Lecture 16

Wednesday, 22 September 2021   3:54 PM

DFT properties

Linearity :

$x_1(n), x_2(n) \rightarrow$ finite length.

$x_3(n) = a\,x_1(n) + b\,x_2(n)$

$$DFT[x_3(n)] = X_3(K) = a X_1(K) + b X_2(K)$$

$$\boxed{\begin{array}{l} x_1(n) \xleftrightarrow{DFT} X_1(K) \\ x_2(n) \xleftrightarrow{DFT} X_2(K) \end{array}}$$

$$a\,x_1(n) + b\,x_2(n) \xleftrightarrow{DFT} a\,X_1(K) + b\,X_2(K).$$

→ If Length of $x_1(n) = N_1$

" " " $x_2(n) = N_2$.

Length of $x_3(n) = N_3 = \max(N_1, N_2)$

If $N_1 < N_2$, then zero padding will be done ($N_2 - N_1$ zeros padded to $x_1(n)$)

→ Length of DFTs $= N \geq N_3$.

Circular Time-shifting.

If $x(n) \xleftrightarrow{DFT} X(K)$ → N-point DFT.

$$x(\langle n-n_0\rangle_N) \longleftrightarrow W_N^{Kn_0} X(K).$$

↳ circular time-shifted sequence.

Circular frequency-shifting.

If $x(n) \xleftrightarrow{DFT} X(K)$

$$W_N^{-K_0 n} x(n) \xleftrightarrow{DFT} X(\langle K-K_0\rangle_N).$$

↓ circular shift in freq.

Duality Theorem :-

If $x(n) \xleftrightarrow{N\text{-point DFT}} X(K)$, then

then N-point DFT $X(n)$ is

$$X(n) \xleftrightarrow{DFT} N\,x(\langle -K\rangle_N)$$

$x(n) = \{1, 2, 1, 1\}$

$X(K) = \{a, b, c, d\}$.

↓ DFT

↓ $N\,x(\langle -K\rangle_N) = 4\,x(\langle -K\rangle_4)$.

Circular Convolution Theorem

$x(n) \rightarrow 0 \leq n \leq N-1$

$h(n) \rightarrow 0 \leq n \leq N-1$

$$x(n) \circledast_N h(n) \xleftrightarrow{DFT} X(K)\,H(K)$$

where $x(n) \xleftrightarrow{N\text{-Point DFT}} X(K)$

$h(n) \xleftrightarrow{N\text{-DFT}} H(K)$.

Proof

Let $y(n) = x(n) \circledast_N h(n)$

Prove $Y(K) = X(K)\,H(K)$.

$$Y(K) = \sum_{n=0}^{N-1} y(n)\,W_N^{Kn}$$

$$= \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} x(m)\,h(\langle n-m\rangle_N)\right) W_N^{Kn}$$

$$= \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} h(\langle n-m\rangle_N)\,W_N^{Kn}.$$

Substitute $n - m = l + Nr$. ① → integer.

↓ $0 \leq l \leq N-1$

$\langle n-m\rangle_N = \langle l+Nr\rangle_N = l.$

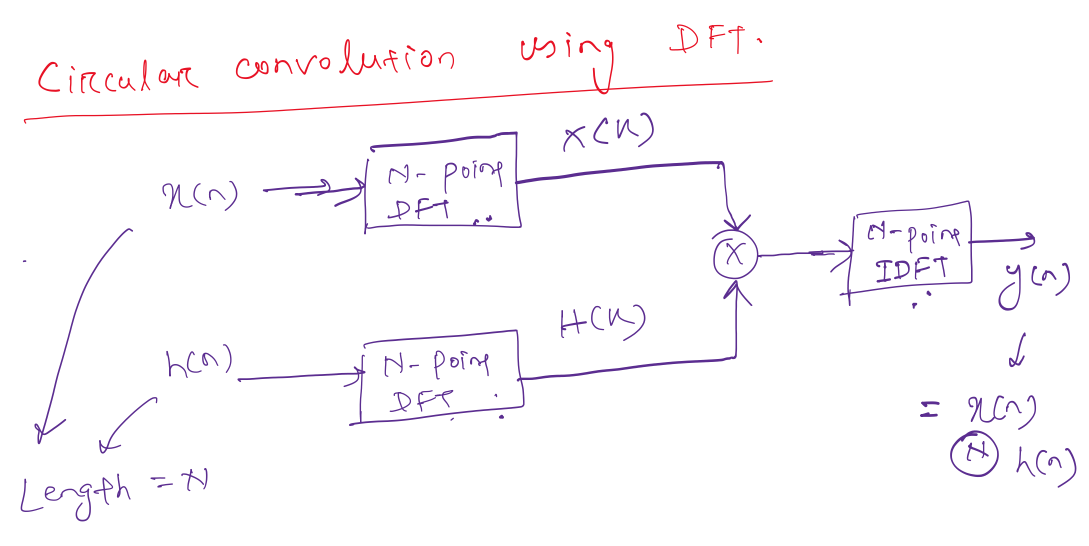
$$\Rightarrow Y(K) = \sum_{m=0}^{N-1} x(m)\left(\sum_{l=0}^{N-1} h(l)\,W_N^{K(m+l+Nr)}\right) \quad (\because n = m+l+Nr)$$

