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THE NUMERICAL RANGE OF A PERIODIC TRIDIAGONAL OPERATOR REDUCES TO THE NUMERICAL RANGE OF A FINITE MATRIX

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Dedicated to the memory of Rudolf Kippenhahn (1926–2020)

Abstract. In this paper we show that the closure of the numerical range of an $n+1$-periodic tridiagonal operator is equal to the numerical range of a $2(n+1)\times 2(n+1)$ complex matrix.

Introduction

Consider $\mathcal{A}$ to be a finite set of complex numbers and let $a = (a_i)_{i\in\mathbb{Z}}$ be a biinfinite sequence in the total shift space $\mathcal{A}^{\mathbb{Z}}$. In [13], the tridiagonal operator $A_a\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ associated to a is defined as

$$A_a = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 0 & 1 & & & & \\ & a_{-2} & 0 & 1 & & & \\ & & a_{-1} & \boxed{0} & 1 & & \\ & & & a_0 & 0 & 1 & \\ & & & & a_1 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix} \tag{1}$$

where the square marks the matrix entry at $(0,0)$. In the particular case of the alphabet $\mathcal{A} = \{-1, 1\}$, the corresponding operator A_a is related to the so called "hopping sign model" introduced in [7] and subsequently studied in many other works, such as [1–6, 9, 10, 13], just to name a few. On the other hand, when the alphabet is $\mathcal{A} = \{0, 1\}$ some results for computing the numerical range of A_a are presented in [13,14]. In particular, work in [14] addresses the case when a is an $n+1$-periodic sequence. Relying on the fact that the closure of the numerical range of A_a may be written as the closure of the convex hull of an uncountable union of numerical ranges of certain matrices, in [14] the closure of the numerical range of the 2-periodic case is computed by substituting such uncountable union of numerical ranges by the convex hull of the union of the numerical ranges of just two 2×2 matrices. In this work, we further contribute to the study of the numerical range of A_a when a is an $n+1$ periodic biinfinite sequence.

Instead of working with the operators A_a, we work with the more general tridiagonal operators $T = T(a, b, c)$ defined in Section 2, since, as can be seen in [14], the computation of the closure of the numerical range of A_a is a particular case of that of T. Using a result of Plaumann and Vinzant [20], we show that the closure of the numerical range of the

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$n+1$ periodic tridiagonal operator T is the numerical range of a $2(n+1)\times 2(n+1)$ matrix (cf. Theorem 2.6).

We divide this work in two sections. In Section 1 we briefly introduce the notation and terminologies needed in the rest of the paper. In Section 2 we develop the required machinery, first by computing the Kippenhahn polynomial of the symbol of $n+1$ periodc tridiagonal operatos T on $\ell^2(\mathbb{N}_0)$ and then by combining our computations with results of Plaumann and Vizant. We will conclude that the closure of the numerical range of T is equal to the numerical range of a $2(n+1)\times 2(n+1)$ matrix A. Furthermore, we provide some examples where A can be explicitly computed and we show that the size of A is optimal.

1. Preliminaries

In this section we introduce the notation required which will be needed in the following sections. As usual, the symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the set of positive integers, the sets of nonnegative integers, the set of integers, the set of real numbers and the set of complex numbers, respectively.

For a given $n\in\mathbb{N}$, let a, b and c be $(n+1)$-periodic infinite sequences in $\mathcal{A}^{\mathbb{N}_0}$. We will denote by $T=T(a,b,c)$ the $(n+1)$-periodic tridiagonal operator on $\ell^2(\mathbb{N}_0)$ given by

$$T=\begin{pmatrix} b_0 & c_0 & & & & & & & & & \\ a_1 & b_1 & c_1 & & & & & & & & \\ & a_2 & b_2 & c_2 & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & a_n & b_n & c_n & & & & & \\ & & & & a_0 & b_0 & c_0 & & & & \\ & & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & a_{n-1} & b_{n-1} & c_{n-1} & & \\ & & & & & & & a_n & b_n & c_n & \\ & & & & & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We should observe that T is a bounded operator since the sum of the moduli of the entries in each column (and in each row) is uniformly bounded (see, e.g., [16, Example 2.3]). The biinfinite matrix A_a is also a bounded operator, as long as the biinfinite sequence a arises from a finite alphabet.

