CS 205
Lecture 14
10/2/21

Decision Problems for Regular Languages

(1) Emptiness of $L(M)$: Check whether any final state is reachable.

Algorithm : BFS or DFS Complexity $O(m+n) \in O(n^2)$
m : # edges (transitions)
n : # vertices (states)

(2) Finiteness of $L(M)$

(1) Run BFS/DFS and delete all unreachable states and all states from which one cannot reach a final state (How ?)

(2) DFA M accepts an infinite language iff the resulting transition diagram has the initial state, at least one final state and a cycle.

(3) Equivalence of DFA If $L_1 = L(M_1)$ and $L_2 = L(M_2)$
then $L_1 = L_2 \Leftrightarrow (L_1 \cap \bar{L}_2) \cup (L_2 \cap \bar{L}_1) = \emptyset$

Myhill - Nerode Theorem

Recap Equivalence relations : $R \subseteq S \times S$ reflexive, symmetric, transitive
xRx $xRy \Rightarrow yRx$ $xRy \wedge yRz \Rightarrow xRz$
for all $x, y, z \in S$

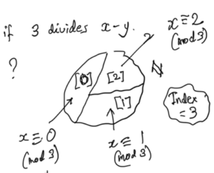
Equivalence class of $x \in S$: $[x] = \{y \mid xRy\}$

Set of equivalence classes of S under R :
$$S/R = \{[x] \mid x \in S\}$$

Notation $\equiv$, $\cong$

Ex (1) Congruence modulo 3
$x \equiv y \pmod 3$ if 3 divides $x - y$.

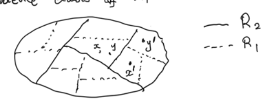
Equivalence classes ?



Def : Given an equivalence relation $\equiv$ on S, the index of $\equiv$ is the cardinality of its equivalence classes (i.e. $|S/\equiv|$)

Def : Given two equivalence relations R_1 and R_2 on S, R_1 is a refinement of R_2 if
$\forall x, y \in S.\ xR_1y \Rightarrow xR_2y$.

Fact : Every equivalence class of R_2 is a union of equivalence classes of R_1.



We define two equivalence on Σ^* as follows.

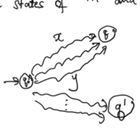
(1) Given an arbitrary $L \subseteq \Sigma^*$ (L need not be regular)
define $R_L \subseteq \Sigma^* \times \Sigma^*$ by
$$\boxed{xR_Ly \text{ iff } \forall z \in \Sigma^*.\ (xz \in L \Leftrightarrow yz \in L)}$$
We will show that L is regular iff the index of R_L is finite.

(2) Given a DFA $M = (Q, \Sigma, \delta, q_0, F)$
define $R_M \subseteq \Sigma^* \times \Sigma^*$ by
$$\boxed{xR_My \text{ iff } \delta(q_0, x) = \delta(q_0, y)}$$

Facts

(1) There is a one-to-one correspondence (bijection) between the reachable states of M and the equivalence classes of R_M.

DFA M



xR_My

$$L(M) = \bigcup_{\delta(q_0, x) \in F} [x]$$

(2) If xR_My then xzR_Myz for all $z \in \Sigma^*$
Since $\delta(q_0, xz) = \delta(\delta(q_0, x), z) = \delta(\delta(q_0, y), z)$
$= \delta(q_0, yz)$

Def : An equivalence relation R on Σ^* satisfying
$xRy \Rightarrow xzRyz$ for all $z \in \Sigma^*$
is said to be right-invariant.

Item (2) above says R_M is right-invariant.