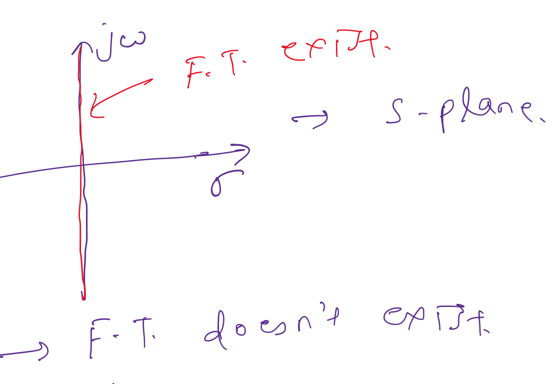
Lecture 20

Wednesday, 29 September 2021 2:26 PM

Z - Transform.

In CT signals



$x(t) \to$ F.T. doesn't exist.

$x(t)\, e^{-\sigma t}\, e^{-j\omega t}.$

→ convergence factor.

It's F.T. exists within a R.O.C.

→ Laplace transformation.

Z- transformation. (DT signals & systems)

→ Some DT exist for which DTFT doesn't exist

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\omega}$$

→ F.T May n't exist

$x(n)\, r^{-n}\, e^{-jn\omega}.$

→ Convergence factor.

It's F.T. exists.

$$\sum_{n=-\infty}^{\infty} x(n)\, r^{-n}\, e^{-jn\omega}.$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (\underbrace{r e^{j\omega}}_{z})^{-n} \to X(re^{j\omega})$$

$$= \sum x(n)\, z^{-n}.$$

$$= X(z) = X(re^{j\omega})$$

$z = re^{j\omega}$, $|z| = r$, $\angle z = \omega$.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}.}$$

$$x(n) \xleftrightarrow{z} X(z)$$

→ If $r = 1$, then $X(z) = X(e^{j\omega})$

$\Rightarrow \boxed{ZT = DTFT.}$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(x(n)\, r^{-n}\right) e^{-jn\omega} \quad \text{—(1)}$$

$$= \text{F.T.}\left\{x(n)\, r^{-n}\right\}.$$

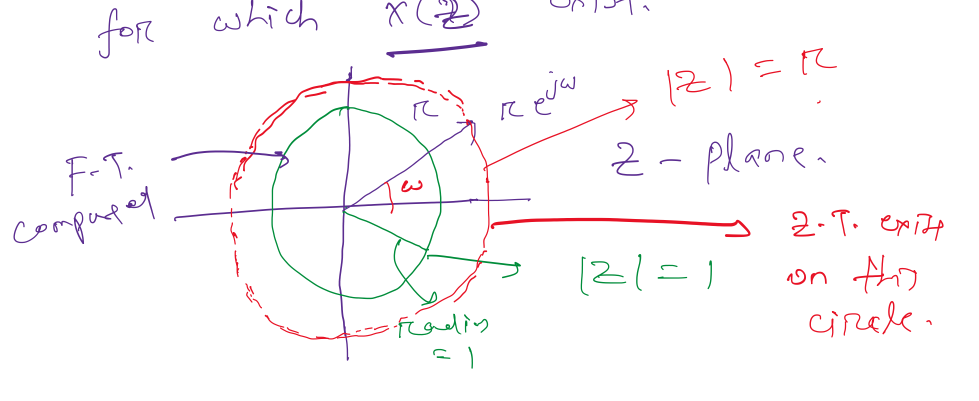
→ $X(z)$ exists if series in (1) will converge. $\Rightarrow x(n)\, r^{-n}$ should be absolutely summable.

$|X(z)| < \infty.$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \left|x(n)\, r^{-n}\right| < \infty.$$

By properly choosing the value of r, the above series can be converged.

→ If there is a specific value of $z = re^{j\omega}$, for which $X(z)$ exist.

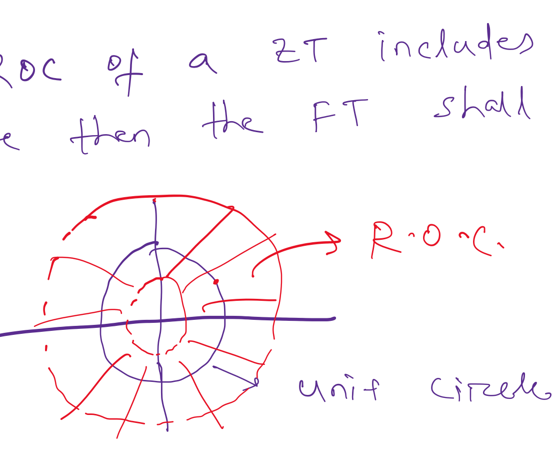


→ The range of z for which Z.T. $X(z)$ exists, is known as Region of Convergence (ROC.