$$= \sum_{m=0}^{N-1} x(m)\underbrace{\left(\sum_{l=0}^{N-1} h(l)\,W_N^{Kl}\right)}_{H(K)} \underbrace{W_N^{K(m+Nr)}}_{= W_N^{Km}}$$

$$= \underbrace{\left(\sum_{m=0}^{N-1} x(m)\,W_N^{Km}\right)}_{X(K)} H(K)$$

$= X(K)\,H(K)$.

Circular convolution using DFT.



Modulation Theorem.

$$x(n)\,h(n) \xleftrightarrow{DFT} \frac{1}{N}\sum_{l=0}^{N-1} X(l)\,H(\langle K-l\rangle_N)$$

↓

$$\frac{1}{N}\left(X(K) \circledast_N H(K)\right)$$

Parseval's Theorem.

$x(n) \xleftrightarrow{DFT} X(K)$

$h(n) \xleftrightarrow{DFT} H(K)$

$$\sum_{n=0}^{N-1} x(n)\,h^*(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\,H^*(K)$$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{K=0}^{N-1} |X(K)|^2$$

↓ Total energy of $x(n)$

↳ proof (Assignment).

Ex Given $g(n) = \{\underset{\uparrow}{1}, 2, 0, 1\}$

$y(n) = g(n) \circledast_4 h(n)$.

$y(n) = \{6, 7, 6, 5\}$.

Find $h(n)$. ?

Sol^n

$Y(K) = G(K)\,H(K)$

$H(K) = \dfrac{Y(K)}{G(K)}$

$h(n) = \{2, 2, 1, 1\}$.

Implementation of Linear Convolution of two finite length sequences using circular convolution :

Ex $g(n) = \{1, 2, 0, 1\} \rightarrow 0 \leq n \leq 3$.

$h(n) = \{2, 2, 1, 1\} \rightarrow 0 \leq n \leq 3$.

Compute $y(n) = g(n) * h(n)$, using the circular conv.

$$\boxed{g(n) * h(n) \neq g(n) \circledast_4 h(n).}$$

Length of $y(n) = L = 4 + 4 - 1 = 7$.

→ $0 \leq n \leq 6$.

$$\tilde{g}(n) = \begin{cases} g(n), & 0 \leq n \leq 3. \\ 0 & 4 \leq n \leq 6. \end{cases}$$

$L - 4$ zeros are padded.

$$\tilde{h}(n) = \begin{cases} h(n), & 0 \leq n \leq 3. \\ 0, & 4 \leq n \leq 6. \end{cases}$$

$$\tilde{y}(n) = \tilde{g}(n) \circledast_7 \tilde{h}(n).$$

$$= \sum_{m=0}^{6} \tilde{g}(m)\,\tilde{h}(\langle n-m\rangle_7)$$

$$\tilde{y}(0) = \tilde{g}(0)\tilde{h}(0) + \tilde{g}(1)\cancel{\tilde{h}(6)} + \tilde{g}(2)\cancel{\tilde{h}(5)} + \tilde{g}(3)\cancel{\tilde{h}(4)} + \cancel{\tilde{g}(4)}\tilde{h}(3) + \cancel{\tilde{g}(5)}\tilde{h}(2) + \cancel{\tilde{g}(6)}\tilde{h}(1)$$

$= g(0)\,h(0) = 2.$

$$\tilde{y}(1) = \tilde{g}(0)\tilde{h}(1) + \tilde{g}(1)\tilde{h}(0) + \tilde{g}(2)\cancel{\tilde{h}(6)} + \tilde{g}(3)\cancel{\tilde{h}(5)} + \cancel{\tilde{g}(4)}\cancel{\tilde{h}(4)} + \cancel{\tilde{g}(5)}\tilde{h}(3) + \cancel{\tilde{g}(6)}\tilde{h}(2)$$

$= g(0)\,h(1) + g(1)\,h(0).$

$= 6.$

$\tilde{y}(2) = 5$

$\tilde{y}(3) = 5$

$\tilde{y}(4) = 4$

$\tilde{y}(5) = 1$

$\tilde{y}(6) = 1$

$\tilde{y}(n) = \{\underset{\uparrow}{2}, 6, 5, 5, 4, 1, 1\}$

$= y(n)$

$$\boxed{g(n) * h(n) = \tilde{g}(n) \circledast_7 \tilde{h}(n)}$$