If $n>1$, for each $\phi\in[0,2\pi)$, following [1,14] we define the symbol of T, as the following $(n+1)\times(n+1)$ matrix

$$T_\phi=\begin{pmatrix} b_0 & c_0 & 0 & & & 0 & a_0e^{-i\phi} \\ a_1 & b_1 & c_1 & 0 & & & 0 \\ 0 & a_2 & b_2 & c_2 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & a_{n-2} & b_{n-2} & c_{n-2} & 0 \\ 0 & & & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ c_ne^{i\phi} & 0 & & & 0 & a_n & b_n \end{pmatrix}; \tag{2}$$

while the symbol of T for $n=1$ is the 2×2 matrix

$$T_\phi = \begin{pmatrix} b_0 & c_0 + a_0 e^{-i\phi} \\ a_1 + c_1 e^{i\phi} & b_1 \end{pmatrix}. \tag{3}$$

Recall that given a Hilbert space $\mathcal{H}$ and a bounded operator A on it, the numerical range is defined as the set

$$W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}.$$

The Toeplitz-Haussdorf Theorem establishes that $W(A)$ is a bounded convex subset of $\mathbb{C}$ (closed, if the Hilbert space is finite dimensional) and hence the closure of the numerical range can be seen as the intersection of the closed half-spaces containing the numerical range.

Kippenhahn [17] (see also [18]) characterized two vertical support lines of $W(A)$ for a given $n\times n$ matrix as $\mathrm{Re}(z) = \lambda_1(A)$ and $\mathrm{Re}(z) = \lambda_n(A)$, where $\lambda_1(A)$ and $\lambda_n(A)$ are the respective largest and least eigenvalues of $\mathrm{Re}(A)$ (recall that $\mathrm{Re}(A) := \frac{1}{2}(A + A^*)$ and $\mathrm{Im}(A) := \frac{1}{2i}(A - A^*)$). In fact, if $\alpha \in W(A)$ then $\lambda_n(A) \le \mathrm{Re}(\alpha) \le \lambda_1(A)$ (and the equalities hold for some points $\alpha_1, \alpha_2 \in W(A)$). Since $e^{i\theta}W(A) = W(e^{i\theta}A)$ for each $\theta \in [0, 2\pi)$, it follows that if $\alpha \in W(A)$, then $e^{-i\theta}\alpha \in W(e^{-i\theta}A)$ and hence $\mathrm{Re}(e^{-i\theta}\alpha) \le \lambda_1(e^{-i\theta}A)$. It follows that the lines $\mathrm{Re}(e^{-i\theta}z) = \lambda_1(e^{-i\theta}A)$ are support lines of $W(A)$. Hence the convex set $W(A)$ is uniquely determined by the numbers $\lambda_1(e^{-i\theta}A)$, as θ varies on the interval $[0, 2\pi)$; i.e. $W(A)$ is determined by the largest eigenvalue of $\mathrm{Re}(e^{-i\theta}A)$, which equals $\cos(\theta)\mathrm{Re}(A) + \sin(\theta)\mathrm{Im}(A)$. Thus the numerical range is determined by the largest roots of the family of characteristic polynomials

$$\det(tI_n - \cos(\theta)\mathrm{Re}(A) - \sin(\theta)\mathrm{Im}(A)).$$

The homogeneous polynomial $F_A(t, x, y) = \det(tI_n + x\mathrm{Re}(A) + y\mathrm{Im}(A))$ is called the Kippenhahn polynomial of the matrix A. It clearly follows that two matrices have the same numerical range if their Kippenhahn polynomials coincide. Furthermore,

$$\max\{t \in \mathbb{R} : F_A(t, -\cos(\theta), -\sin(\theta)) = 0\} = \max\{\mathrm{Re}(e^{-i\theta}z) : z \in W(A)\}$$

for each $\theta \in [0, 2\pi)$.

