Stieltjes constants appearing in the Laurent expansion of the hyperharmonic zeta function

Mümün Can, Ayhan Dil and Levent Kargın

Department of Mathematics, Akdeniz University, TR-07058 Antalya, Turkey

mcan@akdeniz.edu.tr, adil@akdeniz.edu.tr, lkargin@akdeniz.edu.tr,

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Abstract

In this paper, we consider meromorphic extension of the function

$$\zeta_{h^{(r)}}(s) = \sum_{k=1}^{\infty} \frac{h_k^{(r)}}{k^s}, \ \ \mathrm{Re}(s) > r,$$

(which we call *hyperharmonic zeta function*) where $h_n^{(r)}$ are the hyperharmonic numbers. We establish certain constants, denoted $\gamma_{h^{(r)}}(m)$, which naturally occur in the Laurent expansion of $\zeta_{h^{(r)}}(s)$. Moreover, we show that the constants $\gamma_{h^{(r)}}(m)$ and integrals involving generalized exponential integral can be written as a finite combination of some special constants.

1   Introduction

The Riemann zeta function which is initially defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ \ \mathrm{Re}(s) > 1,$$

has an analytic continuation to the whole complex s plane except for a simple pole $s = 1$ with residue 1. It is well known that around this simple pole, $\zeta(s)$ has the following Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{m=0}^{\infty} (-1)^m \frac{\gamma(m)}{m!} (s-1)^m, \tag{1}$$

where the coefficients $\gamma(m)$ are called *Stieltjes constants*. It is shown by various authors that these constants can be alternatively presented by the limit

$$\gamma(m) = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{\ln^m k}{k} - \int_1^n \frac{\ln^m x}{x} dx \right), \ \ m = 0, 1, 2, \dots. \tag{2}$$

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(see for example [3, 9, 23, 26] and for an extensive literature information see [5]). The special case $m = 0$ is the famous Euler-Mascheroni constant $\gamma = \gamma(0) = 0.577\,215\,664\,9\ldots$. Besides, the constant γ has relation $\psi(1) = \Gamma'(1) = -\gamma$, where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \ x > 0$$

is Euler's gamma function and $\psi(x) = \Gamma'(x) / \Gamma(x)$ is the digamma function (e.g. [32]). It is well-known that for a positive integer n, $\psi(n+1) + \gamma = H_n = 1 + 1/2 + \cdots + 1/n$, where H_n are the harmonic numbers.

There is comprehensive literature on deriving series and integral representations for the Stieltjes constants and their extensions (see for example [4, 5, 13, 14, 15, 33, 22]). These representations usually allow a more accurate estimation of mentioned constants (see for example [1, 3, 5]).

The Dirichlet series associated with harmonic numbers, so called *harmonic zeta function*, is defined by

$$\zeta_H(s) = \sum_{k=1}^\infty \frac{H_k}{k^s}, \ \ \mathrm{Re}(s) > 1.$$

The harmonic zeta function has been studied by many authors. Euler [21, pp. 217–264] gives a closed form formula for $\zeta_H(s)$ in terms of the Riemann zeta values when $s \in \mathbb{N} \setminus \{1\}$. Apostol and Vu [2] and Matsuoka [27] show that the function ζ_H can be continued meromorphically to the whole complex s plane except for the poles $s = 1$, $s = 0$ and $s = 1 - 2j$, $j \in \mathbb{N}$. Later, Boyadzhiev et al. [7] deal with the Laurent expansion of the harmonic zeta function

$$\zeta_H(s) = \frac{a_{-1}}{s - b} + a_0 + O(s - b), \tag{3}$$

and give explicitly the coefficient a_0 when $b = 0$ and $b = 1 - 2j$, $j \in \mathbb{N}$. Recently, using the Ramanujan summation method, Candelpergher and Coppo [12] record that the *harmonic Stieltjes constants* $\gamma_H(m)$ defined by the Laurent expansion

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{s-1} + \sum_{m=0}^\infty (-1)^m \frac{\gamma_H(m)}{m!} (s-1)^m, \ 0 < |s - 1| < 1$$

can be presented as

$$\gamma_H(m) = \lim_{x \to \infty} \left(\sum_{n \le x} \frac{H_n \ln^m n}{n} - \frac{\ln^{m+2} x}{m+2} - \gamma \frac{\ln^{m+1} x}{m+1} \right).$$

Besides they also present $\gamma_H(0)$ explicitly and rediscover the coefficient a_0 in (3).

We now introduce the main object of this study, *the hyperharmonic zeta function*:

$$\zeta_{h^{(r)}}(s) = \sum_{k=1}^\infty \frac{h_k^{(r)}}{k^s}, \ \ \mathrm{Re}(s) > r,$$

where $h_n^{(r)}$ are the hyperharmonic numbers defined by [16]

$$h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)} \text{ with } h_n^{(0)} = \frac{1}{n}, \ n, r \in \mathbb{N}.$$

It is clear that $\zeta_{h^{(0)}}(s) = \zeta(s+1)$ and $\zeta_{h^{(1)}}(s) = \zeta_H(s)$. The Dirichlet series $\sum_{k=1}^{\infty} h_k^{(r)}/k^s$ converges absolutely and represents an analytic function of s for $\text{Re}(s) > r$ since $h_n^{(r)} = O\left(n^{r-1}\ln n\right)$. Kamano [24] has shown that the function $\zeta_{h^{(r)}}(s)$ can be continued meromorphically to the whole complex s plane except for the double poles at $s = 1, 2, \ldots, r$ and an infinite number of simple poles at $s = -k$, $k \in \mathbb{N} \cup \{0\}$. For positive integer values of s, in which case it is called Euler sum of the hyperharmonic numbers [29], it has enjoyed considerable attention in a number of publications during the last decade. For instance, evaluations of $\zeta_{h^{(r)}}(m)$, $m \in \mathbb{N}$ and their extensions in terms of the Riemann zeta values and some other special constants can be found in [11, 18, 25, 29, 36].

The aim of this paper is to determine certain constants which naturally occur in the Laurent expansion of hyperharmonic zeta function.

