CS 205
Lecture 12
8/2/21

Pumping Lemma : If L is regular then there is a constant n s.t. $\forall w \in L$ with $|w| \geq n$, $\exists x, y, z \in \Sigma^*$ s.t.

1. $w = xyz$
2. $y \neq \varepsilon$
3. $|xy| \leq n$
4. $\forall k \geq 0.\ xy^kz \in L$

Ex 4 $L = \{0^i \mid i \text{ prime}\}$ is not regular.

Proof : Suppose L is regular. Let n be the constant from the PL.

Consider $w = 0^m \in L$ where $m \geq \boxed{n+2} \xrightarrow{} \text{why?}$ and m is prime.

By the PL there exist x, y, z s.t. (1) ~ (4) are satisfied. Since $|xy| \leq n$, $x = 0^r$, $y = 0^s$ and $z = 0^t$, where $r, t \geq 0$ and $s \geq 1$.

Now $xyz = 0^r \underbrace{0^s}_{y} 0^t$ Pump y $(r+t)$ times.

So $xy^{r+t}z = 0^r (0^s)^{r+t} 0^t = 0^{(r+t)(s+1)}$

Now $s+1 > 1$ since $s \geq 1$.

$r+t = m - s \geq 2$ since $m \geq n+2$ and $s \leq n$ since $|xy| \leq n$.

Hence $(r+t)(s+1)$ is not a prime. Contradiction.

Ex 4 $L = \{xx \mid x \in \{0,1\}^*\}$ is not regular.

Proof : Suppose L is not regular, Let n be the constant in the PL.

Choice of w :

1. $w = 0^n 0^n$ ✗ Bad choice
2. $w = (01)^n (01)^n$ ✗ Bad choice
3. $w = 0^n 1 0^n 1$ ✓ Will work.

Closure Properties of Regular Languages

If L_1 and L_2 are regular then so are

✓✓(1) $L_1 \cup L_2$ (Immediate : Suppose $L_1 = L(r_1)$, $L_2 = L(r_2)$. Then $L_1 \cup L_2 = L(r_1 + r_2)$)

✓(2) $L_1 \cap L_2$

✓(3) $\bar{L_1} \triangleq \Sigma^* \setminus L_1$

✓(4) $L_1 \setminus L_2$

(5) Reversal ; L_1^R

✓✓(6) Kleene cloure : L_1^*

✓✓(7) $L_1 L_2$

Complement : Suppose L is regular and the DFA M accepts L where $M = (Q, \Sigma, \delta, q_0, F)$.

Then the DFA $\hat{M} = (Q, \Sigma, \delta, q_0, Q \setminus F)$.

Will it work for an NFA M?

Intersection $L_1 \cap L_2 = \overline{(\bar{L_1} \cup \bar{L_2})}$ de Morgan

Direct Construction : Product construction

Given DFA $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$

and $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$

define DFA $M = (Q_1 \times Q_2, \Sigma, \delta, (q_{01}, q_{02}), F_1 \times F_2)$

where $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$

Set difference $L_1 \setminus L_2 = L_1 \cap \bar{L_2}$.

Reversal $L^R = \{w \in \Sigma^* \mid \text{the reverse } w^R \text{ of } w \text{ belongs to } L\}$

Exercise : Starting with a DFA M that accepts L, define an ϵ-NFA M^R that accepts L^R.

Homomorphisms

Def : A homomorphism is a function $h : \Sigma \to \Delta^*$ that replaces a symbol from Σ by a string over Δ.

Fact :

1. Any homomorphism $h : \Sigma \to \Delta^*$ can be extended to a function $\bar{h} : \Sigma^* \to \Delta^*$ by
   (i) $\bar{h}(\varepsilon) = \epsilon$
   (ii) $\bar{h}(ax) = h(a)\bar{h}(x)$ $a \in \Sigma$, $x \in \Sigma^*$

   i.e. $h(a_1 \ldots a_n) = h(a_1) h(a_2) \ldots h(a_n)$
2. Can extend h to languages :

   $\underbrace{h(L)}_{\text{homomorphic image of language } L} = \{h(w) \mid w \in L\}$ where $L \subseteq \Sigma^*$

Ex $\Sigma = \{0,1\}$ $\Delta = \{a,b\}$

$h(0) = ab$ $h(1) = \varepsilon$ homomorphism from Σ to Δ

$h(\underbrace{10^*1}_{\text{language over } \Sigma}) = \underbrace{(ab)^*}_{\text{language over } \Delta}$