2. The Kippenhahn polynomial of the symbol T_ϕ

In this section, after some preliminary work, we show that the closure of the numerical range of a $n+1$-periodic tridiagonal operator T is the numerical range of a $2(n+1)\times 2(n+1)$ matrix.

We will need the following lemma.

Lemma 2.1. Consider the $(n+1)\times(n+1)$ "almost tridiagonal" matrix

$$\Lambda = \begin{pmatrix}
\lambda_{1,1} & \lambda_{1,2} & 0 & 0 & \dots & 0 & 0 & \lambda_{1,n+1} \\
\lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & 0 & \dots & 0 & 0 & 0 \\
0 & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \dots & 0 & 0 & 0 \\
0 & 0 & \lambda_{4,3} & \lambda_{4,4} & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & \lambda_{n-1,n-1} & \lambda_{n-1,n} & 0 \\
0 & 0 & 0 & 0 & \dots & \lambda_{n,n-1} & \lambda_{n,n} & \lambda_{n,n+1} \\
\lambda_{n+1,1} & 0 & 0 & 0 & \dots & 0 & \lambda_{n+1,n} & \lambda_{n+1,n+1}
\end{pmatrix},$$

where every $\lambda_{i,j} \in \mathbb{C}$. Then, $\det(\Lambda)$ equals

$$\det\begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & 0 & \dots & 0 & 0 \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \dots & 0 & 0 \\ 0 & \lambda_{3,2} & \lambda_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{n-1,n} & 0 \\ 0 & 0 & 0 & \dots & \lambda_{n,n} & \lambda_{n,n+1} \\ 0 & 0 & 0 & \dots & \lambda_{n+1,n} & \lambda_{n+1,n+1} \end{pmatrix} - \lambda_{1,n+1}\lambda_{n+1,1}\det\begin{pmatrix} \lambda_{2,2} & \lambda_{2,3} & \dots & 0 & 0 \\ \lambda_{3,2} & \lambda_{3,3} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1,n-1} & \lambda_{n-1,n} \\ 0 & 0 & \dots & \lambda_{n,n-1} & \lambda_{n,n} \end{pmatrix}$$

$$+ (-1)^n \lambda_{n+1,1}\lambda_{1,2}\lambda_{2,3}\cdots\lambda_{n-1,n}\lambda_{n,n+1} + (-1)^n \lambda_{1,n+1}\lambda_{2,1}\lambda_{3,2}\cdots\lambda_{n,n-1}\lambda_{n+1,n}.$$

Proof. This follows by a long (but straightforward) application of the multilinearity of the determinant function and the Laplace Expansion Theorem. □

Let us set the following notation for the rest of this paper. For $0 \le j < n$ we define

$$\alpha_j = \frac{c_j + \overline{a_{j+1}}}{2}, \quad \gamma_j = \frac{c_j - \overline{a_{j+1}}}{2i}$$

and

$$\alpha_n = \frac{a_0 + \overline{c_n}}{2}, \quad \gamma_n = \frac{a_0 - \overline{c_n}}{2i}.$$

We now find an expression for the Kippenhahn polynomial F_{T_ϕ} of the symbol matrix T_ϕ of an arbitrary $n+1$-periodic tridiagonal matrix T acting on $\ell^2(\mathbb{N}_0)$, involving the determinants of some tridiagonal matrices. This expression will be useful in what follows.