The organization of the paper is as follows. In Section 2 we prove that the function $\zeta_{h^{(r)}}$ has a meromorphic continuation (Theorem 1). In Section 3 we present the Laurent expansion of the hyperharmonic zeta function in the region $0 < |s - r| < 1$ (Theorem 3). For appearing coefficients, denoted $\gamma_{h^{(r)}}(m)$, we obtain the following limit representation

$$\gamma_{h^{(r)}}(m) = \lim_{x \to \infty} \left(\sum_{n \leq x} \frac{h_n^{(r)} \log^m n}{n^r} - \frac{1}{\Gamma(r)} \frac{\ln^{m+2} x}{m+2} + \frac{\psi(r)}{\Gamma(r)} \frac{\ln^{m+1} x}{m+1} \right), \quad (4)$$

by modifying the method of Briggs and Buschman [8]. It is clear that the coefficients $\gamma_{h^{(r)}}(m)$, which we call *hyperharmonic Stieltjes constants*, reduce to the Stieltjes constants $\gamma(m)$ when $r = 0$ (with the assumption $\psi(0)/\Gamma(0) = \lim_{r \to 0} \psi(r)/\Gamma(r) = -1$) and to the harmonic Stieltjes constants $\gamma_H(m)$ when $r = 1$. In Section 4 we confer two more representations for $\gamma_{h^{(r)}}(m)$ in addition to (4). The first one is in terms of Stieltjes constants $\gamma_{h^{(r-1)}}(m)$, $\gamma(m)$ and values related to the zeta function (Theorem 8). The second one is in terms of Stieltjes constants $\gamma_H(m)$, $\gamma(m)$ and some other special values related to the Riemann zeta function (Theorem 10). In Section 5, we consider the limit

$$\gamma_{h^{(r)}}^*(m) = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} \ln^m k - \int_1^n \frac{h_x^{(r)}}{x^r} \ln^m x \, dx \right).$$

which is motivated by the interpretation of (2) as

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} f(k) - \int_1^n f(x) \, dx \right). \quad (5)$$

(If $f : (0, \infty) \to (0, \infty)$ is continuous, strictly decreasing and $\lim_{x \to \infty} f(x) = 0$, then the limit (5) exists (cf. [35]).) Here $h_x^{(r)}$ is an analytic extension of $h_n^{(r)}$,

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defined by [28]

$$h_x^{(r)} = \frac{x^{\overline{r}}}{x\Gamma(r)}\left(\psi(x+r) - \psi(r)\right), \; r, x, r + x \in \mathbb{C}\setminus\{0, -1, -2, ...\}, \qquad (6)$$

where

$$x^{\overline{r}} = x(x+1)\cdots(x+r-1) = \sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}x^j \qquad (7)$$

and $\begin{bmatrix}r\\j\end{bmatrix}$ are the Stirling numbers of the first kind, with $\begin{bmatrix}r\\r\end{bmatrix} = 1$ for $r \geq 0$ and $\begin{bmatrix}r\\0\end{bmatrix} = 0$ for $r > 0$.

Moreover, we give a representation for $\gamma_{h^{(r)}}^*(m)$ (Theorem 12) which leads us to show that integrals involving the generalized exponential function

$$E_s^0(t) = \int_1^{\infty} \frac{e^{-xt}}{x^s}dx$$

can be written in terms of some special constants (Theorem 13). The following examples demonstrate Theorem 13:

$$\int_0^{\infty}\left(\frac{1}{t} - \frac{1}{1 - e^{-t}}\right)E_1^0(t)\,dt = -\gamma(1) - \sigma_1 + \zeta(2) - 1,$$

$$\int_0^{\infty}\left(\frac{1}{t} - \frac{1}{1 - e^{-t}}\right)E_2^0(t)\,dt = -\gamma + \sigma_2 - \zeta'(2) - \frac{3}{2},$$

where

$$\sigma_k = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\zeta(k+j), \; k \geq 1 \text{ (see [6, 12, 17, 19])}.$$

2 Meromorphic continuation

The following theorem, which was firstly proven by Kamano [24, Theorem 1.1], gives the meromorphic continuation of the function $\zeta_{h^{(r)}}$. Here we give a slightly different proof.

Theorem 1 *Let r be a non-negative integer. The function $\zeta_{h^{(r)}}$ has a meromorphic continuation to the region $\mathbb{C}\setminus\{k \in \mathbb{Z} : k \leq r\}$.*

Proof. We first recall the following expansion of the harmonic numbers (see for example [2, Eq. (2)])

$$H_n = \ln n + \gamma + \frac{1}{2n} + \sum_{m=1}^{k}\frac{\zeta(1-2m)}{n^{2m}} + \int_n^{\infty}\frac{P_{2k+1}(x)}{x^{2k+2}}dx, \qquad (8)$$

where $P_k(x) = B_k(x - \lfloor x \rfloor)$ is the periodic extension of the Bernoulli polynomial $B_k(x)$ given by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{k}B_k(x)\frac{t^k}{k!}, \; |t| < 2\pi$$

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with $\lfloor x \rfloor$ being the largest integer $\leq x$.

Considering the well-known relation $\psi(n) + \gamma = H_{n-1}$ for $n \in \mathbb{N}$, (6) becomes

$$h_n^{(r)} = \frac{n^{\overline{r}}}{n\Gamma(r)}\left(H_{n+r-1} - \psi(r) - \gamma\right). \tag{9}$$

For $\mathrm{Re}(s) = \sigma > r$, from (8) and (9), we deduce the following representation for $\zeta_{h^{(r)}}(s)$:

$$\begin{aligned}
\zeta_{h^{(r)}}(s) = &-\frac{1}{\Gamma(r)}\zeta'(s+1-r) - \frac{\psi(r)}{\Gamma(r)}\zeta(s+1-r) \\
&+ \frac{1}{\Gamma(r)}\sum_{j=1}^{r-1}\left\{\begin{bmatrix}r\\j\end{bmatrix}\left(-\zeta'(s+1-j) - \psi(r)\zeta(s+1-j)\right)\right. \\
&\left.+\begin{bmatrix}r\\j+1\end{bmatrix}\left(\frac{1}{2}\zeta(s+1-j) + \sum_{v=1}^{r-1}\sum_{n=1}^{\infty}\frac{1}{n^{s-j}(n+v)}\right)\right\} \\
&+ \frac{1}{\Gamma(r)}\sum_{j=1}^{r}\begin{bmatrix}r\\j\end{bmatrix}\left(\sum_{m=1}^{k}\zeta(1-2m)\zeta(s+1-j+2m) + R(s,k,j)\right),
\end{aligned} \tag{10}$$

where

$$R(s,k,j) = \sum_{n=1}^{\infty}\frac{1}{n^{s+1-j}}\int_{n}^{\infty}\frac{P_{2k+1}(x)}{x^{2k+2}}dx.$$

For each $k \in \mathbb{N}$ the function $P_k(x)$ is bounded, say $|P_k(x)| \leq K$, then we have

$$\left|\sum_{n=1}^{\infty}\frac{1}{n^{s+1-j}}\int_{n}^{\infty}\frac{P_{2k+1}(x)}{x^{2k+2}}dx\right| \leq \frac{K}{2k+1}\sum_{n=1}^{\infty}\frac{1}{n^{\sigma-j+2k+2}}.$$

Therefore $R(s,k,j)$ is analytic in the half-plane $\sigma > r-2k-1$, and (10) provides the analytic continuation of $\zeta_{h^{(r)}}(s)$ in the half-plane $\sigma > r - 2k - 1$.

It is seen from (10) that the singularities of $\zeta_{h^{(r)}}(s)$ arise from the poles of $\zeta'(s+1-j)$, $\zeta(s+1-j)$, $\sum_{n=1}^{\infty}\frac{1}{n^{s-j}(n+v)}$ and $\zeta(s+1-j+2m)$.