→ Z.T. will be computed over the complete z- plane. Therefore R.O.C. exists.

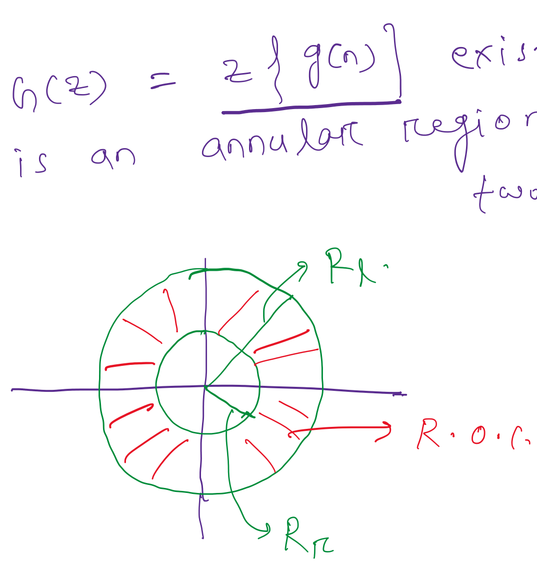
$$\boxed{X(e^{j\omega}) = X(z)\Big|_{|z|=1}}$$

⇒ If the ROC of a ZT includes the unit circle then the FT shall also exist.



Note

If $G(z) = Z\{g(n)\}$ exists then R.O.C. is an annular region (Region betn two circles).



$R_r < |z| < R_l.$

R_r can be zero.

R_l " " ∞.

Examples.

1) $x(n) = \delta(n)$

$X(z) = 1, \quad \forall z.$

R.O.C. → Complete z- plane.

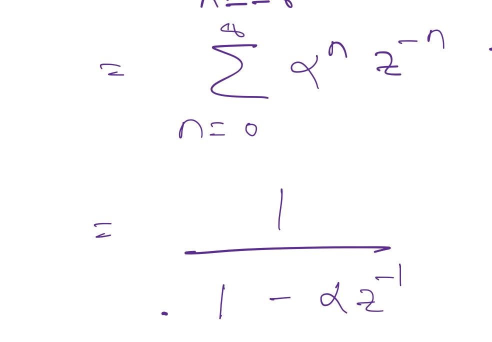
$\Rightarrow R_l = \infty, \quad R_r = 0.$

2) $x(n) = u(n)$

$$X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}$$

$|z^{-1}| < 1$

$\Rightarrow |z| > 1$ → R.O.C.



3) $x(n) = \alpha^n u(n)$ → (causal exponential seq.)

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$= \frac{1}{1 - \alpha z^{-1}}, \quad |\alpha z^{-1}| < 1.$$

$\Rightarrow |z| > |\alpha|$ → R.O.C.

4) $x(n) = -\alpha^n u(-n-1)$ → Anticausal exponential seq.

$$X(z) = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n}.$$

$$= -\sum_{m=1}^{\infty} \alpha^{-m} z^m$$

$$= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m \to (\alpha^{-1} z)^m$$

$$= \frac{-\alpha^{-1} z}{1 - \alpha^{-1} z}$$

if $|\alpha^{-1} z| < 1$

$\Rightarrow \boxed{|z| < |\alpha|}$ → R.O.C.

$$\Rightarrow X(z) = \frac{1}{1 - \alpha z^{-1}}$$

Note

If ROC is not given, the Z.T. is not one-to-one.

⇒ Z.T. always defined with R.O.C.

Ex

$x(n) = r^n \cos(n\omega_0)\, u(n)$

$$X(z) = \frac{1 - (r\cos\omega_0)\, z^{-1}}{1 + (2r\cos\omega_0)\, z^{-1} + r^2 z^{-2}}.$$

R.O.C. → $|z| > r.$ → $|re^{j\omega_0}|$

Ex

$x(n) = r^n \sin(n\omega_0)\, u(n)$

$$X(z) = \frac{(r\sin\omega_0)\, z^{-1}}{1 - (2r\cos\omega_0)\, z^{-1} + r^2 z^{-2}}.$$