Proposition 2.2. Let $n \in \mathbb{N}$. Consider the symbol T_ϕ, that is, the $(n+1)\times(n+1)$ matrix defined as in (2) for $n \ge 2$ and as in (3) for $n = 1$. Then the Kippenhahn polynomial of T_ϕ is equal to

$$F_{T_\phi}(t,x,y) = G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y) + 2(-1)^n \operatorname{Re}\left((\overline{\alpha_n}x + \overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x + \gamma_j y)\right)\cos\phi$$
$$- 2(-1)^n \operatorname{Im}\left((\overline{\alpha_n}x + \overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x + \gamma_j y)\right)\sin\phi,$$

where $G_n(t,x,y)$ is the determinant of the tridiagonal $(n+1)\times(n+1)$ matrix

$$\begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & 0 & 0 & \dots & 0 & 0 \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,3} & \cdots & 0 & 0 \\ 0 & 0 & \lambda_{4,3} & \lambda_{4,4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{n,n} & \lambda_{n,n+1} \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{n+1,n} & \lambda_{n+1,n+1} \end{pmatrix},$$

and, where we set $H_n(t,x,y) = 1$ when $n = 1$, and, for $n \geq 2$, we set $H_n(t,x,y)$ to be the determinant of $(n-1)\times(n-1)$ tridiagonal matrix

$$\begin{pmatrix} \lambda_{2,2} & \lambda_{2,3} & 0 & \cdots & 0 & 0 \\ \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \cdots & 0 & 0 \\ 0 & \lambda_{4,3} & \lambda_{4,4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1,n-1} & \lambda_{n-1,n} \\ 0 & 0 & 0 & \cdots & \lambda_{n,n-1} & \lambda_{n,n} \end{pmatrix}.$$

Here we have set, for $1 \leq j \leq n+1$,

$$\lambda_{j,j} = t + \mathrm{Re}(b_{j-1})x + \mathrm{Im}(b_{j-1})y,$$

and for $1 \leq j \leq n$,

$$\lambda_{j,j+1} = \alpha_{j-1}x + \gamma_{j-i}y \quad \text{ and } \quad \lambda_{j+1,j} = \overline{\alpha_{j-1}}x + \overline{\gamma_{j-1}}y.$$

Proof. We divide the proof in two cases. For $n+1 = 2$, by computing the real and imaginary parts of the matrix T_ϕ in (3), we obtain that the 2×2 matrix $tI_2 + x\mathrm{Re}(T_\phi) + y\mathrm{Im}(T_\phi)$ is given by

$$\begin{pmatrix} t + \mathrm{Re}(b_0)x + \mathrm{Im}(b_0)y & \alpha_0 x + \gamma_0 y + (\alpha_1 x + \gamma_1 y)e^{-i\phi} \\ (\overline{\alpha_0}x + \overline{\gamma_0}y) + (\overline{\alpha_1}x + \overline{\gamma_1}y)e^{i\phi} & t + \mathrm{Re}(b_1)x + \mathrm{Im}(b_1)y \end{pmatrix},$$

where α_0, α_1, γ_0 and γ_1 are as defined above. The determinant of this matrix can be simplified to

$$\begin{aligned} F_{T_\phi}(t,x,y) &= (t + \mathrm{Re}(b_0)x + \mathrm{Im}(b_0)y)(t + \mathrm{Re}(b_1)x + \mathrm{Im}(b_1)y) - |\alpha_0 x + \gamma_0 y|^2 - |\alpha_1 x + \gamma_1 y|^2 \\ &\quad - 2\mathrm{Re}\Big((\alpha_0 x + \gamma_0 y)(\overline{\alpha_1}x + \overline{\gamma_1}y)e^{i\phi}\Big) \\ &= (t + \mathrm{Re}(b_0)x + \mathrm{Im}(b_0)y)(t + \mathrm{Re}(b_1)x + \mathrm{Im}(b_1)y) - |\alpha_0 x + \gamma_0 y|^2 - |\alpha_1 x + \gamma_1 y|^2 \\ &\quad - 2\mathrm{Re}\left((\alpha_0 x + \gamma_0 y)(\overline{\alpha_1}x + \overline{\gamma_1}y)\right)\cos\phi + 2\mathrm{Im}\left((\alpha_0 x + \gamma_0 y)(\overline{\alpha_1}x + \overline{\gamma_1}y)\right)\sin\phi \\ &= G_1(t,x,y) - |\alpha_1 x + \gamma_1 t|^2 H_1(t,x,y) \\ &\quad - 2\mathrm{Re}\left((\alpha_0 x + \gamma_0 y)(\overline{\alpha_1}x + \overline{\gamma_1}y)\right)\cos\phi + 2\mathrm{Im}\left((\alpha_0 x + \gamma_0 y)(\overline{\alpha_1}x + \overline{\gamma_1}y)\right)\sin\phi, \end{aligned}$$

