Question 1:
Calculator Allowed: Yes

[Maximum mark: 8]

It is known that the number of fish in a given lake will decrease by 7% each year unless some new fish are added. At the end of each year, 250 new fish are added to the lake. At the start of 2018, there are 2500 fish in the lake.

(a) Show that there will be approximately 2645 fish in the lake at the start of 2020. [3]

(b) Find the approximate number of fish in the lake at the start of 2042. [5]

Question 2:
Calculator Allowed: Yes

5. [Maximum mark: 9]

The sum of the first n terms of a geometric sequence is given by $S_n = \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, u_1. [2]

(b) Find S_∞. [3]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

Question 3:
Calculator Allowed: Yes

3. [Maximum mark: 5]

A geometric sequence has a first term of 50 and a fourth term of 86.4.

The sum of the first n terms of the sequence is S_n.

Find the smallest value of n such that $S_n > 33\,500$.

Question 4:
Calculator Allowed: Yes

[Maximum mark: 18]

On the day of her birth, 1st January 1998, Mary's grandparents invested $\$x$ in a savings account. They continued to deposit $\$x$ on the first day of each month thereafter.
The account paid a fixed rate of 0.4% interest per month. The interest was calculated on the last day of each month and added to the account.
Let $\$A_n$ be the amount in Mary's account on the last day of the nth month, immediately after the interest had been added.

(a) Find an expression for A_1 and show that $A_2 = 1.004^2x + 1.004x$. [2]

(b) (i) Write down a similar expression for A_3 and A_4.

(ii) Hence show that the amount in Mary's account the day before she turned 10 years old is given by $251(1.004^{120} - 1)x$. [6]

(c) Write down an expression for A_n in terms of x on the day before Mary turned 18 years old showing clearly the value of n. [1]

(d) Mary's grandparents wished for the amount in her account to be at least $\$20\,000$ the day before she was 18. Determine the minimum value of the monthly deposit $\$x$ required to achieve this. Give your answer correct to the nearest dollar. [4]

(e) As soon as Mary was 18 she decided to invest $\$15\,000$ of this money in an account of the same type earning 0.4% interest per month. She withdraws $\$1000$ every year on her birthday to buy herself a present. Determine how long it will take until there is no money in the account. [5]

Question 5:
Calculator Allowed: Yes

10. *[Maximum mark: 6]*

Find the set of values of x for which $\left|0.1x^2 - 2x + 3\right| < \log_{10} x$.

Question 6:
Calculator Allowed: No

Let $\{u_n\}$, $n \in \mathbb{Z}^+$, be an arithmetic sequence with first term equal to a and common difference of d, where $d \neq 0$. Let another sequence $\{v_n\}$, $n \in \mathbb{Z}^+$, be defined by $v_n = 2^{u_n}$.

(a) (i) Show that $\frac{v_{n+1}}{v_n}$ is a constant.

(ii) Write down the first term of the sequence $\{v_n\}$.

(iii) Write down a formula for v_n in terms of a, d and n. [4]

Let S_n be the sum of the first n terms of the sequence $\{v_n\}$.

(b) (i) Find S_n, in terms of a, d and n.

(ii) Find the values of d for which $\sum_{i=1}^{\infty} v_i$ exists.

You are now told that $\sum_{i=1}^{\infty} v_i$ does exist and is denoted by S_∞.

(iii) Write down S_∞ in terms of a and d.

(iv) Given that $S_\infty = 2^{a+1}$ find the value of d. [8]

Let $\{w_n\}$, $n \in \mathbb{Z}^+$, be a geometric sequence with first term equal to p and common ratio q, where p and q are both greater than zero. Let another sequence $\{z_n\}$ be defined by $z_n = \ln w_n$.

(c) Find $\sum_{i=1}^{n} z_i$ giving your answer in the form $\ln k$ with k in terms of n, p and q. [6]

Question 7:
Calculator Allowed: No

8. [Maximum mark: 6]

Consider integers a and b such that $a^2 + b^2$ is exactly divisible by 4. Prove by contradiction that a and b cannot both be odd.

Question 8:
Calculator Allowed: No

[Maximum mark: 5]

Prove by contradiction that the equation $2x^3 + 6x + 1 = 0$ has no integer roots.

Question 9:
Calculator Allowed: No

9. [Maximum mark: 7]

Consider the expression $\frac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q}$, $a \neq 0$.

The binomial expansion of this expression, in ascending powers of x, as far as the term in x^2 is $4bx + bx^2$, where $b \in \mathbb{Q}$.

(a) Find the value of a and the value of b. [6]

(b) State the restriction which must be placed on x for this expansion to be valid. [1]

Question 10:
Calculator Allowed: Yes

[Maximum mark: 10]

A function f is defined by $f(x) = (x+1)(x-1)(x-5)$, $x \in \mathbb{R}$.

