CS 205

Lecture 13

9/2/21

Homomorphism $h : \Sigma \to \Delta^*$

can be extended to $h : \Sigma^* \to \Delta^*$ $\quad h(a_1 \ldots a_n) = h(a_1)\,h(a_2) \ldots h(a_n)$

can be extended to $h : 2^{\Sigma^*} \to 2^{\Delta^*}$ $\quad h(L) = \{h(w) \mid w \in L\}$

Thm If $L \subseteq \Sigma^*$ is regular and $h : \Sigma \to \Delta^*$ is a homomorphism then $h(L)$ is also regular.

Proof : Let $L = L(r)$ for some r.e. r.

If e is a r.e. over Σ, let $h(e)$ be the r.e. over Δ obtained by replacing each $a \in \Sigma$ by $h(a)$.

Claim :
1. $h(e)$ is a r.e. over Δ
2. $L(h(e)) = h(L)$

The proof is by structural induction on r.

Inverse homomorphism

Let $h : \Sigma \to \Delta^*$ be a homomorphism and $L \subseteq \Delta^*$.

Define $h^{-1}(L) = \{w \in \Sigma^* \mid h(w) \in L\}$.

Fact : $w \in h^{-1}(L)$ iff $h(w) \in L$.

Ex Let $L = (00+1)^*$ $\quad \Sigma = \{a, b\}$ $\quad \Delta = \{0, 1\}$

$h : \Sigma \to \Delta^*$ homom. defined by $h(a) = 01$ and $h(b) = 10$

Then $h^{-1}(L) \stackrel{?}{=} (ba)^*$

Not the image of any elt in Σ^* under h

$L =$	$\{\varepsilon,$	$1,$	$11,$	$111,$	$00,$	$001,$	$100,$	$1001,$	$0110,$
$h^{-1} \downarrow$	$\{\varepsilon\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\{ba\}$	$\emptyset$

$h^{-1}(L)$ $\quad \ldots\}$

Thm If $h : \Sigma \to \Delta^*$ is a homomorphism and $L \subseteq \Delta^*$ is regular then $h^{-1}(L)$ is also regular.

Proof : Let $L = L(M)$ for DFA $M = (Q, \Delta, \delta_M, q_0, F)$.

Define a DFA $N = (Q, \Sigma, \delta_N, q_0, F)$

where δ_N is defined by

$$\delta_N(q, a) \stackrel{?}{=} \hat{\delta}_M(q, \underbrace{h(a)}_{\in \Delta^*})$$

By induction on $|w|$, can show $\hat{\delta}_N(q_0, w) = \hat{\delta}_M(q_0, h(w))$.

Since the final states are the same in M and N

$w \in L(N)$ iff $h(w) \in L(M) = L$

i.e., $L(N) = h^{-1}(L)$.

Decision Problems of Regular Languages

Thm : The set of strings accepted by a DFA M with n states is

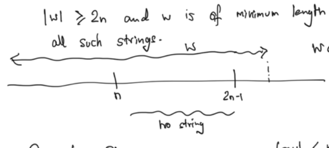
✓ (1) nonempty iff M accepts a string of length $< n$.

(2) infinite iff M accepts some string of length ℓ where $n \le \ell < 2n$.

Proof of (2) (Idea)

($\Leftarrow$) If $w \in L(M)$ and $n \le |w| < 2n$ by the P.L. $\exists x, y, z \in \Sigma^*$. $|y| \ne 0$, $w = xyz$ and $xy^{K}z \in L$ for all $K \geq 0$. Hence L is infinite.

($\Rightarrow$) Suppose $L(M)$ is infinite. So there is a $w \in L$ s.t. $|w| \geq n$. If $|w| < 2n$ we are done. Assume there is no string in $L(M)$ of length between n and $2n-1$ and let $w \in L(M)$ be such that $|w| \geq 2n$ and w is of minimum length among all such strings.



By the PL $w = xyz$ $\quad |xy| \le n$, $|y| \geq 1$, $xy^K z \in L$ for all $K \geq 0$.

Take $K = 0$

$\boxed{\text{Claim } \; n \le |xz| < 2n.}$

Since $xz \in L$ this is a contradiction.