Lecture 19

Monday, 27 September 2021 2:58 PM

Eigen functions for LTI system.

Consider I/P seq. $x(n) = e^{j\omega n}$ $-\infty < n < \infty$

$\downarrow$ Complex exponential.

$h(n) \rightarrow$ Impulse response

$x(n) = e^{j\omega n} \longrightarrow \boxed{LTI} \longrightarrow y(n) = ?$ (LTI box labelled $h(n)$)

$$y(n) = x(n) * h(n)$$
$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$
$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$
$$= e^{j\omega n} \underbrace{\left(\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k} \right)}_{H(e^{j\omega})}$$
$$= e^{j\omega n}\, H(e^{j\omega})$$

$$\Rightarrow y(n) = \underbrace{H(e^{j\omega})}_{\text{Associated eigenvalue}}\ \underbrace{e^{j\omega n}}_{\text{eigenfunction of the system}}$$

No change in freq.

$$x(n) = \sum_{k=-\infty}^{\infty} \alpha_k\, e^{j\omega_k n}$$

periodic. $\hookrightarrow$ DT fourier series.

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$ (LTI)

$$y(n) = \sum_{k=-\infty}^{\infty} \alpha_k\, H(e^{j\omega_k})\, e^{j\omega_k n}$$

Ex

$$x(n) = A\cos(\omega_0 n + \phi)$$

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = ?$ (LTI)

$$x(n) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

$$y(n) = \frac{A}{2}\left[H(e^{j\omega_0})\, e^{j\phi} e^{j\omega_0 n} + H(e^{-j\omega_0})\, e^{-j\phi} e^{-j\omega_0 n} \right]$$

Reduction of Computational Complexity of DFT.

1) Decimation-in-time (DIT) FFT Algorithm

→ $N = 2^\nu$ → even integer.

→ Decomposing the seq. $x(n)$ into successively smaller sequences.

$\hookrightarrow$ DIT.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad 0 \le k \le N-1$$

$$= \sum_{n\ \text{even}} x(n)\, W_N^{nk} + \sum_{n\ \text{odd}} x(n)\, W_N^{nk}$$

$$= \sum_{r=0}^{(\frac{N}{2})-1} x(2r)\, W_N^{2rk} + \sum_{r=0}^{(\frac{N}{2})-1} x(2r+1)\, W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{(\frac{N}{2})-1} x(2r) \underbrace{(W_N^2)}^{rk} + W_N^k \sum x(2r+1)\, (W_N^2)^{rk}$$

$$\boxed{W_N^2 = e^{-2j(2\pi/N)} = e^{-j2\pi/(N/2)} = W_{N/2}}$$

$$\Rightarrow X(k) = \underbrace{\sum_{r=0}^{(\frac{N}{2})-1} x(2r)\, W_{N/2}^{rk}}_{\frac{N}{2}\text{-point DFT}} + W_N^k \underbrace{\sum_{r=0}^{(\frac{N}{2})-1} x(2r+1)\, W_{N/2}^{rk}}_{\frac{N}{2}\text{-point DFT}}$$

$$= G(k) + W_N^k\, H(k).$$

$$0 \le k \le N-1$$

$G(k)$, $H(k)$ → periodic with period $\frac{N}{2}$.

→ We can consider two $(\frac{N}{4})$-point DFTs to compute each $\frac{N}{2}$-point DFT.

$$G(k) = \sum_{r=0}^{(\frac{N}{2})-1} g(r)\, W_{N/2}^{rk}$$

$$= \sum_{l=0}^{(\frac{N}{4})-1} g(2l)\, \underbrace{W_{N/2}^{2lk}}_{W_{N/4}^{lk}} + \sum_{l=0}^{(\frac{N}{4}-1)} g(2l+1)\, W_{N/2}^{(2l+1)k}$$

$$= \sum_{l=0}^{(\frac{N}{4})-1} g(2l)\, W_{N/4}^{lk} + W_{N/2}^{k} \sum_{l=0}^{\frac{N}{4}-1} g(2l+1)\, W_{N/4}^{lk}$$

Similarly.

$$H(k) = \sum_{l=0}^{(\frac{N}{4})-1} h(2l)\, W_{N/4}^{lk} + W_{N/2}^{k} \sum_{l=0}^{k} h(2l+1)\, W_{N/4}^{lk}$$

2-point DFT

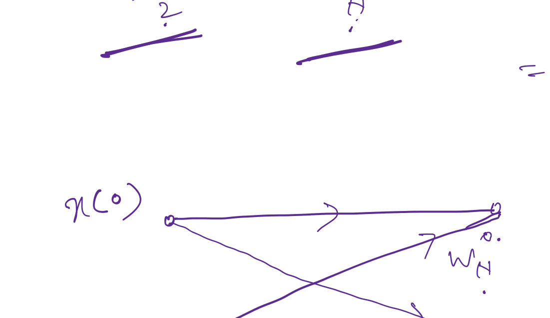
$x(0), x(1) \longrightarrow \boxed{\text{2-point DFT}} \longrightarrow X(0), X(1)$

