

Topic: Proofs

Total Marks: 47

Total Time : 50 mins

Important instructions :

1. When you submit your answer sheet, please put it in one PDF file. Make sure to clearly show the page number and the question number. Also, on the first page, include your name, the time you started, and the time you finished.
2. If you come across a question that hasn't been covered in your class yet, no need to worry. Just skip that question and mention "NA", and your grade will be based on the questions you did attempt.
3. Use of GDC is not allowed.
4. Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.

1. Prove by mathematical induction that, for $n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \dots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

(Total 8 marks)

2. Prove by mathematical induction $\sum_{r=1}^n r(r!) = (n+1)! - 1$, $n \in \mathbb{Z}^+$.

(Total 8 marks)

3. (a) Consider the following sequence of equations.

$$\begin{aligned}1 \times 2 &= \frac{1}{3} (1 \times 2 \times 3), \\1 \times 2 + 2 \times 3 &= \frac{1}{3} (2 \times 3 \times 4), \\1 \times 2 + 2 \times 3 + 3 \times 4 &= \frac{1}{3} (3 \times 4 \times 5), \\&\dots\end{aligned}$$

- (i) Formulate a conjecture for the n^{th} equation in the sequence.

- (ii) Verify your conjecture for $n = 4$.

(2)

- (b) A sequence of numbers has the n^{th} term given by $u_n = 2^n + 3$, $n \in \mathbb{Z}^+$. Bill conjectures that all members of the sequence are prime numbers. Show that Bill's conjecture is false.

(2)

- (c) Use mathematical induction to prove that $5 \times 7^n + 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

(6)

(Total 10 marks)

4. (a) The sum of the first six terms of an arithmetic series is 81. The sum of its first eleven terms is 231. Find the first term and the common difference. (6)
- (b) The sum of the first two terms of a geometric series is 1 and the sum of its first four terms is 5. If all of its terms are positive, find the first term and the common ratio. (5)
- (c) The r^{th} term of a new series is defined as the product of the r^{th} term of the arithmetic series and the r^{th} term of the geometric series above. Show that the r^{th} term of this new series is $(r + 1)2^{r-1}$. (3)
- (d) Using mathematical induction, prove that

$$\sum_{r=1}^n (r+1)2^{r-1} = n2^n, n \in \mathbb{Z}^+.$$

(7)
(Total 21 marks)