Type 1 Linear phase FIR filters.

$h(n) = h(N-n)$
length = $N+1$ = odd.
order = N = even.

$N = 8$

$$H(z) = h_0\left(1+z^{-8}\right) + h_1\left(z^{-1}+z^{-7}\right) + h_2\left(z^{-2}+z^{-6}\right) + h_3\left(z^{-3}+z^{-5}\right) + h_4 z^{-4}$$

$$H(z) = z^{-4}\Big[h_0\left(z^{4}+z^{-4}\right) + h_1\left(z^{3}+z^{-3}\right) + h_2\left(z^{2}+z^{-2}\right) + h_3\left(z+z^{-1}\right) + h_4\Big]$$

$$H(e^{j\omega}) = e^{-j4\omega}\Big[2h_0\cos 4\omega + 2h_1\cos 3\omega + 2h_2\cos 2\omega + 2h_3\cos\omega + h_4\Big]$$

$\tilde{H}(\omega)$

$\tilde{H}(\omega)$ → real fn with zero phase.
(zero-reference)

$\phi = -4\omega + \beta$. α = 4 n.s.

$\tilde{H}(\omega) = \tilde{H}$

→ Flat response.

In general:

$h(n) = h(N-n)$
$N+1$ → odd.

$$H(z) = z^{-N/2}\, \tilde{H}(z)$$

↓

Also a linear phase.

$$H(e^{j\omega}) = e^{-j\omega\frac{N}{2}}\, \tilde{H}(\omega)$$

$\tilde{H}(\omega)$ = $h(\frac{N}{2}) + \sum_{n=1}^{N/2} h(\frac{N}{2}-n)\cos n\omega$

$\phi = -\omega\frac{N}{2} + \beta$

If $N+1 = 3$

$\{h_0, h_1, h_2\}$

$$H(z) = h_1 + h_0 z^{-1} + h_2 z^{-2}$$
$$= z^{-1}\Big(h_0 z + h_1 + h_2 z^{-1}\Big)$$
$$= z^{-1}\, H(z^{-1})$$

$H(z)$

$$\boxed{H_1(z) = z^{-N}\, H_1(z^{-1})}$$

↳ N is even
(minimum length polynomial)

$H(z) = H(1)$
$H(-z) = H(-z)$

If $H_1(z_0) = 0$ then
$H_1(z_0^{-1}) = 0$

⇒ $z = z_0$ is a zero, then
$z = 1/z_0$ ↑

⇒ Zeros occur in reciprocal pairs.

Type 2 FIR filters. (Symmetry, sym length)

$h(n) = h(N-n)$.
length = $N+1$ = even.
order = N = odd.

Let $N = 7$

$\pi - \frac{\pi}{2} = \frac{\pi}{2}$

Symmetry around $\pi/2$.

$$H(z) = h_0\left(1+z^{-7}\right) + h_1\left(z^{-1}+z^{-6}\right) + h_2\left(z^{-2}+z^{-5}\right) + h_3\left(z^{-3}+z^{-4}\right)$$
$$= z^{-3}\Big[h_0\left(z^{3}+z^{-3}\right) + h_1\left(z^{2}+z^{-2}\right) + h_2\left(z+z^{-1}\right)\Big]$$

$$H(e^{j\omega}) = e^{-j3\omega}\Big[2h_0\cos 3\omega + 2h_1\cos 2\omega + 2h_2\cos\omega\Big]$$

$\tilde{H}(\omega)$

In general:

$$H(e^{j\omega}) = e^{-j\frac{N}{2}\omega}\sum_{n=1}^{(N+1)/2} h\left(\tfrac{N}{2}-n\right)\cos\left(\omega\left(\tfrac{N}{2}-n\right)\right)$$

$$\phi = -\frac{N\omega}{2} + \beta$$

$$\tilde{H}(\omega) = \tilde{H}$$

$$\phi(\omega) = \frac{N}{2}$$

$$\boxed{H_2(z) = z^{-N}\, H_2(z^{-1})}$$

→ $N+1$ = odd.

$H(z) = H(1)$
$H(-z) = -H(-z)$

$\boxed{H(-1) = 0}$

One zero at $z = -1$

$H(z) = (1+z^{-1})\, H_1(z)$ → It should also be a linear phase.

Note

Since $H(-1) = 0$, type 4 is not possible
4. design HPF and BSF using type 2.
FIR filters.

Type 3 Linear phase FIR filters.

(Antisymmetry, odd length)

$h(n) = -h(N-n)$
length = $N+1$ = odd
order = N = even

$N = 8$

$h(\frac{N}{2}) = 0$

$$H(z) = h_0\left(1-z^{-8}\right) + h_1\left(z^{-1}-z^{-7}\right) + h_2\left(z^{-2}-z^{-6}\right) + h_3\left(z^{-3}-z^{-5}\right)$$

$$H(e^{j\omega}) = e^{-j4\omega}\Big[2j\,h_0\sin 4\omega + 2j\,h_1\sin 3\omega + 2j\,h_2\sin 2\omega + 2j\,h_3\sin\omega\Big]$$

$e^{j\pi/2} \to j$

$$= 2e^{-j4\omega}\, e^{j\pi/2}\sum_{n=1}^{4} h(n)\sin n\omega$$

In general

$$H(e^{j\omega}) = 2e^{-j\frac{N}{2}\omega}\, e^{j\pi/2}\sum_{n=1}^{N/2} h\left(\tfrac{N}{2}-n\right)\sin n\omega$$

$$= 2e^{-j\frac{N\omega}{2}}\, e^{j\pi/2}\, \tilde{H}_3(\omega)$$

$$\phi = -\frac{N\omega}{2} + \frac{\pi}{2} + \beta$$

$$\boxed{H_3(z) = -z^{-N}\, H_3(z^{-1})}$$

↳ N is even.

$$\tilde{H}_3(\omega) = \tilde{H}$$

$$\phi(\omega) = \frac{\pi}{2}$$

$H(1) = -H(1)$ ⇒ $H(1) = 0$
$H(-1) = -H(-1)$ ⇒ $H(-1) = 0$

$z = \pm 1$ ⇒ zeros.

$$H(z) = (1+z^{-1})\,(1-z^{-1})\, H_1(z)$$

→ only BPF is possible using type 3

Type 4

$h(n) = -h(N-n)$
$N+1$ = even

$$H_4(e^{j\omega}) = 2e^{-j\frac{N\omega}{2}}\, e^{j\pi/2}\sum_{n=1}^{(N+1)/2} h\left(\tfrac{N+1}{2}-n\right)\sin\left(\omega\left(n-\tfrac{1}{2}\right)\right)$$

$$\phi = -\frac{N\omega}{2} + \frac{\pi}{2} + \beta$$

$$\phi(\omega) = \frac{\pi}{2}$$

$$\boxed{H_4(z) = -z^{-N}\, H_4(z^{-1})}$$

↳ N is odd.

$H(1) = -H(1)$ ⇒ $H(1) = 0$
$H(-1) = -H(-1)$ ⇒ $H(-1) = 0$

$z = \pm 1$ ⇒ zeros.

$$H(z) = (1+z^{-1})\,(1-z^{-1})\, H_1(z)$$

→ LPF & BPF ⇒ not possible.

Eg

Differentiable:

$$H(e^{j\omega}) = \begin{cases} j\omega & -\pi < \omega < \pi \\ 0 & \end{cases}$$

$$\left|H(e^{j\omega})\right|_{\omega=0} = 0$$

⇒ $H(z) = 0$ at $z = 1$

Type-4 is most suitable for differentiator.