Lecture 24

Monday, 29 October 2012 10:00

Parseval's Theorem:

$$\sum_{n=-\infty}^{\infty} g(n)\, h^*(n) = \frac{1}{2\pi j}\oint G(z)\, H^*\!\left(\frac{1}{z^*}\right) z^{-1}\, dz$$

Proof:

We know

$$\underbrace{g(n)\, h^*(n)}_{} \longrightarrow \underbrace{\frac{1}{2\pi j}\oint G(v)\, H^*\!\left(\frac{1}{v^*}\right) v^{-1}\, dv}_{}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} g(n)\, h^*(n) z^{-n}$$

$$= \frac{1}{2\pi j}\oint G(v)\, H^*\!\left(\frac{z^*}{v^*}\right) v^{-1}\, dv$$

Set: $z = 1$

$$\Rightarrow \sum_{n=-\infty}^{\infty} g(n)\, h^*(n) = \frac{1}{2\pi j}\oint G(v)\, H^*\!\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint G(z)\, H^*\!\left(\frac{1}{z^*}\right) z^{-1}\, dz$$

$$E_g = \sum_{n=-\infty}^{\infty} |g(n)|^2 = \frac{1}{2\pi j}\oint \frac{G(z)\, H^*\!\left(\frac{1}{z^*}\right)}{z^{-1}}\, dz$$

If $G(z)$ of $H(z)$ include unit circle.

$z = e^{j\omega}$

$$\therefore \sum_{n=-\infty}^{\infty} g(n)\, h^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})\, H^*(e^{j\omega})\, d\omega$$

$$E_g = \sum |g(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2\, d\omega$$

If $g(n)$ is real value.

$$E_g = \sum_{n=-\infty}^{\infty} |g(n)|^2$$

$$= \frac{1}{2\pi j}\oint G(z)\, G\!\left(\frac{1}{z^*}\right) z^{-1}\, dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2\, d\omega.$$

Ex

$$x(n) = r^n \cos(n\omega_0)\, u(n)$$

$$x(n) = r^n \frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right) u(n)$$

$$= \underbrace{\frac{1}{2} r^n e^{j\omega_0 n} u(n)}_{w(n)} + \underbrace{\frac{1}{2} r^n e^{-j\omega_0 n} u(n)}_{w^*(n)}$$

$$W(z) = \frac{1}{2}\left(\frac{1}{1 - re^{j\omega_0} z^{-1}}\right), \quad |z| > |r|$$

$$w^*(n) \longleftrightarrow W^*(z^*)$$

$$W(z^*) = \frac{1}{2}\left(\frac{1}{1 - re^{j\omega_0} (z^*)^{-1}}\right)$$

$$W^*(z^*) = \frac{1}{2}\left(\frac{1}{1 - re^{-j\omega_0} z^{-1}}\right)$$

$$\longrightarrow |z| > |r|$$

$$X(z) = \frac{1}{2}\left(\frac{1}{1 - re^{j\omega_0} z^{-1}} + \frac{1}{1 - re^{-j\omega_0} z^{-1}}\right)$$

$$= \frac{1 - (r\cos\omega_0)\, z^{-1}}{1 - (2r\cos\omega_0)\, z^{-1} + r^2 z^{-2}}.$$

ROC: $|z| > |r|$

Ex

$$y(n) = (n+1)\, r^n\, u(n)$$

$$y(n) = n r^n u(n) + \underbrace{r^n u(n)}_{x(n)}$$

$$= n\, x(n) + x(n).$$

$$x(n) = r^n u(n)$$

$$X(z) = \frac{1}{1 - rz^{-1}}, \quad |z| > |r|$$

$$n\, x(n) \longrightarrow -z \frac{dX(z)}{dz}$$

$$= -z \frac{d}{dz}\left(\frac{1}{1 - rz^{-1}}\right) = \frac{rz^{-2}}{(1 - rz^{-1})^2}$$

$$\longrightarrow |z| > |r|$$

ordinary z

$$Y(z) = X(z) - z\frac{dX(z)}{dz}$$

$$= \frac{1}{1 - rz^{-1}} + \frac{rz^{-1}}{(1 - rz^{-1})^2}$$

$$= \frac{1}{(1 - rz^{-1})^2} \longrightarrow |z| > |r|$$

$$(n+1)\, r^n u(n) \longleftrightarrow \frac{1}{(1 - rz^{-1})^2}$$

$$|z| > |r|$$

Ex

$$G(z) = \frac{z^3}{(z - \frac{1}{2})(z + \frac{1}{3})}, \quad |z| > \frac{1}{2}$$

