Topic:
Total Time:
Name of student

General Instructions

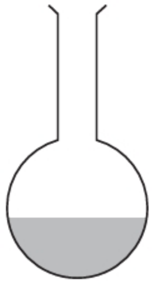
- On the first page, write your **name**, **start time**, and **end time** clearly.
- Submit your answers as a **single PDF** file.
- During the test, stay **live on Zoom** and **share your entire desktop**.
- If a question hasn't been covered in class or is outside the syllabus, just skip it and write **'NA'**. It won't affect your grade.
- Unless mentioned otherwise, give all numerical answers **exactly** or correct to **three significant figures**.
- You may use the **official IB formula booklet** during the test.

Total Questions: 6
Total Marks: 71

Question 1:
Calculator Allowed: Yes

8. [Maximum mark: 6]

The following diagram shows liquid in a round-bottomed glass flask, which is made of a sphere and a cylindrical neck.



Initially, the flask is empty. Liquid is poured into the flask at a rate of $2\,\mathrm{cm}^3\mathrm{s}^{-1}$. You may assume that the liquid does not reach the cylindrical neck.

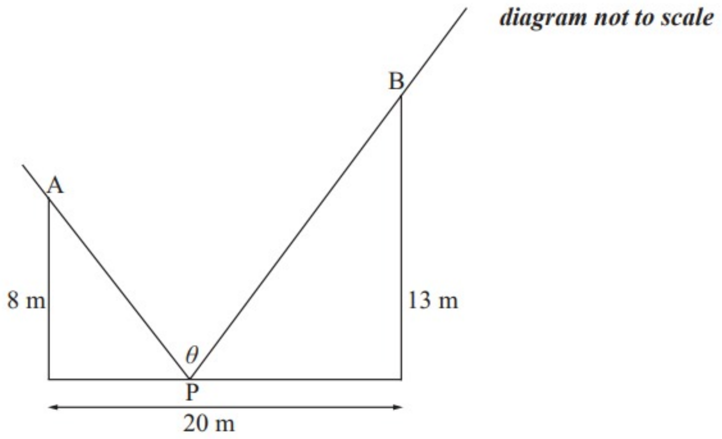
The volume $V\,\mathrm{cm}^3$ and the height $h\,\mathrm{cm}$ of the liquid in the flask satisfy the equation $V = 5\pi h^2 - \frac{1}{3}\pi h^3$.

Find the rate of change of the height of the liquid in the flask at the instant when the volume of the liquid is $200\,\mathrm{cm}^3$.

Question 2:
Calculator Allowed: Yes

[Maximum mark: 19]

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle θ where $\theta = \mathrm{A}\hat{\mathrm{P}}\mathrm{B}$, as shown in the diagram.



(a) Find an expression for θ in terms of x, where x is the distance of P from the base of the wall of height 8m. *[2 marks]*

(b) (i) Calculate the value of θ when $x = 0$.

(ii) Calculate the value of θ when $x = 20$. *[2 marks]*

(c) Sketch the graph of θ, for $0 \leq x \leq 20$. *[2 marks]*

(d) Show that $\dfrac{d\theta}{dx} = \dfrac{5(744 - 64x - x^2)}{(x^2 + 64)(x^2 - 40x + 569)}$. *[6 marks]*

(e) Using the result in part (d), or otherwise, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures. *[3 marks]*

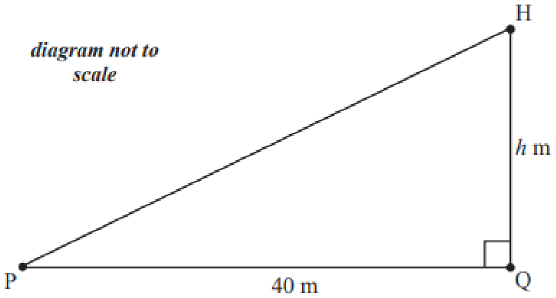
(f) The point P moves across the street with speed $0.5\ \mathrm{ms}^{-1}$. Determine the rate of change of θ with respect to time when P is at the midpoint of the street. *[4 marks]*

Question 3:

Calculator Allowed: Yes

10. *[Maximum mark: 7]*

A helicopter H is moving vertically upwards with a speed of 10 m s^{-1}. The helicopter is h m directly above the point Q which is situated on level ground. The helicopter is observed from the point P which is also at ground level and $PQ = 40$ m. This information is represented in the diagram below.