as desired.

Now, for the case $n+1 \geq 3$, by computing the real and imaginary parts of the matrix T_ϕ in (2), we can observe that $tI_{n+1} + x\mathrm{Re}(T_\phi) + y\mathrm{Im}(T_\phi)$ is the matrix

$$\begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & 0 & 0 & \dots & 0 & \lambda_{1,n+1} \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,3} & \cdots & 0 & 0 \\ 0 & 0 & \lambda_{4,3} & \lambda_{4,4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{n,n} & \lambda_{n,n+1} \\ \lambda_{n+1,1} & 0 & 0 & 0 & \cdots & \lambda_{n+1,n} & \lambda_{n+1,n+1} \end{pmatrix},$$

where we have now set

$$\lambda_{1,n+1} = (\alpha_n x + \gamma_n y)e^{-i\phi} \quad \text{ and } \quad \lambda_{n+1,1} = (\overline{\alpha_n}x + \overline{\gamma_n}y)e^{i\phi}.$$

The above matrix is tridiagonal, except for the upper-right and bottom-left corners.

We can compute the determinant of the matrix polynomial $tI_{n+1}+x\mathrm{Re}(T_\phi)+y\mathrm{Im}(T_\phi)$ by using Lemma 2.1 obtaining

$$\begin{aligned}
F_{T_\phi}(t,x,y) &= \det(tI_{n+1}+x\mathrm{Re}(T_\phi)+y\mathrm{Im}(T_\phi)) \\
&= G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2\, H_n(t,x,y) \\
&\quad + (-1)^n(\overline{\alpha_n}x+\overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x+\gamma_j y)e^{i\phi} + (-1)^n(\alpha_n x+\gamma_n y)\prod_{j=0}^{n-1}(\overline{\alpha_j}x+\overline{\gamma_j}y)e^{-i\phi} \\
&= G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2\, H_n(t,x,y) + 2(-1)^n \mathrm{Re}\left((\overline{\alpha_n}x+\overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x+\gamma_j y)e^{i\phi}\right).
\end{aligned}$$

Computing the real part of the last term above, we obtain the equation

$$\begin{aligned}
F_{T_\phi}(t,x,y) &= G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y) + 2(-1)^n\ \mathrm{Re}\left((\overline{\alpha_n}x+\overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x+\gamma_j y)\right)\cos\phi \\
&\quad - 2(-1)^n\ \mathrm{Im}\left((\overline{\alpha_n}x+\overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x+\gamma_j y)\right)\sin\phi,
\end{aligned}$$

which completes the proof. □

For every $n\in\mathbb{N}$ and for a fixed point $(x,y)\in\mathbb{R}^2$, the angle $\phi\in[0,2\pi)$ is involved only in the constant term (with respect to the variable t) of the polynomial $F_{T_\phi}(t,x,y)$. Furthermore, for every $(x,y)\in\mathbb{R}^2$ and for every $\phi\in[0,2\pi)$, the polynomial $F_{T_\phi}(t,x,y)$, seen as a polynomial in t, has $n+1$ real roots, counting multiplicities, as it is the characteristic polynomial of the Hermitian matrix $-x\mathrm{Re}(T_\phi)-y\mathrm{Im}(T_\phi)$. The following lemma will be useful later when applied to the polynomial F_{T_ϕ}.