nonzero poles at $z = -\frac{1}{3}$

$$G(z) = \frac{0.36}{1 - \frac{1}{2} z^{-1}} + \frac{0.27}{1 + \frac{1}{3} z^{-1}}$$

$$\longrightarrow 0.36\left(\tfrac{1}{2}\right)^n u(n)$$

$$+ \frac{0.27}{1 + \frac{1}{3} z^{-1}}$$

$$\longrightarrow 0.27\left(-\tfrac{1}{3}\right)^n u(n)$$

$$0.36\left(\tfrac{1}{2}\right)^n u(n)$$

$$g(n) = \left(0.36\left(\tfrac{1}{2}\right)^n + 0.27\left(-\tfrac{1}{3}\right)^n\right) u(n)$$

Fridge 2-8

$$X(z) = z^2\left(1 - \tfrac{1}{2} z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

$$= z^2 - \tfrac{1}{2} z - 1 + \tfrac{1}{2} z^{-1}$$

$$x(n) = \left\{1, -\tfrac{1}{2}, 1, \tfrac{1}{2}\right\}$$

Linear Convolution using ZT

$$y(n) = x(n) * h(n)$$

$$Y(z) = X(z)\, H(z).$$

$$x(n) = \left\{-2, 0, 1, 0, -1, 3\right\}$$

$$h(n) = \left\{1, 2, 0, -1\right\}$$

$$X(z) = -2 + z^{-2} - z^{-4} + 3z^{-5}$$

$$H(z) = 1 + 2z^{-1} - z^{-3}$$

$$Y(z) = -2 - 4z^{-1} + z^{-2} + 2z^{-3} + z^{-4} - 3z^{-5} + 5z^{-6} + z^{-7} - 3z^{-8}$$

$$y(n) = \left\{-2, -4, 1, 2, 1, 5, 1, -3\right\}$$

Circular Convolution using ZT

$$y_c(n) = x(n) \circledast h(n)$$

Find $Y_c(z) = X(z)\, H(z)$.

$$Y_c(z) = \langle X(z) H(z) \rangle_{(z^N - 1)}$$

compute in y_c using z^{N-1}

$$\langle z^{n-1} \rangle_{(z^N - 1)} = z^{-1}$$

$$\langle z^{-1} X(z) \rangle$$

$$\langle z^{-n-1} \rangle_{(z^N - 1)} = z^{-2}$$

Ex

$$y_c(n) = g(n) \circledast h(n) \longrightarrow \text{Length } 4.$$

$$G(z) = g(0) + g(1) z^{-1} + g(2) z^{-2} + g(3) z^{-3}$$

$$H(z) = h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}.$$

$$Y_c(z) = G(z)\, H(z)$$

$$= g(0) + g(1) z^{-1} + g(2) z^{-2} + \cdots + g(3) z^{-3}.$$

$$Y_c(z) = \langle Y(z) \rangle_{(z^4 - 1)}$$

$$= \left(y_c(0) + y_c(1) z^{-1} + y_c(2) z^{-2} + y_c(3) z^{-3}\right) + \left(y_c(4) + y_c(5) z^{-1} + y_c(6) z^{-2} + y_c(7) z^{-3}\right) z^{-4}$$

$$= \left(y_c(0) + y_c(4)\right) + \left(y_c(1) + y_c(5)\right) z^{-1} + \left(y_c(2) + y_c(6)\right) z^{-2} + \left(y_c(3) + y_c(7)\right) z^{-3}$$

Characterization DT systems using ZT

LTI system

Any DT signal $= \sum$ Sinusoidals.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

$$x(n) = e^{j\omega n}$$

$$y(n) = H(e^{j\omega})\, e^{j\omega n}.$$

eigenvalue. eigenfunction.

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k} = \text{FT}\left[h(n)\right]$$

$H(e^{j\omega}) \longrightarrow$ Frequency response.

$$e^{j\omega n} \longrightarrow H(e^{j\omega})\, e^{j\omega n}.$$

$$\cos\omega n \longrightarrow \left|H(e^{j\omega})\right| \cos(\omega n + \angle H(e^{j\omega}))$$

$$\sin\omega n \longrightarrow \left|H(e^{j\omega})\right| \sin(\omega n + \angle H(e^{j\omega}))$$