CS 205

Lecture # 6

19-01-21

NFA $\quad M = (Q, \Sigma, \delta, q_0, F)$

$\delta : Q \times \Sigma \to 2^Q$

Extend $\delta : Q \times \Sigma \to 2^Q$

to $\hat{\delta} : Q \times \Sigma^* \to 2^Q$

Inductive defn.

$\hat{\delta}(q, \epsilon) = \{q\}$

$$\boxed{\begin{aligned}\hat{\delta}(q, ua) &= \bigcup_{r \in \hat{\delta}(q,u)} \delta(r, a) \\ &= \{p \mid \exists r \in \hat{\delta}(q,u) \text{ and } p \in \delta(r,a)\}\end{aligned}} \qquad u \in \Sigma^*, \; a \in \Sigma$$

$\hat{\delta}(q, w)$: set of all states that NFA M can possibly reach after reading the input string w starting from state q

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Def M accepts w if $\underbrace{\delta(q_0, w)}_{\text{set of states}} \cap F \neq \emptyset$

$L(M) = \{w \mid \delta(q_0, w) \cap F \neq \emptyset\}$

Expressive Power of DFAs & NFAs

Prop For every DFA M there is an NFA N s.t. $L(M) = L(N)$.

Proof : Easy $\quad$ DFA $\delta_M(q, a) = q'$

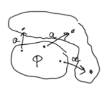
NFA $\delta_N(q, a) = \{q'\}$

Prop : For every NFA N there is a DFA M s.t. $L(M) = L(N)$.

Proof : (Subset Construction / Powerset construction)

Let $N = (Q_N, \Sigma, \delta_N, q_{0N}, F_N)$ be an NFA.

Let $M = (\underbrace{2^{Q_N}}_{Q_M}, \Sigma, \delta_M, q_{0M}, F_M)$ be a DFA



where $\boxed{\delta_M(P, a) = \bigcup_{q \in P} \underbrace{\delta_N(q, a)}_{\text{set}}} \in Q_M$

$P \subseteq Q_N$

$\in Q_M$

$q_{0M} \stackrel{?}{=} \{q_{0N}\}$

$F_M \stackrel{?}{=} \{P \subseteq Q_N \mid P \cap F_N \neq \emptyset\}$

$P \in Q_M$

Any subset of states of N that contains a final state of N is a final state of M.

Claim $L(M) = L(N)$.

We will show a stronger result ;

$$\boxed{\forall w. \; \hat{\delta}_M(\{q_0\}, w) = \hat{\delta}_N(q_0, w).}$$

Proof: By induction on $|w|$.

Base Case $|w| = 0$, i.e. $w = \epsilon$.

LHS $= \hat{\delta}_M(\{q_0\}, \epsilon) = \{q_0\}$ $\quad$ By defn. of $\hat{\delta}$ for DFAs

RHS $= \hat{\delta}_N(q_0, \epsilon) = \{q_0\}$

Inductive Case : $\boxed{\text{Assume that the stmt. holds for all } w \text{ with } |w| = n.}$ IH

For any $w' = wa$ where $|w| = n$, $a \in \Sigma$

$$\begin{aligned}
\text{LHS} = \hat{\delta}_M(\{q_0\}, wa) &= \delta_M(\hat{\delta}_M(\{q_0\}, w), a) && \text{by defn of } \hat{\delta} \text{ for DFAs} \\
&= \delta_M(\underbrace{\hat{\delta}_N(q_0, w)}_{\text{set of states in } N}, a) && \text{by IH} \\
&= \bigcup_{q \in \hat{\delta}_N(q_0, w)} \delta_N(q, a) && \text{by defn. of } \delta_M \\
&= \hat{\delta}_N(q_0, wa) && \text{by defn. of } \hat{\delta}_N \\
&= \text{RHS}
\end{aligned}$$

$$\begin{aligned}
w \in L(M) \text{ iff } & \hat{\delta}_M(\{q_0\}, w) \in F_M && \text{defn of acceptance by DFA } M \\
\text{iff } & \hat{\delta}_N(q_0, w) \in F_M && \text{by the above result} \\
\text{iff } & \hat{\delta}_N(q_0, w) \cap F_N \neq \emptyset && \text{by defn. of } F_M \\
\text{iff } & w \in L(N) && \text{by defn. of acceptance by NFA } N
\end{aligned}$$

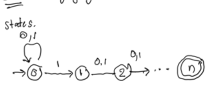
Q Why use NFAs at all ?

A Often the NFA is much smaller.

Ex $L_n = \{w \in \{0,1\}^* \mid$ the n^{th} letter from the end of w is a $1\}$

Any DFA recognizing L_n must have at least 2^n states.

NFA :



Succinct automata