For $r > 0$ there is a second-order pole at $s = r$ arising from the term $-\frac{1}{\Gamma(r)}\zeta'(s+1-r) - \frac{\psi(r)}{\Gamma(r)}\zeta(s+1-r)$ with residue $-\frac{\psi(r)}{\Gamma(r)}$. Moreover, there are second-order poles at $s = j$ for $1 \leq j \leq r-1$, and simple poles at $s = j - 2m$ with $j - 2m < 1$ for $1 \leq j \leq r$ and $1 \leq m \leq k$. Since k is an arbitrary positive integer, this implies that $\zeta_{h^{(r)}}(s)$ has a simple pole at every non-positive integer. ∎

Remark 2 *Interested readers can find the residues of $\zeta_{h^{(r)}}(s)$ at $s = k$, $k \in \mathbb{Z}$ with $k < r$ in [24, Theorem 1.1]. However, there are little misprints; the factors $r!$ in equations (1.6), (1.7) and (1.8) of [24, Theorem 1.1] should be $(r-1)!$.*

3   Laurent expansion and hyperharmonic Stieltjes constants

This section is devoted to determine the hyperharmonic Stieltjes constants, the main theme of this study.

Theorem 3 *Let r be a non-negative integer. The hyperharmonic zeta function has the following Laurent expansion in the annulus $0 < |s - r| < 1$*

$$\zeta_{h^{(r)}}(s) = \frac{a_{-2}}{(s-r)^2} + \frac{a_{-1}}{s-r} + \sum_{m=0}^{\infty} \frac{(-1)^m \gamma_{h^{(r)}}(m)}{m!}(s-r)^m,$$

where $a_{-2} = 1/\Gamma(r)$, $a_{-1} = -\psi(r)/\Gamma(r)$ and

$$\gamma_{h^{(r)}}(m) = \lim_{x\to\infty}\left(\sum_{n\le x}\frac{h_n^{(r)}\log^m n}{n^r} - a_{-2}\frac{\ln^{m+2}x}{m+2} - a_{-1}\frac{\ln^{m+1}x}{m+1}\right).$$

It is clear from the proof of Theorem 1 that, $\zeta_{h^{(r)}}(s)$ has the Laurent series

$$\zeta_{h^{(r)}}(s) = \frac{a_{-2}}{(s-r)^2} + \frac{a_{-1}}{s-r} + \sum_{m=0}^{\infty} a_m(s-r)^m$$

in the annulus $0 < |s - r| < 1$, where $a_{-2} = 1/\Gamma(r)$ and $a_{-1} = -\psi(r)/\Gamma(r)$.

To complete the proof we must show that

$$a_m = \frac{(-1)^m}{m!}\lim_{x\to\infty}\left(\sum_{n\le x}\frac{h_n^{(r)}\log^m n}{n^r} - a_{-2}\frac{\ln^{m+2}x}{m+2} - a_{-1}\frac{\ln^{m+1}x}{m+1}\right).$$

For this purpose we give some lemmas and a theorem.

Lemma 4 (Abel summation formula) *If $b_1, b_2, b_3, \ldots$ is a sequence of complex numbers and $v(x)$ has a continuous derivative for $x > 1$, then*

$$\sum_{n\le x} b_n v(n) = \left(\sum_{n\le x} b_n\right) v(x) - \int_1^x \left(\sum_{n\le t} b_n\right) v'(t)\,dt.$$

For $n \le x < n+1$ we see from (8) and (9) that

$$\sum_{k\le x} h_k^{(r)} = h_n^{(r+1)} = \frac{n^{\overline{r+1}}}{n\Gamma(r+1)}\left(\log n - \psi(r+1) + O\left(\frac{1}{n}\right)\right)$$

$$= \frac{1}{\Gamma(r+1)}n^r \log n - \frac{\psi(r+1)}{\Gamma(r+1)}n^r + O\left(n^{r-1}\log n\right).$$

Setting $b_n = h_n^{(r)}$ and $v(x) = x^{-s}$ in Lemma 4 gives the following result.

Lemma 5 *Let r be a non-negative integer. For $\mathrm{Re}\,(s) > r$ we have*

$$\zeta_{h^{(r)}}\,(s) = s \int_1^\infty x^{-s-1} \left(\sum_{k \leq x} h_k^{(r)} \right) dx.$$

Lemma 6 *Let r be a non-negative integer. Let*

$$E\,(x) = \sum_{k \leq x} h_k^{(r)} - \frac{1}{\Gamma\,(r+1)} x^r \log x + \frac{\psi\,(r+1)}{\Gamma\,(r+1)} x^r = O\left(x^{r-1} \log x \right). \qquad (11)$$

Then for $\mathrm{Re}\,(s) > r - 1$ we have

$$f\,(s) := s \int_1^\infty x^{-s-1} E\,(x)\,dx = \frac{\psi\,(r+1)}{\Gamma\,(r+1)} + \sum_{n=0}^\infty a_n\,(s-r)^n.$$

Proof. For $\mathrm{Re}\,(s) > r - 1$ the aforementioned integral is an analytic function. Moreover for $\mathrm{Re}\,(s) > r$ we have

$$s \int_1^\infty x^{-s-1} E\,(x)\,dx = \zeta_{h^{(r)}}\,(s) - \frac{1}{\Gamma\,(r+1)} s \int_1^\infty x^{r-s-1} \log x\,dx + \frac{\psi\,(r+1)}{\Gamma\,(r+1)} s \int_1^\infty x^{r-s-1}\,dx$$

$$= \frac{a_{-2}}{(s-r)^2} + \frac{a_{-1}}{s-r} + \sum_{n=0}^\infty a_n\,(s-r)^n - \frac{1}{\Gamma\,(r+1)} \frac{s}{(r-s)^2} + \frac{\psi\,(r+1)}{\Gamma\,(r+1)} \frac{s}{s-r}$$

$$= \frac{1}{r^2 \Gamma\,(r)} + \frac{\psi\,(r)}{r\Gamma\,(r)} + \sum_{n=0}^\infty a_n\,(s-r)^n$$

from (11) and Lemma 5. The proof is then completed. ∎

Theorem 7 *Let m and r be non-negative integers. Let $u < -\,(r-1)$. Then,*

$$\sum_{n \leq x} h_n^{(r)} n^u \log^m n = \frac{1}{\Gamma\,(r)} \int_1^x t^{r+u-1} \log^{m+1} t\,dt - \frac{\psi\,(r)}{\Gamma\,(r)} \int_1^x t^{r+u-1} \log^m t\,dt$$

$$- \frac{\psi\,(r+1)}{\Gamma\,(r+1)} \frac{d^m}{du^m} 1 + o\,(1) + (-1)^m f^{(m)}\,(-u),$$

where $f\,(s)$ is given in Lemma 6.

