CS 205

Lecture 15

16-02-21

Myhill - Nerode Theorem

Recap

- For any $L \subseteq \Sigma^*$, $R_L \subseteq \Sigma^* \times \Sigma^*$ is an equivalence relation, where R_L is defined by:

$$\boxed{x R_L y \text{ iff } xz \in L \Leftrightarrow yz \in L}$$

- For for DFA $M = (Q, \Sigma, \delta, q_0, F)$ define the equivalence relation $R_M \subseteq \Sigma^* \times \Sigma^*$ by:

$$\boxed{x R_M y \text{ iff } \delta(q_0, x) = \delta(q_0, y)}$$

- Reachable states of $M \Longleftrightarrow$ equivalence classes of R_M

- Right-invariant equivalence relation R on Σ^*:

$$xRy \Rightarrow xzRyz \text{ for all } z \in \Sigma^*.$$

Ex (1) R_M is right-invariant : $\delta(q_0, x) = \delta(q_0, y)$

$\Rightarrow \delta(q_0, xz) = \delta(q_0, yz)$

(2) R_L is also invariant (proved later)

- Index of an equivalence relation : cardinality of the set of equiv. classes

- Refinement of equivalence relations: R_1 refines R_2 if $xR_1y \Rightarrow xR_2y$

i.e. every equiv. class of R_1 is contained in an equiv. class of R_2

Thm (Myhill - Nerode) The following three statements about any $L \subseteq \Sigma^*$ are equivalent:

(1) L is regular, i.e. it is accepted by some DFA M.

(2) L is union of some of the equivalence classes of a right invariant equivalence reln. of finite index.

(3) R_L is of finite index.

Proof : We show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2) We know that R_M is right invariant and it has finite index.

$$L = \bigcup_{\delta(q_0, x) \in F} [x]$$

Recall xR_My iff $\delta(q_0, x) = \delta(q_0, y)$

(2) $\Rightarrow$ (3) : We show that any equiv. reln. E satisfying (2) is a refinement of R_L. (In particular, since R_M satisfies (2) R_M refines R_L). Hence R_L is of finite index.

Assume xEy. Since E is right invariant, $xzEyz$ for all $z \in \Sigma^*$. Since L is the union of some of the equiv. classes of E, this implies $xz \in L$ iff $yz \in L$, i.e., xR_Ly. Thus, E is a refinement of R_L and hence R_L is of finite index.

(3) $\Rightarrow$ (1) : We first show that R_L is right invariant.

Suppose xR_Ly and $w \in \Sigma^*$. We have to show that xwR_Lyw, i.e., $xwz \in L \Leftrightarrow ywz \in L$.

Now take $v = wz$. By the defn. of R_L, $xv \in L \Leftrightarrow yv \in L$. $\boxed{\text{Hence } R_L \text{ is right-invariant}}$

We now define a DFA M_L from R_L as follows.

Let Q_L be the finite set of equivalence classes of R_L. Let $[x]_L$ be the element of Q_L containing $x \in \Sigma^*$.

Define $\boxed{\delta_L([x]_L, a) = [xa]_L}$.

Need to show that the defn. is independent of the choice of x, i.e., if $[x]_L = [y]_L$ then $[xa]_L = [ya]_L$.

But this follows from the right-invariance of R_L (simply take $z = a$).

Define $\boxed{\begin{aligned} q_{0L} &= [\varepsilon]_L \text{ and} \\ F_L &= \{[x]_L \mid x \in L\} \end{aligned}}$

Claim : The DFA $M_L = (Q_L, \Sigma, \delta_L, q_{0L}, F_L)$ accepts L, since

$$\begin{aligned} \delta_L(q_0, x) &= \delta_L([\varepsilon]_L, x) \quad \text{by defn. of } q_0 \\ &= [\varepsilon x]_L \\ &= [x]_L \end{aligned}$$

and thus

$$\begin{aligned} x \in L(M_L) &\text{ iff } [x]_L \in F_L \\ &\text{ iff } x \in L. \end{aligned}$$