Lemma 2.3. Let $F(t:\phi)$ be a family of polynomials in $\mathbb{R}[t]$ given by the expression

$$F(t:\phi) = t^{n+1} + p_n t^n + \ldots + p_1 t + p_0 - u\cos\phi - v\sin\phi,$$

where $\phi\in[0,2\pi)$. Assume that the polynomial $F(t:\phi)$ has $n+1$ real roots counting multiplicities for any angle $\phi\in[0,2\pi)$. Let $\phi_0,\phi_1\in[0,2\pi)$ be such that

$$u\cos\phi_0 + v\sin\phi_0 = -\sqrt{u^2+v^2} \quad \text{and} \quad u\cos\phi_1 + v\sin\phi_1 = \sqrt{u^2+v^2}.$$

Then

$$\max\{\max\{t\in\mathbb{R} : F(t:\phi)=0\} : 0\le\phi<2\pi\} = \max\{t\in\mathbb{R} : F(t:\phi_1)=0\},$$

and

$$\min\{\max\{t\in\mathbb{R} : F(t:\phi)=0\} : 0\le\phi<2\pi\} = \max\{t\in\mathbb{R} : F(t:\phi_0)=0\}.$$

Proof. Define $p(t)$ as

$$p(t) = t^{n+1} + p_n t^n + \ldots + p_1 t + p_0.$$

Observe that, by assumption, the equation

$$p(t) = u\cos\phi + v\sin\phi$$

has $n+1$ real solutions (counting multiplicities) for every $\phi \in [0, 2\pi)$. For some $\phi \in [0, 2\pi)$, we have $u\cos\phi + v\sin\phi = 0$, and hence p has $n+1$ real roots (counting multiplicities) and the derivative of p has n real roots (counting multiplicities). Let r_0 be the largest root of $p'(t)$. Hence, p is increasing on the interval $[r_0, \infty)$ and the equations

$$p(t) = u\cos\phi + v\sin\phi$$

have a unique solution on the interval $[r_0, \infty)$.

Observe that for every $\phi \in [0, 2\pi)$

$$-\sqrt{u^2+v^2} \le u\cos\phi + v\sin\phi \le \sqrt{u^2+v^2};$$

equality occurs on the left-hand-side inequality at ϕ_0 while equality occurs on the right-hand-side inequality at ϕ_1.

For each $\phi \in [0, 2\pi)$, consider the number

$$\max\{t \in \mathbb{R} : p(t) = u\cos\phi + v\sin\phi\}.$$

Since the function p is increasing on $[r_0, \infty)$, the largest of these numbers, when ϕ varies, occurs when t is the largest solution of the equation

$$p(t) = \sqrt{u^2+v^2}.$$

Hence we have

$$\max\{\max\{t \in \mathbb{R} : F(t,\phi) = 0\} : 0 \le \phi < 2\pi\} = \max\{t \in \mathbb{R} : F(t,\phi_1) = 0\}.$$

Analogously, the smallest, when ϕ varies in $[0, 2\pi)$, among the largest solutions t of the equations

$$p(t) = u\cos\phi + v\sin\phi,$$

occurs when t is the largest solution of the equation

$$p(t) = -\sqrt{u^2+v^2}.$$

Hence we have

$$\min\{\max\{t \in \mathbb{R} : F(t,\phi) = 0\} : 0 \le \phi < 2\pi\} = \max\{t \in \mathbb{R} : F(t,\phi_0) = 0\}. \qquad \square$$

In Theorem 2.7, we will show that the closure of the numerical range of T is the numerical range of a single matrix. One of the key steps in the proof of said theorem will be to use the following proposition, which computes the closure of the numerical range of T by using a single homogeneous polynomial, instead of the uncountable number of Kippenhahn polynomials of the symbols T_ϕ, which Theorem 2.8 in [14] would suggest: this is achieved by getting rid of the parameter ϕ in the expression of the Kippenhahn polynomial of the symbol T_ϕ in Proposition 2.2.

