquad \omega_K = \frac{2\pi K}{N}$$

$x(n) \to$



$X(K) \to$



It is a sufficient description of $X(e^{j\omega})$

How to get $X(e^{j\omega})$ from $X(K)$. ?

Computation of DTFT from DFT.

Interpolation method.

$x(n) \to 0 \le n \le N-1$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \underbrace{\frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi nK}{N}}}_{x(n)} e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) \underbrace{\sum_{n=0}^{N-1} \underbrace{e^{-jn\left(\omega - \frac{2\pi K}{N}\right)}}_{\text{continuous fun}^n}}_{\text{Simplify.}}$$

$$\sum_{n=0}^{N-1} e^{-jn\left(\omega - \frac{2\pi K}{N}\right)}$$

$$= \frac{1 - e^{-j(\omega N - 2\pi K)}}{1 - e^{-j(\omega - 2\pi K/N)}}.$$

$$= \frac{e^{-j[(\omega N - 2\pi K)/2]}}{e^{-j[(\omega N - 2\pi K)/(2N)]}} \times \frac{\sin\left(\frac{\omega N - 2\pi K}{2}\right)}{\sin\left(\frac{\omega N - 2\pi K}{2N}\right)}$$

$$= \frac{\sin\left(\frac{\omega N - 2\pi K}{2}\right)}{\sin\left(\frac{\omega N - 2\pi K}{2N}\right)} \times e^{-j\left(\omega - \frac{2\pi K}{N}\right)\left(\frac{N-1}{2}\right)}.$$

$$\Rightarrow X(e^{j\omega}) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \frac{\sin\left(\frac{\omega N - 2\pi K}{2}\right)}{\sin\left(\frac{\omega N - 2\pi K}{2N}\right)} \times e^{-j\left(\omega - \frac{2\pi K}{N}\right)\left(\frac{N-1}{2}\right)}$$

This can be re-written as.

$$X(e^{j\omega}) = \sum_{K=0}^{N-1} X(K) \, \phi\left(\omega - \frac{2\pi K}{N}\right)$$

$$\phi(\omega) = \frac{\sin\left(\frac{\omega N}{2}\right)}{N \sin\left(\frac{\omega}{2}\right)} e^{-j\omega\left(\frac{N-1}{2}\right)}.$$

$\hookrightarrow$ Interpolating polynomial.