Proof. Set $v\,(x) = x^u \log^m x$, $u < -r$ and $b_n = h_n^{(r)}$ in Lemma 4. Then

$$S := \sum_{n \leq x} h_n^{(r)} n^u \log^m n = x^u \log^m x \sum_{n \leq x} h_n^{(r)} - \int_1^x \sum_{n \leq t} h_n^{(r)} \frac{d}{dt}\,(t^u \log^m t)\,dt.$$

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We now use the equality

$$\frac{d}{dt}\left(t^u \log^m t\right) = \frac{d^m}{du^m}\left(ut^{u-1}\right)$$

to see that

$$S = x^u \log^m x \sum_{n \leq x} h_n^{(r)} - \frac{d^m}{du^m} u \int\limits_1^x \sum_{n \leq t} h_n^{(r)} t^{u-1} dt$$

$$= x^u \log^m x \left(\frac{1}{\Gamma(r+1)} x^r \log x - \frac{\psi(r+1)}{\Gamma(r+1)} x^r + O\left(x^{r-1} \log x\right)\right)$$

$$- \frac{d^m}{du^m} u \int\limits_1^x \left(E(t) + \frac{1}{\Gamma(r+1)} t^r \log t - \frac{\psi(r+1)}{\Gamma(r+1)} t^r\right) t^{u-1} dt$$

$$= \frac{1}{\Gamma(r+1)} \left(x^{u+r} \log^{m+1} x - \frac{d^m}{du^m} u \int\limits_1^x t^{r+u-1} \log t \, dt\right)$$

$$- \frac{\psi(r+1)}{\Gamma(r+1)} \left(x^{u+r} \log^m x - \frac{d^m}{du^m} u \int\limits_1^x t^{r+u-1} dt\right)$$

$$- \frac{d^m}{du^m} u \int\limits_1^x E(t) t^{u-1} dt + O\left(x^{r-1+u} \log^{m+1} x\right).$$

Here

$$x^{u+r} \log^m x - \frac{d^m}{du^m} u \int\limits_1^x t^{r+u-1} dt = \frac{d^m}{du^m} \left(x^{u+r} - 1 - u \int\limits_1^x t^{r+u-1} dt\right) + \frac{d^m}{du^m} 1$$

$$= \frac{d^m}{du^m} \left(r \int\limits_1^x t^{r+u-1} dt\right) + \frac{d^m}{du^m} 1$$

$$= r \int\limits_1^x t^{r+u-1} \log^m t \, dt + \frac{d^m}{du^m} 1.$$

Since

$$\int\limits_1^x t^{r+u-1} \log t \, dt = \frac{x^{u+r} \log x}{u+r} - \frac{x^{u+r}}{(u+r)^2} + \frac{1}{(u+r)^2},$$

we obtain

$$x^{u+r} \log x - u \int\limits_1^x t^{r+u-1} \log t \, dt = r \int\limits_1^x t^{r+u-1} \log t \, dt + \frac{x^{u+r}}{u+r} - \frac{1}{u+r}.$$

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Hence,

$$x^{u+r}\log^{m+1}x - \frac{d^m}{du^m}u\int_1^x t^{r+u-1}\log t\,dt = \frac{d^m}{du^m}\left(x^{u+r}\log x - u\int_1^x t^{r+u-1}\log t\,dt\right)$$

$$= \frac{d^m}{du^m}\left(r\int_1^x t^{r+u-1}\log t\,dt + \int_1^x t^{r+u-1}dt\right)$$

$$= r\int_1^x t^{r+u-1}\log^{m+1}t\,dt + \int_1^x t^{r+u-1}\log^m t\,dt.$$

These complete the proof. ∎

Now we are ready to complete the proof of Theorem 3.

Proof of Theorem 3. From Lemma 6 we have

$$f(r) = \frac{\psi(r+1)}{\Gamma(r+1)} + a_0 \text{ and } a_m = \frac{f^{(m)}(r)}{m!} \text{ for } m > 0.$$

Setting $u = -r$ in Theorem 7 yields

$$a_m = \frac{f^{(m)}(r)}{m!} = \frac{(-1)^m}{m!}\lim_{x\to\infty}\left(\sum_{n\le x}\frac{h_n^{(r)}\log^m n}{n^r} - a_{-2}\frac{\ln^{m+2}x}{m+2} - a_{-1}\frac{\ln^{m+1}x}{m+1}\right),$$

which gives the desired result. ∎

4 Alternative representations for hyperharmonic Stieltjes constants

Recall that $\gamma_{h^{(r)}}(m)$ reduces to $\gamma(m)$ for $r = 0$. Now we are going to analyze the case $r > 0$ in the following theorems.

Theorem 8 *Let m be a non-negative integer and r be a positive integer. Then,*

$$\gamma_{h^{(r+1)}}(m) = \frac{1}{r}\gamma_{h^{(r)}}(m) - \frac{1}{r\Gamma(r+1)}\gamma(m) + \zeta_{h^{(r)}}^{(m)}(r+1) - \frac{1}{r\Gamma(r+1)}\sum_{j=0}^{r-1}\begin{bmatrix}r\\j\end{bmatrix}\zeta^{(m)}(r+1-j),$$

where $\zeta_{h^{(r)}}^{(m)}(r+1) = \left.\frac{d^m}{ds^m}\zeta_{h^{(r)}}(s)\right|_{s=r+1}$ *and* $\zeta^{(m)}(r) = \left.\frac{d^m}{ds^m}\zeta(s)\right|_{s=r}.$

Proof. We employ the following equation [20, Proposition 3]

$$h_n^{(r+1)} = \left(1 + \frac{n}{r}\right)h_n^{(r)} - \frac{n}{r(n+r)}\binom{n+r}{r}$$

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$$= \left(1 + \frac{n}{r}\right) h_n^{(r)} - \frac{1}{r\Gamma(r+1)} \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} n^j$$

in (4). After some manipulations, we deduce that

$$\gamma_{h^{(r+1)}}(m) = \lim_{x \to \infty} \left(\frac{1}{r} \sum_{n \leq x} \frac{h_n^{(r)} \log^m n}{n^r} - \frac{1}{r\Gamma(r)} \frac{\ln^{m+2} x}{m+2} + \frac{\psi(r)}{r\Gamma(r)} \frac{\ln^{m+1} x}{m+1} \right.$$

$$- \frac{1}{r\Gamma(r+1)} \sum_{n \leq x} \frac{\log^m n}{n} + \frac{1}{r\Gamma(r+1)} \frac{\ln^{m+1} x}{m+1}$$

$$\left. + \sum_{n \leq x} \frac{h_n^{(r)} \log^m n}{n^{r+1}} - \frac{1}{r\Gamma(r+1)} \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \sum_{n \leq x} \frac{\log^m n}{n^{r+1-j}} \right),$$

which is the desired result. ■

Now we are going to give a representation for the constants $\gamma_{h^{(r)}}(m)$ in terms of Stieltjes constants $\gamma_H(m)$, $\gamma(m)$ and some other special values related to the Riemann zeta function. For this purpose we need the following lemma:

Lemma 9 *Let m and r be non-negative integers. Then,*

$$\sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} \ln^m k = \frac{1}{\Gamma(r)} \left(\sum_{k=1}^{n} \frac{H_k}{k} \ln^m k - \sum_{j=1}^{r-1} \frac{1}{j} \sum_{k=1}^{n+j} \frac{\ln^m k}{k} \right) + \delta(r) \sum_{k=1}^{n} \frac{\ln^m k}{k}$$

$$+ \frac{1}{\Gamma(r)} \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \left(\sum_{k=1}^{n} \frac{H_{k+r-1}}{k^{r+1-j}} \ln^m k - H_{r-1} \sum_{k=1}^{n} \frac{\ln^m k}{k^{r+1-j}} \right)$$

$$+ \frac{1}{\Gamma(r)} \sum_{j=1}^{r-1} \frac{1}{j} \left(\sum_{k=1}^{j} \frac{\ln^m k}{k} - \sum_{k=j+1}^{n+j} \frac{\ln^m (1 - j/k)}{k} \right), \tag{12}$$

where

$$\delta(r) = \frac{1}{\Gamma(r)} \sum_{j=1}^{r-1} \frac{1}{j} - \frac{\psi(r) + \gamma}{\Gamma(r)} = \begin{cases} 1, & r = 0, \\ 0, & r > 0. \end{cases}$$

Proof. In the light of (7) and (9) we have

$$h_k^{(r)} = \frac{1}{\Gamma(r)} \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} k^{j-1} (H_{k+r-1} - \psi(r) - \gamma),$$

from which we obtain that

$$\Gamma(r) \sum_{k=1}^{n} \frac{h_k^{(r)}}{k^s} = \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \sum_{k=1}^{n} \frac{H_{k+r-1}}{k^{s+1-j}} + \sum_{k=1}^{n} \frac{H_{k+r-1}}{k^{s+1-r}}$$

$$- \left(\psi\left(r\right) + \gamma\right)\left(\sum_{k=1}^{n}\frac{1}{k^{s+1-r}} + \sum_{j=0}^{r-1}\begin{bmatrix}r\\j\end{bmatrix}\sum_{k=1}^{n}\frac{1}{k^{s+1-j}}\right).$$

Differentiating both sides m times with respect to s at $s = r$ gives

$$\Gamma\left(r\right)\sum_{k=1}^{n}\frac{h_k^{(r)}}{k^s}\ln^m k = \sum_{j=0}^{r-1}\begin{bmatrix}r\\j\end{bmatrix}\sum_{k=1}^{n}\frac{H_{k+r-1}}{k^{r+1-j}}\ln^m k + \sum_{k=1}^{n}\frac{H_{k+r-1}}{k}\ln^m k$$

$$- \left(\psi\left(r\right) + \gamma\right)\left(\sum_{k=1}^{n}\frac{\ln^m k}{k} + \sum_{j=0}^{r-1}\begin{bmatrix}r\\j\end{bmatrix}\sum_{k=1}^{n}\frac{\ln^m k}{k^{r+1-j}}\right). \quad (13)$$

It is easy to see that

$$\sum_{k=1}^{n}\frac{H_{k+r-1}}{k}\ln^m k = \sum_{k=1}^{n}\frac{H_k}{k}\ln^m k + \sum_{j=1}^{r-1}\frac{1}{j}\sum_{k=1}^{n}\frac{\ln^m k}{k} - \sum_{k=1}^{n}\sum_{j=1}^{r-1}\frac{\ln^m k}{j\left(k+j\right)}$$

and

$$\sum_{k=1}^{n}\sum_{j=1}^{r-1}\frac{\ln^m k}{j\left(k+j\right)} = \sum_{j=1}^{r-1}\sum_{k=j+1}^{n+j}\frac{\ln^m k + \ln^m\left(1-j/k\right)}{jk}$$

$$= \sum_{j=1}^{r-1}\frac{1}{j}\left(\sum_{k=1}^{n+j}\frac{\ln^m k}{k} - \sum_{k=1}^{j}\frac{\ln^m k}{k} + \sum_{k=j+1}^{n+j}\frac{\ln^m\left(1-j/k\right)}{k}\right).$$

Then, we have

$$\sum_{k=1}^{n}\frac{H_{k+r-1}}{k}\ln^m k = \sum_{k=1}^{n}\frac{H_k}{k}\ln^m k + H_{r-1}\sum_{k=1}^{n}\frac{\ln^m k}{k} - \sum_{j=1}^{r-1}\frac{1}{j}\sum_{k=1}^{n+j}\frac{\ln^m k}{k}$$

$$+ \sum_{j=1}^{r-1}\frac{1}{j}\sum_{k=1}^{j}\frac{\ln^m k}{k} - \sum_{j=1}^{r-1}\sum_{k=j+1}^{n+j}\frac{\ln^m\left(1-j/k\right)}{jk} \quad (14)$$

Hence, (12) follows from (13) and (14). ∎

We are ready to give the aforementioned representation for the constants $\gamma_{h^{(r)}}\left(m\right)$.

Theorem 10 *Let m and r be positive integers. Then,*

$$\Gamma\left(r\right)\gamma_{h^{(r)}}\left(m\right) = \gamma_H\left(m\right) - H_{r-1}\gamma\left(m\right)$$

$$+ \sum_{j=0}^{r-1}\begin{bmatrix}r\\j\end{bmatrix}\left(\widetilde{\zeta}_H^{(m)}\left(r+1-j, r-1\right) - H_{r-1}\zeta^{(m)}\left(r+1-j\right)\right) + \sum_{j=1}^{r-1}\frac{1}{j}\left(\sum_{k=1}^{j}\frac{\ln^m k}{k} - C\left(j, m\right)\right),$$

where

$$\widetilde{\zeta}_H\left(s, a\right) = \sum_{k=1}^{\infty}\frac{H_{k+a}}{k^s} = \sum_{k=a+1}^{\infty}\frac{H_k}{\left(k-a\right)^s}, \quad \widetilde{\zeta}_H^{(m)}\left(r, a\right) = \left.\frac{d^m}{ds^m}\widetilde{\zeta}_H\left(s, a\right)\right|_{s=r}$$

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and the constants $C(j, m)$

$$C(j, m) = \sum_{k=j+1}^{\infty} \frac{\ln^m (1 - j/k)}{k}.$$

Proof. From Lemma 9 we have

$$\sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} \ln^m k - \frac{1}{\Gamma(r)} \frac{\ln^{m+2} n}{m+2} + \frac{\psi(r)}{\Gamma(r)} \frac{\ln^{m+1} n}{m+1}$$

$$= \frac{1}{\Gamma(r)} \left(\sum_{k=1}^{n} \frac{H_k}{k} \ln^m k - \sum_{j=1}^{r-1} \frac{1}{j} \sum_{k=1}^{n+j} \frac{\ln^m k}{k} - \frac{\ln^{m+2} n}{m+2} + \psi(r) \frac{\ln^{m+1} n}{m+1} \right)$$

$$+ \frac{1}{\Gamma(r)} \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \left(\sum_{k=1}^{n} \frac{H_{k+r-1}}{k^{r+1-j}} \ln^m k - H_{r-1} \sum_{k=1}^{n} \frac{\ln^m k}{k^{r+1-j}} \right)$$

$$+ \frac{1}{\Gamma(r)} \sum_{j=1}^{r-1} \frac{1}{j} \left(\sum_{k=1}^{j} \frac{\ln^m k}{k} - \sum_{k=j+1}^{n+j} \frac{\ln^m (1 - j/k)}{k} \right).$$