Proposition 2.4. Let $n \in \mathbb{N}$. Suppose that $T(a,b,c)$ is an $n+1$-periodic tridiagonal operator acting on $\ell^2(\mathbb{N}_0)$. Let G_n and H_n be as in Proposition 2.2 and let P be the real homogeneous polynomial of degree $2(n+1)$ given by

$$P(t,x,y) = \Big(G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y)\Big)^2 - 4\prod_{j=0}^{n} \big|\alpha_j x + \gamma_j y\big|^2.$$

Then $P(t,0,0) = t^{2(n+1)}$ and

$$\sup\Big\{\mathrm{Re}(e^{-i\theta}z):\ z \in W(T(a,b,c))\Big\} = \max\{t \in \mathbb{R}:\ P(t,-\cos\theta,-\sin\theta) = 0\},$$

for each $\theta \in [0, 2\pi)$.

Proof. It is trivial to check that $P(t,0,0) = t^{2(n+1)}$. Now, let $F(t:\phi) = F_{T_\phi}(t,x,y)$, where we know by Proposition 2.2 that

$$F_{T_\phi}(t,x,y) = G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y) - u\cos\phi - v\sin\phi,$$

where

$$u = -2(-1)^n \operatorname{Re}\left((\overline{\alpha_n}x + \overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x + \gamma_j y)\right)$$

and

$$v = 2(-1)^n \operatorname{Im}\left((\overline{\alpha_n}x + \overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x + \gamma_j y)\right).$$

Notice that

$$\begin{aligned} u^2 + v^2 &= 4\mathrm{Re}^2\left((\overline{\alpha_n}x + \overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x + \gamma_j y)\right) + 4\mathrm{Im}^2\left((\overline{\alpha_n}x + \overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x + \gamma_j y)\right) \\ &= 4\left|(\overline{\alpha_n}x + \overline{\gamma_n}y)\prod_{j=0}^{n-1}(\alpha_j x + \gamma_j y)\right|^2 \\ &= 4\prod_{j=0}^{n}\left|\alpha_j x + \gamma_j y\right|^2. \end{aligned}$$

The polynomial $F(t:\phi)$ has the form outlined in Lemma 2.3 and, as was mentioned before Lemma 2.3, it has $n+1$ real roots, counting multiplicities. Hence, by Lemma 2.3, for ϕ_0 and ϕ_1 satisfying

$$u\cos(\phi_0) + v\sin(\phi_0) = -\sqrt{u^2+v^2}, \qquad u\cos(\phi_1) + v\sin(\phi_1) = \sqrt{u^2+v^2},$$

we have that

$$\max\{\max\{t : F(t:\phi) = 0\} : 0 \le \phi < 2\pi\} = \max\{t : F(t:\phi_1) = 0\},$$

and

$$\min\{\max\{t : F(t:\phi) = 0\} : 0 \le \phi < 2\pi\} = \max\{t : F(t:\phi_0) = 0\}.$$

Notice that

$$\begin{aligned} F(t:\phi_0)\cdot F(t:\phi_1) &= \Big(G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y) - (u\cos(\phi_0) + v\sin(\phi_0))\Big) \\ &\quad \cdot \Big(G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y) - (u\cos(\phi_1) + v\sin(\phi_1))\Big) \\ &= \Big(G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y)\Big)^2 - \Big(\sqrt{u^2+v^2}\Big)^2 \\ &= \Big(G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y)\Big)^2 - (u^2+v^2) \\ &= P(t,x,y). \end{aligned}$$