(a) Find the values of x for which $f(x) < |f(x)|$. [3]

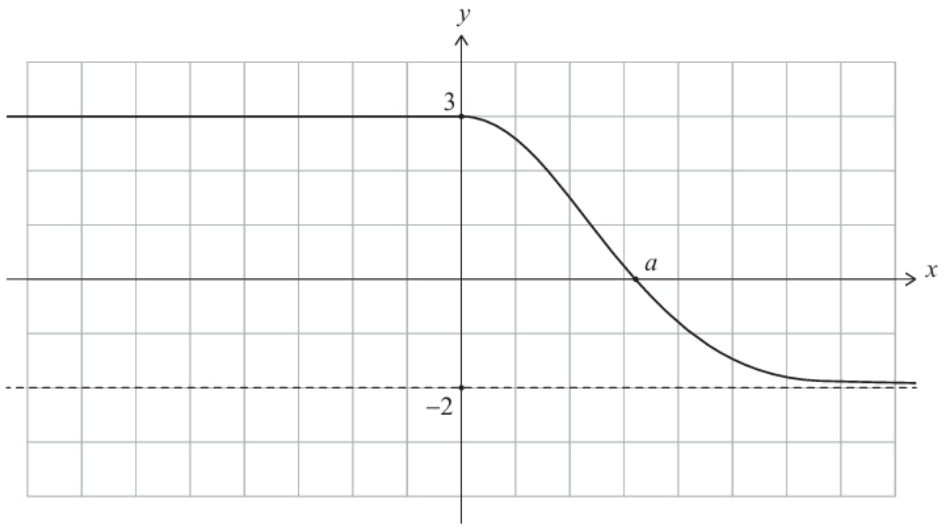
A function g is defined by $g(x) = x^2 + x - 6$, $x \in \mathbb{R}$.

(b) Find the values of x for which $g(x) < \frac{1}{g(x)}$. [7]

Question 11:
Calculator Allowed: No

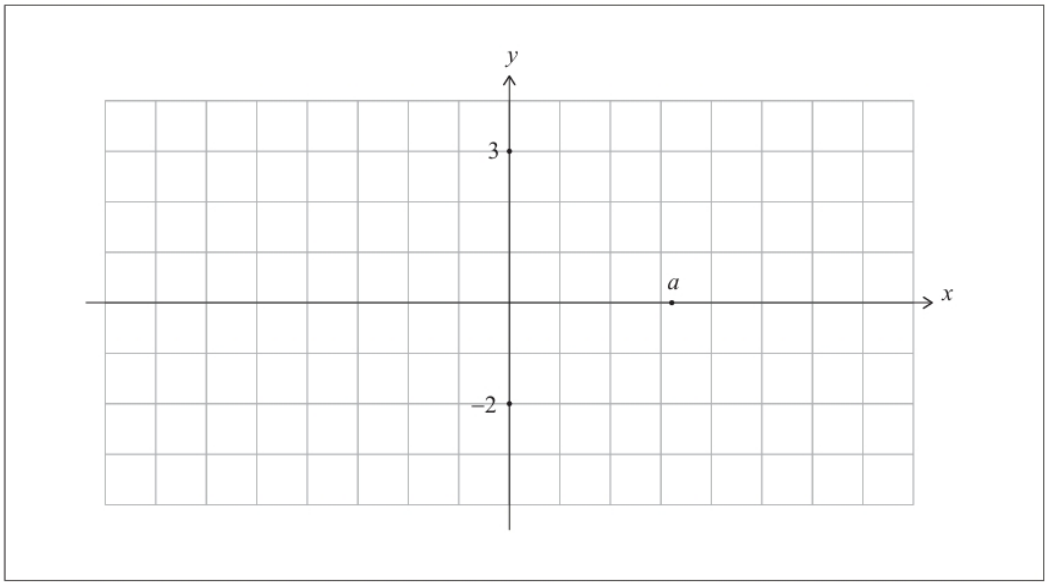
8. [Maximum mark: 7]

Part of the graph of a function, f, is shown in the following diagram. The graph of $y = f(x)$ has a y-intercept at $(0, 3)$, an x-intercept at $(a, 0)$ and a horizontal asymptote $y = -2$.



Consider the function $g(x) = |f(|x|)|$.