By letting n tend to infinity we see that

$$\gamma_{h^{(r)}}(m) = \frac{1}{\Gamma(r)} \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{H_k}{k} \ln^m k - \sum_{j=1}^{r-1} \frac{1}{j} \sum_{k=1}^{n+j} \frac{\ln^m k}{k} - \frac{\ln^{m+2} n}{m+2} + \psi(r) \frac{\ln^{m+1} n}{m+1} \right)$$

$$+ \frac{1}{\Gamma(r)} \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \left(\widetilde{\zeta}_H^{(m)}(r+1-j, r-1) - H_{r-1} \zeta^{(m)}(r+1-j) \right)$$

$$+ \frac{1}{\Gamma(r)} \sum_{j=1}^{r-1} \frac{1}{j} \left(\sum_{k=1}^{j} \frac{\ln^m k}{k} - \sum_{k=j+1}^{\infty} \frac{\ln^m (1 - j/k)}{k} \right).$$

Then the desired result follows from

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{H_k}{k} \ln^m k - \sum_{j=1}^{r-1} \frac{1}{j} \sum_{k=1}^{n+j} \frac{\ln^m k}{k} - \frac{\ln^{m+2} n}{m+2} + \psi(r) \frac{\ln^{m+1} n}{m+1} \right)$$

$$= \lim_{n \to \infty} \left(\gamma_{h^{(1)}}(m) + H_{r-1} \frac{\ln^{m+1} n}{m+1} - \sum_{j=1}^{r-1} \frac{1}{j} \sum_{k=1}^{n+j} \frac{\ln^m k}{k} \right)$$

$$= \gamma_{h^{(1)}}(m) - \sum_{j=1}^{r-1} \frac{1}{j} \lim_{n \to \infty} \left(\frac{\ln^{m+1}(1 + j/n)}{m+1} + \sum_{k=1}^{n+j} \frac{\ln^m k}{k} - \frac{\ln^{m+1}(n+j)}{m+1} \right)$$

$$= \gamma_{h^{(1)}}(m) - H_{r-1} \gamma_{h^{(0)}}(m).$$

∎

5   A formal extension of the Stieltjes constants

In this section, we give a presentation for the constants

$$\gamma^*_{h^{(r)}}\left(m\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{h_k^{(r)}}{k^r}\ln^m k - \int_1^n\frac{h_x^{(r)}}{x^r}\ln^m xdx\right),$$

which are a formal extension of the Stieltjes constants. As a result we show that integrals involving generalized exponential function can be written in terms of some special constants.

We first analyze the integral in the limit above.

Lemma 11 *Let m and r be non-negative integers. Then,*

$$\Gamma\left(r\right)\int_1^n\frac{h_x^{(r)}}{x^r}\ln^m xdx = \frac{\ln^{m+2}n}{m+2} - \psi\left(r\right)\frac{\ln^{m+1}n}{m+1} + \frac{n^{\overline{r}}}{n^{r+1}}\ln^m n$$

$$+ \int_1^n\left(\sum_{j=0}^{r-1}\begin{bmatrix}r\\j\end{bmatrix}\frac{\ln^{m+1}x - \psi\left(r\right)\ln^m x}{x^{r+1-j}} + \sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}\frac{\left(r+1\right)\ln^m x - m\ln^{m-1}x}{x^{r+2-j}}\right)dx$$

$$+ \sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}\int_0^\infty\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)\left(\int_1^n\frac{\ln^m x}{x^{r+1-j}}e^{-xt}dx\right)dt. \tag{15}$$

Proof. Utilizing the identity $\psi\left(x+1\right) = \psi\left(x\right) + 1/x$ in (6) we have

$$\Gamma\left(r\right)\int_1^n\frac{h_x^{(r)}}{x^s}dx = \int_1^n\frac{x^{\overline{r}}}{x^{s+1}}\psi\left(x+1\right)dx$$

$$+ \int_1^n\frac{1}{x^s}\frac{d}{dx}\left(x+1\right)^{\overline{r-1}}dx - \psi\left(r\right)\int_1^n\frac{x^{\overline{r}}}{x^{s+1}}dx. \tag{16}$$

In view of (7) the first integral on the RHS of (16) becomes

$$\int_1^n\frac{x^{\overline{r}}}{x^{s+1}}\psi\left(x+1\right)dx = \sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}\int_1^n\frac{\psi\left(x+1\right)}{x^{s+1-j}}dx.$$

Thanks to the expression [32, p. 26]

$$\psi\left(z\right) = \ln z + \int_0^\infty\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)e^{-zt}dt, \ \ \text{Re}\left(z\right) > 0,$$

we write

$$\int_1^n\frac{\psi\left(x+1\right)}{x^{s+1-j}}dx = \int_1^n\frac{\ln x}{x^{s+1-j}}dx + \int_1^n\frac{1}{x^{s+2-j}}dx$$

$$+ \int_0^\infty\left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)\left(\int_1^n\frac{e^{-xt}}{x^{s+1-j}}dx\right)dt.$$

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We then deduce that

$$\frac{\partial^m}{\partial s^m} \int_1^n \frac{x^{\overline{r}}}{x^{s+1}} \psi\left(x+1\right) dx \bigg|_{s=r} = \sum_{j=0}^r \begin{bmatrix} r \\ j \end{bmatrix} \int_1^n \frac{\psi\left(x+1\right)}{x^{r+1-j}} \ln^m x\, dx$$

$$= \int_1^n \frac{\ln^{m+1} x}{x} dx + \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \int_1^n \frac{\ln^{m+1} x}{x^{r+1-j}} dx + \sum_{j=0}^r \begin{bmatrix} r \\ j \end{bmatrix} \int_1^n \frac{\ln^m x}{x^{r+2-j}} dx$$

$$+ \sum_{j=0}^r \begin{bmatrix} r \\ j \end{bmatrix} \int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right) \left(\int_1^n \frac{\ln^m x}{x^{r+1-j}} e^{-xt} dx\right) dt.$$

On the other hand, the second and the third integrals on the RHS of (16) can be obtained as

$$\frac{\partial^m}{\partial s^m} \int_1^n \frac{1}{x^s} \frac{d}{dx} \left(x+1\right)^{\overline{r-1}} dx \bigg|_{s=r} = \int_1^n \frac{\ln^m x}{x^r} \frac{d}{dx} \left(x+1\right)^{\overline{r-1}} dx$$

$$= (n+1)^{\overline{r-1}} \frac{\ln^m n}{n^r} - \sum_{j=0}^r \begin{bmatrix} r \\ j \end{bmatrix} \int_1^n \frac{\left(-r \ln^m x + m \ln^{m-1} x\right)}{x^{r+2-j}} dx$$

and

$$\psi\left(r\right) \frac{\partial^m}{\partial s^m} \int_1^n \frac{x^{\overline{r}}}{x^{s+1}} dx \bigg|_{s=r} = \psi\left(r\right) \int_1^n \frac{\ln^m x}{x} dx + \psi\left(r\right) \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \int_1^n \frac{\ln^m x}{x^{r+1-j}} dx.$$

Combining the results above yields (15). ∎

The next theorem gives the aforementioned representation for the constants $\gamma_{h^{(r)}}^*(m)$, which involves the generalized integro-exponential function (e.g. [31])

$$E_s^m\left(t\right) = \frac{1}{\Gamma\left(m+1\right)} \int_1^\infty \frac{e^{-xt}}{x^s} \ln^m x\, dx$$

(see [30, 34] for different fields where the integral arises).