When $h = 30$,

(a) show that the rate of change of $\text{H}\hat{\text{P}}\text{Q}$ is 0.16 radians per second; *[3 marks]*

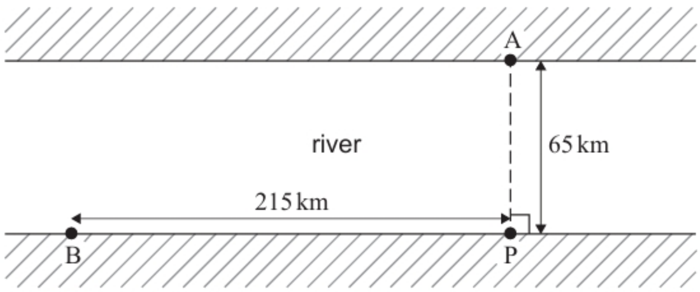
(b) find the rate of change of PH. *[4 marks]*

Question 4:
Calculator Allowed: Yes

7. [Maximum mark: 14]

Points A and P lie on opposite banks of a river, such that AP is the shortest distance across the river. Point B represents the centre of a city which is located on the riverbank. $PB = 215 \text{ km}$, $AP = 65 \text{ km}$ and $\text{A}\hat{\text{P}}\text{B} = 90^\circ$.

The following diagram shows this information.



A boat travels at an average speed of $42\,\mathrm{km\,h^{-1}}$. A bus travels along the straight road between P and B at an average speed of $84\,\mathrm{km\,h^{-1}}$.

(a) Find the travel time, in hours, from A to B given that

(i) the boat is taken from A to P, and the bus from P to B;

(ii) the boat travels directly to B. [4]

There is a point D, which lies on the road from P to B, such that $\mathrm{BD} = x\,\mathrm{km}$. The boat travels from A to D, and the bus travels from D to B.

(b) (i) Find an expression, in terms of x for the travel time T, from A to B, passing through D.

(ii) Find the value of x so that T is a minimum.

(iii) Write down the minimum value of T. [6]

(c) An excursion involves renting the boat and the bus. The cost to rent the boat is $\$\,200$ per hour, and the cost to rent the bus is $\$\,150$ per hour.

(i) Find the new value of x so that the total cost C to travel from A to B via D is a minimum.

(ii) Write down the minimum total cost for this journey. [4]

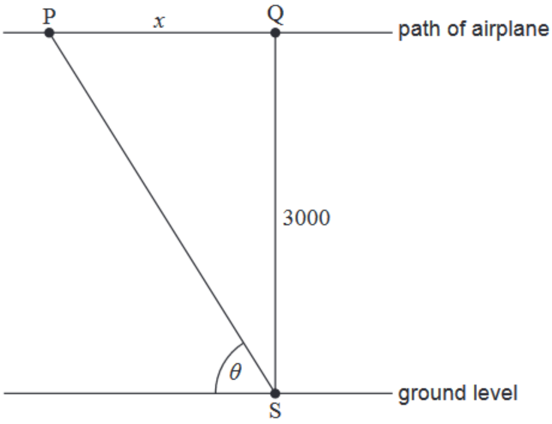
Question 5:

Calculator Allowed: Yes

17. [Maximum mark: 9]

An airplane, P, is flying at a constant altitude of $3000\,\mathrm{m}$ at a speed of $250\,\mathrm{m\,s^{-1}}$. Its path passes over a tracking station, S, at ground level. Let Q be the point $3000\,\mathrm{m}$ directly above the tracking station.

At a particular time, T, as the airplane is flying towards Q, the angle of elevation, θ, of the airplane from S is increasing at a rate of 0.075 radians per second. The distance from Q to P is given by x.



(a) Use related rates to show that, at time T, $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -\dfrac{10\,000}{3}$. [2]

(b) Find $x(\theta)$, x as a function of θ. [1]

(c) Find an expression for $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ in terms of $\sin\theta$. [3]

(d) Hence find the horizontal distance from the station to the plane at time T. [3]

Question 6:

Calculator Allowed: Yes

5. [Maximum mark: 16]

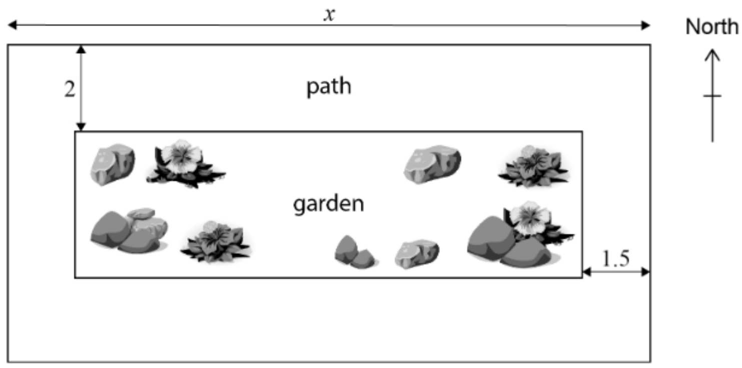
A particular park consists of a rectangular garden, of area $A\,\text{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\text{m}^2$.

The width of the path at the north and south side of the park is $2\,\text{m}$.

The width of the path at the west and east side of the park is $1.5\,\text{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a) (i) Write down the length of the garden in terms of x.

(ii) Find an expression for the width of the garden in terms of x.

(iii) Hence show that $A = 1212 - 4x - \dfrac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\text{m}^2$. [4]

(c) Find an expression for $\dfrac{\text{d}A}{\text{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]