(a) On the following grid, sketch the graph of $y = g(x)$, labelling any axis intercepts and giving the equation of the asymptote. [4]



(b) Find the possible values of k such that $(g(x))^2 = k$ has exactly two solutions. [3]

Question 12:

Calculator Allowed: No

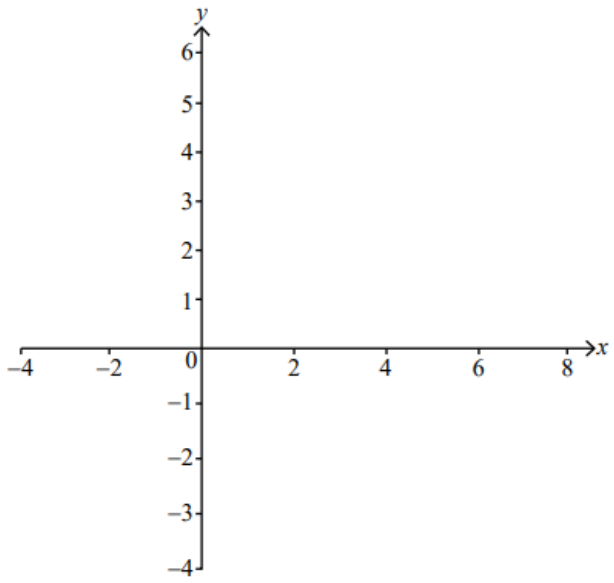
[Maximum mark: 8]

(a) Sketch on the same axes the curve $y = \left|\frac{7}{x-4}\right|$ and the line $y = x + 2$, clearly indicating any axes intercepts and any asymptotes. [3]

(b) Find the exact solutions to the equation $x + 2 = \left|\frac{7}{x-4}\right|$. [5]

Question 13:
Calculator Allowed: No

(a) Sketch the graphs of $y = \frac{x}{2} + 1$ and $y = |x - 2|$ on the following axes. [3]



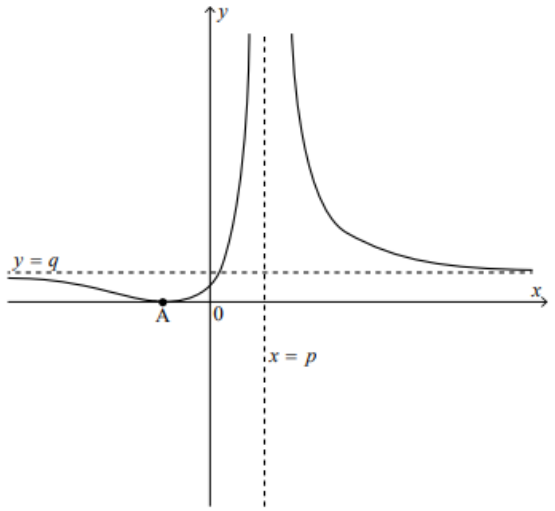
(b) Solve the equation $\frac{x}{2} + 1 = |x - 2|$. [4]

Question 14:
Calculator Allowed: Yes

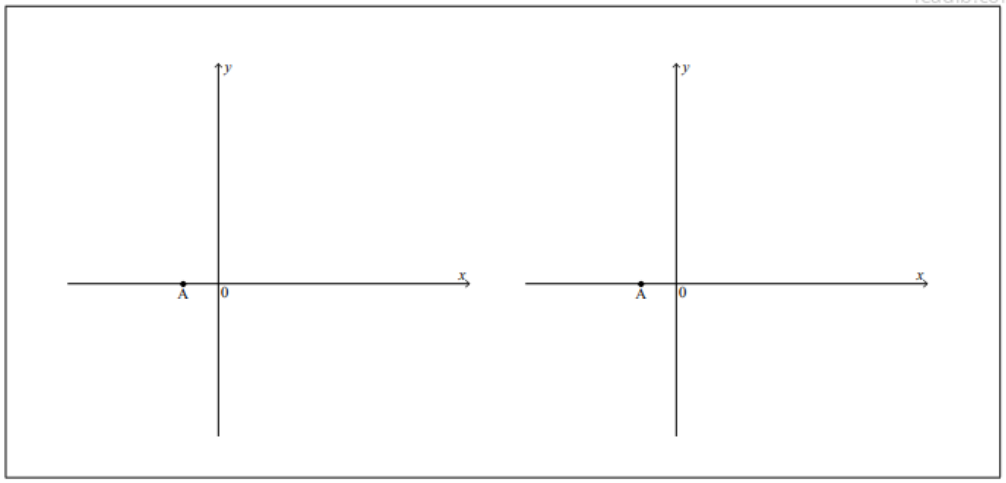
[Maximum mark: 8]

Consider the function $f(x) = \frac{ax+1}{bx+c}$, $x \neq -\frac{c}{b}$, where $a, b, c \in \mathbb{Z}$.

The following graph shows the curve $y = (f(x))^2$. It has asymptotes at $x = p$ and $y = q$ and meets the x-axis at A.



(a) On the following axes, sketch the two possible graphs of $y = f(x)$ giving the equations of any asymptotes in terms of p and q. [4]



(b) Given that $p = \frac{4}{3}$, $q = \frac{4}{9}$ and A has coordinates $\left(-\frac{1}{2}, 0\right)$, determine the possible sets of values for a, b and c. [4]