Theorem 12 *Let m and r be non-negative integers. Then,*

$$\gamma_{h^{(r)}}^*\left(m\right) = \gamma_{h^{(r)}}\left(m\right) - \frac{m!}{\Gamma\left(r\right)} \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \left(\frac{m+1-\left(r-j\right)\psi\left(r\right)}{\left(r-j\right)^{m+2}} + \frac{j}{\left(r+1-j\right)^{m+1}}\right)$$

$$- r \frac{m!}{\Gamma\left(r\right)} - \frac{m!}{\Gamma\left(r\right)} \sum_{j=0}^r \begin{bmatrix} r \\ j \end{bmatrix} \int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right) E_{r+1-j}^m\left(t\right) dt.$$

Proof. In the light of Lemma 11, we have

$$\gamma_{h^{(r)}}^*\left(m\right) = \lim_{n \to \infty} \left(\sum_{k=1}^n \frac{h_k^{(r)}}{k^r} \ln^m k - \frac{1}{\Gamma\left(r\right)} \frac{\ln^{m+2} n}{m+2} + \frac{\psi\left(r\right)}{\Gamma\left(r\right)} \frac{\ln^{m+1} n}{m+1}\right)$$

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$$-\frac{1}{\Gamma\left(r\right)}\sum_{j=0}^{r-1}\begin{bmatrix}r\\j\end{bmatrix}\int_{1}^{\infty}\frac{\ln^{m+1}x-\psi\left(r\right)\ln^{m}x}{x^{r+1-j}}dx$$

$$-\frac{1}{\Gamma\left(r\right)}\sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}\int_{1}^{\infty}\frac{\left(r+1\right)\ln^{m}x-m\ln^{m-1}x}{x^{r+2-j}}dx$$

$$-\frac{1}{\Gamma\left(r\right)}\sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}\int_{0}^{\infty}\left(\frac{1}{t}-\frac{1}{1-e^{-t}}\right)\left(\int_{1}^{\infty}\frac{\ln^{m}x}{x^{r+1-j}}e^{-xt}dx\right)dt. \quad (17)$$

Using the following reduction formula

$$I\left(\mu,k\right)=\int_{1}^{\infty}\frac{\ln^{\mu}x}{x^{k}}dx=\frac{\mu}{k-1}I\left(\mu-1,k\right)$$

we deduce that

$$I\left(\mu,k\right)=\frac{\mu!}{\left(k-1\right)^{\mu}}I\left(0,k\right)=\frac{\mu!}{\left(k-1\right)^{\mu+1}}.$$

Hence after some rearrangements, (17) becomes

$$\gamma_{h^{(r)}}^{*}\left(m\right)=\gamma_{h^{(r)}}\left(m\right)-\frac{m!}{\Gamma\left(r\right)}\sum_{j=0}^{r-1}\begin{bmatrix}r\\j\end{bmatrix}\left(\frac{m+1}{\left(r-j\right)^{m+2}}-\frac{\psi\left(r\right)}{\left(r-j\right)^{m+1}}\right)$$

$$-\frac{m!}{\Gamma\left(r\right)}\sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}\left(\frac{r+1}{\left(r+1-j\right)^{m+1}}-\frac{1}{\left(r+1-j\right)^{m}}\right)$$

$$-\frac{m!}{\Gamma\left(r\right)}\sum_{j=0}^{r}\begin{bmatrix}r\\j\end{bmatrix}\int_{0}^{\infty}\left(\frac{1}{t}-\frac{1}{1-e^{-t}}\right)E_{r+1-j}^{m}\left(t\right)dt,$$

which completes the proof. $\blacksquare$

We conclude the paper with result on evaluation of the integrals involving generalized exponential function.

Theorem 13 *Let $p\in\mathbb{N}$. Then the integral*

$$\int_{0}^{\infty}\left(\frac{1}{t}-\frac{1}{1-e^{-t}}\right)E_{p}^{0}\left(t\right)dt$$

can be written as a finite combination of the Riemann zeta values $\zeta\left(l\right)$ and $\zeta'\left(k\right)$, Euler-Mascheroni constant γ, Stieltjes constant $\gamma\left(1\right)$ and the constants σ_{k}.

Proof. In special case $m=0$, the constants $\gamma_{h^{(r)}}^{*}\left(0\right)$ reduce to the constants

$$\gamma_{h^{(r)}}^{*}=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{h_{k}^{(r)}}{k^{r}}-\int_{1}^{n}\frac{h_{x}^{(r)}}{x^{r}}dx\right)$$

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introduced in recent paper [10]. Accordingly, we have two alternative representations for $\gamma^*_{h^{(r)}}(0) = \gamma^*_{h^{(r)}}$. The first one appears from Theorem 12:

$$\Gamma(r)\,\gamma^*_{h^{(r)}} = \Gamma(r)\,\gamma_{h^{(r)}}(0) - \sum_{j=0}^{r-1}\begin{bmatrix} r \\ j \end{bmatrix}\left(\frac{1-(r-j)\,\psi(r)}{(r-j)^2} + \frac{j}{(r+1-j)}\right)$$

$$- r - \sum_{j=0}^{r}\begin{bmatrix} r \\ j \end{bmatrix}\int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right) E^0_{r+1-j}(t)\,dt,$$

and the second one is given in Theorem 8 of [10]:

$$\Gamma(r)\,\gamma^*_{h^{(r)}} = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \sigma_1 + \gamma(1) + \sum_{j=1}^{r-1}\frac{H_j}{j} - (\psi(r)+\gamma)\gamma + r!$$

$$- r + \sum_{j=1}^{r-1}\begin{bmatrix} r \\ j \end{bmatrix}\left\{\begin{array}{l} \frac{r+3-j}{2}\zeta(r+2-j) - \frac{1}{2}\sum_{v=1}^{r-j-1}\zeta(r+1-j-v)\,\zeta(v+1) \\[2mm] -\sum_{v=2}^{r-j}(-1)^v\,\zeta(r+2-j-v)\left(H^{(v)}_{r-1} + \frac{(-1)^{r-j}}{v-1}\right) + \frac{H_{r-1}}{r-j} \\[2mm] +(-1)^{r-j}\left(\sigma_{r+1-j} - \zeta'(r+1-j) + \sum_{v=1}^{r-1}\frac{H_v}{v^{r+1-j}}\right) - \frac{r}{r+1-j} \end{array}\right\}$$

where $H^{(v)}_p = \sum_{k=1}^p k^{-v}$ are the generalized harmonic numbers. As a consequence of these two representations we have

$$\sum_{j=0}^{r}\begin{bmatrix} r \\ j \end{bmatrix}\int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right) E^0_{r+1-j}(t)\,dt$$