$$X(0) = x(0) \times 1 + x(1) \times \underset{W_N^0}{1}$$

$$X(1) = x(0) \times 1 + x(1) \times \underset{W_N^1}{W_2^1}$$

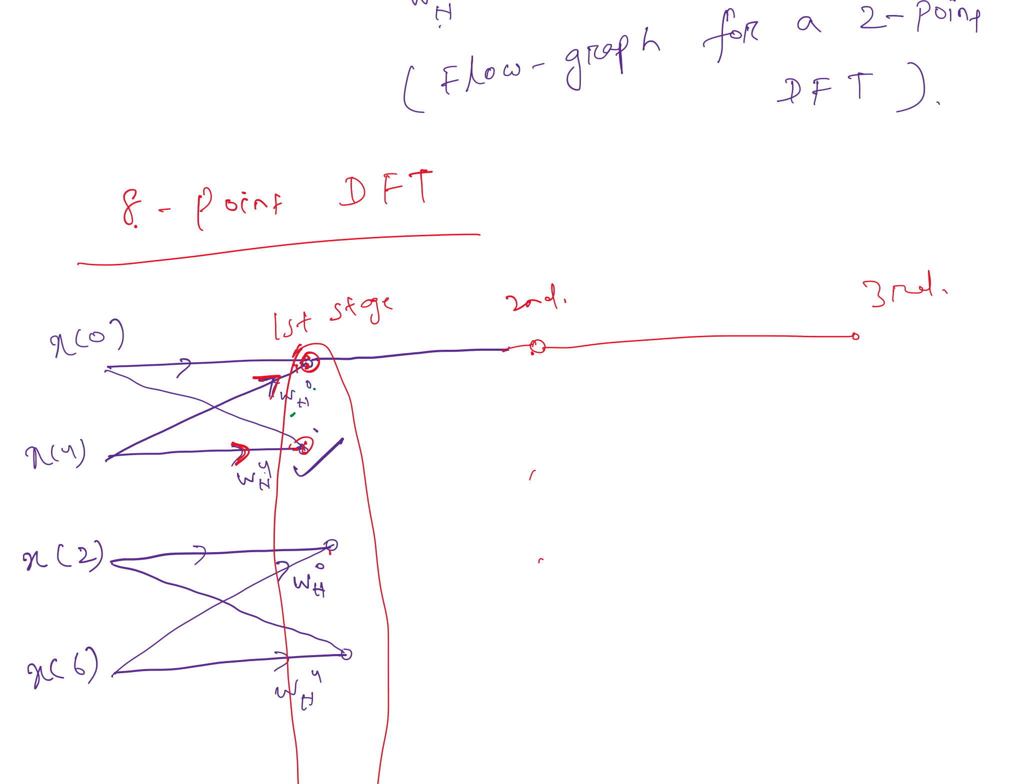
$$W_2^1 = e^{-j2\pi/2} = e^{-j\pi} = -1.$$

$$W_2^1 = W_N^{N/2} = e^{-j\frac{2\pi}{N} \times \frac{N}{2}} = e^{-j\pi} = -1$$



(Flow-graph for a 2-point DFT).

8-point DFT

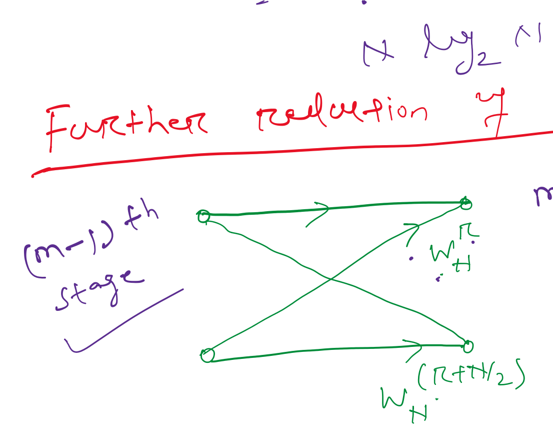


N complex multiplications & N complex additions.

$\log_2 N$ → No. of stages.

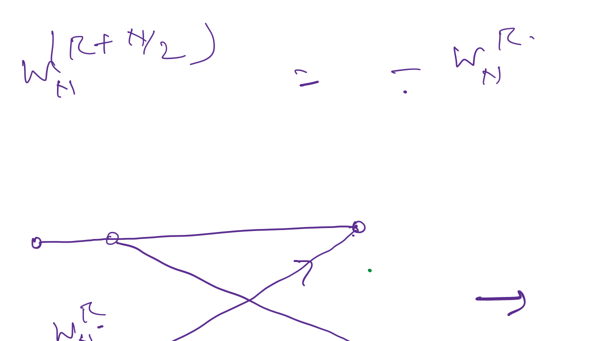
$\Rightarrow$ Total computation $= N\log_2 N$ complex multiplications
$N \log_2 N$ " additions.

Further reduction of computational complexity.



→ 2 complex multiplications

$$W_N^{N/2} = e^{-j\pi} = -1$$
$$W_N^{(r+N/2)} = -W_N^r.$$



→ only one complex multiplication

$$\boxed{\begin{array}{ll} \frac{N}{2}\log_2 N \rightarrow & \text{Complex multiplications} \\ N\log_2 N \rightarrow & \text{" additions} \end{array}}$$

Decimation-in-Frequency (DIF) FFT Algorithm

$\hookrightarrow$ Assignment.