$$= \Gamma(r)\,\gamma_{h^{(r)}}(0) - \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) - \sigma_1 - \gamma(1) - \sum_{j=1}^{r-1}\frac{H_j}{j} + (\psi(r)+\gamma)\gamma - r!$$

$$- \sum_{j=1}^{r-1}\begin{bmatrix} r \\ j \end{bmatrix}\left\{\begin{array}{l} \frac{r+3-j}{2}\zeta(r+2-j) - \frac{1}{2}\sum_{v=1}^{r-j-1}\zeta(r+1-j-v)\,\zeta(v+1) \\[2mm] -\sum_{v=2}^{r-j}(-1)^v\,\zeta(r+2-j-v)\left(H^{(v)}_{r-1} + \frac{(-1)^{r-j}}{v-1}\right) + \frac{H_{r-1}}{r-j} \\[2mm] +(-1)^{r-j}\left(\sigma_{r+1-j} - \zeta'(r+1-j) + \sum_{v=1}^{r-1}\frac{H_v}{v^{r+1-j}}\right) - \frac{r}{r+1-j} \end{array}\right\}$$

$$- \sum_{j=0}^{r-1}\begin{bmatrix} r \\ j \end{bmatrix}\left(\frac{1-(r-j)\,\psi(r)}{(r-j)^2} + \frac{j}{r+1-j}\right). \tag{18}$$

To complete the proof it is enough to show that the constants

$$\gamma_{h^{(r)}} := \gamma_{h^{(r)}}(0) = \lim_{x\to\infty}\left(\sum_{n\le x}\frac{h^{(r)}_n}{n^r} - \frac{1}{2\Gamma(r)}\ln^2 x + \frac{\psi(r)}{\Gamma(r)}\ln x\right)$$

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can be written in terms of zeta values, Euler-Mascheroni constant and harmonic numbers. To achieve this we appeal the following expression [10, Lemma 4]:

$$\sum_{k=1}^{n} \frac{h_k^{(r)}}{k^r} = \frac{(H_n)^2 + H_n^{(2)}}{2\Gamma(r)} - \frac{\psi(r) + \gamma}{\Gamma(r)} H_n$$
$$+ \frac{1}{\Gamma(r)} \sum_{j=0}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \left(\sum_{k=1}^{n} \frac{H_{k+r-1}}{k^{r+1-j}} - H_{r-1} H_n^{(r+1-j)} \right)$$
$$+ \frac{1}{\Gamma(r)} \sum_{j=1}^{r-1} \frac{H_j}{j} - \frac{H_{r-1}}{\Gamma(r)} (H_{n+r-1} - H_n) + \frac{1}{\Gamma(r)} \sum_{j=1}^{r-1} \frac{H_{j-1}}{j+n},$$

where r is a non-negative integer.

We now utilize the asymptotic expression (cf. (8))

$$H_n = \ln n + \gamma + O\left(n^{-1}\right)$$

to find that

$$\lim_{x \to \infty} \left(\frac{(H_n)^2}{2} - (\psi(r) + \gamma) H_n - \frac{\ln^2 n}{2} + \psi(r) \ln n \right) = -\frac{\gamma^2}{2} - \gamma\psi(r).$$

Hence,

$$\Gamma(r) \gamma_{h^{(r)}} = -\frac{\gamma^2}{2} - \gamma\psi(r) + \frac{\zeta(2)}{2} + \frac{(H_{r-1})^2 + H_{r-1}^{(2)}}{2}$$
$$+ \sum_{j=1}^{r-1} \begin{bmatrix} r \\ j \end{bmatrix} \left(\widetilde{\zeta}_H(r+1-j, r-1) - H_{r-1}\zeta(r+1-j) \right). \qquad (19)$$

On the other hand, it is known that the values $\widetilde{\zeta}_H(p, r-1)$ can be written as [37, Theorem 2.1]

$$\widetilde{\zeta}_H(p, r-1) = \frac{1}{2}(p+2)\zeta(p+1) - \frac{1}{2}\sum_{v=1}^{p-2} \zeta(p-v)\zeta(v+1)$$
$$- \sum_{v=1}^{p-1} (-1)^v \zeta(p+1-v) H_{r-1}^{(v)} - (-1)^p \sum_{v=1}^{r} \frac{H_v}{v^p} \qquad (20)$$

for $p \in \mathbb{N}\backslash\{1\}$. Combining (19) and (20) we accomplish that the constants $\gamma_{h^{(r)}}$ can be written in terms of zeta values, Euler-Mascheroni constant and harmonic numbers.

Therefore it can be seen from (18), (19) and (20) that integrals involving $E_p^0(t)$ can be written in terms of some special constants which is the assertion of Theorem 13. ∎

Remark 14 *1) It follows from* (19) *and* (20) *that the constants* $\gamma_{h^{(r)}}$ *can be explicitly written as*

$$\Gamma(r)\,\gamma_{h^{(r)}} = -\frac{\gamma^2}{2} - \gamma\psi(r) + \frac{\zeta(2)}{2} + \frac{(H_{r-1})^2 + H_{r-1}^{(2)}}{2}$$
$$+ \frac{1}{2}\sum_{j=1}^{r-1}\begin{bmatrix} r \\ j \end{bmatrix}(r+3-j)\,\zeta(r+2-j) - \sum_{v=1}^{r}H_v\sum_{j=1}^{r-1}\begin{bmatrix} r \\ j \end{bmatrix}\frac{(-1)^{r+1-j}}{v^{r+1-j}}$$
$$+ \sum_{v=2}^{r-1}\left((-1)^{v-1}H_{r-1}^{(v)} - \frac{1}{2}\zeta(v)\right)\sum_{j=v}^{r-1}\begin{bmatrix} r \\ r-j \end{bmatrix}\zeta(2+j-v).$$

In particular, when $r=1$ *it reduces to*

$$\gamma_{h^{(1)}} = \frac{\gamma^2}{2} + \frac{\zeta(2)}{2}$$

and coincides with the constant $\gamma_H(0)$ *recorded in* [12, Eq. (6)].

2) Equation (18) *can be used recursively to find exact formulas for the integrals mentioned in Theorem* 13. *For instance, we have*

$$\int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)E_1^0(t)\,dt = \gamma_{h^{(1)}} - \frac{1}{2}\gamma^2 - \gamma(1) - \sigma_1 + \frac{1}{2}\zeta(2) - 1,$$
$$\int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)E_2^0(t)\,dt = \gamma_{h^{(2)}} - \gamma_{h^{(1)}} - 2\zeta(3) - \zeta'(2) + \sigma_2 - \frac{3}{2}$$

and

$$\int_0^\infty \left(\frac{1}{t} - \frac{1}{1-e^{-t}}\right)E_3^0(t)\,dt = \gamma_{h^{(3)}} - \frac{3}{2}\gamma_{h^{(2)}} + \gamma_{h^{(1)}} - \frac{5}{4}\gamma - \sigma_3 + \zeta'(3) - \frac{\pi^4}{72} + \frac{3}{8}\pi^2 - \frac{7}{12}.$$

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