We also have, for each $\theta \in [0, 2\pi)$, that

$$
\begin{aligned}
&\max\{t \in \mathbb{R}\colon F_{T_\phi}(t, -\cos\theta, -\sin\theta) = 0,\ 0 \le \phi < 2\pi\} \\
&\qquad = \max\Big\{\max\Big\{t \in \mathbb{R}\colon F_{T_\phi}(t, -\cos\theta, -\sin\theta) = 0\Big\}\colon\ 0 \le \phi < 2\pi\Big\} \\
&\qquad = \max\Big\{t \in \mathbb{R}\colon F_{T_{\phi_1}}(t, -\cos\theta, -\sin\theta) = 0\Big\} \\
&\qquad = \max\{t \in \mathbb{R}\colon P(t, -\cos\theta, -\sin\theta) = 0\}.
\end{aligned}
\tag{4}
$$

The last equality follows since the roots of $P(t, -\cos\theta, -\sin\theta)$ are those of $F(t : \phi_1) = F_{T_{\phi_1}}(t, -\cos\theta, -\sin\theta)$ and $F(t : \phi_0) = F_{T_{\phi_0}}(t, -\cos\theta, -\sin\theta)$, so by the choice of ϕ_0 and ϕ_1, the largest root of $P(t, -\cos\theta, -\sin\theta)$ is the largest root of $F_{T_{\phi_1}}(t, -\cos\theta, -\sin\theta)$.

By the definition of the Kippenhahn polynomial, we have

$$
\max\Big\{t \in \mathbb{R}\colon F_{T_\phi}(t, -\cos(\theta), -\sin(\theta)) = 0\Big\} = \max\Big\{\mathrm{Re}(e^{-i\theta}z) : z \in W(T_\phi)\Big\}.
$$

and hence we obtain

$$
\begin{aligned}
&\max\Big\{t \in \mathbb{R} : F_{T_\phi}(t, -\cos(\theta), -\sin(\theta)) = 0,\ 0 \le \phi < 2\pi\Big\} \\
&\qquad = \max\Big\{\mathrm{Re}(e^{-i\theta}z) : z \in W(T_\phi),\ 0 \le \phi < 2\pi\Big\}.
\end{aligned}
\tag{5}
$$

Lastly, the equality

$$
\sup\Big\{\mathrm{Re}(e^{-i\theta}z) : z \in W(T(a,b,c))\Big\} = \max\Big\{\mathrm{Re}(e^{-i\theta}z) : z \in W\Big(T_\phi\Big),\ 0 \le \phi < 2\pi\Big\}
\tag{6}
$$

follows from Theorem 2.8 in [14]. Putting together equations (4), (5) and (6), we obtain the desired conclusion. $\square$

The following definition will be useful.

Definition 2.5. Suppose that $Q(t,x,y)$ is a real homogeneous polynomial in 3 variables t,x,y of degree m with $Q(1,0,0) > 0$. If the equation $Q(t,x_0,y_0) = 0$ in t has m real solutions counting multiplicities for any $(x_0,y_0) \in \mathbb{R}^2$ with $x_0^2 + y_0^2 > 0$, we say that Q is *hyperbolic* (with respect to $(1,0,0)$).

The above condition may also be formulated as: "the equation $Q(t, -\cos\theta, -\sin\theta) = 0$ in t has m real solutions for any angle $0 \le \theta < 2\pi$".

Theorem 2.6 (Plaumann and Vinzant [20])**.** Suppose that $Q(t,x,y)$ is a real homogeneous hyperbolic polynomial of degree m with $Q(1,0,0) = 1$. Then there exists an $m \times m$ complex matrix A satisfying

$$
Q(t,x,y) = \det(tI_m + x\mathrm{Re}(A) + y\mathrm{Im}(A)).
$$

Remark. Helton and Vinnikov [12] (cf. [11]) proved a result stronger than the above theorem which guarantees that we can construct an $m \times m$ complex *symmetric* matrix A satisfying a similar property. In this paper we do not use the symmetry of the matrix A.

Depending on the above Theorem 2.6, we obtain the main theorem of this paper.

Theorem 2.7. Suppose that $T(a,b,c)$ is an $n+1$-periodic tridiagonal operator acting on $\ell^2(\mathbb{N}_0)$. Then there exists a $2(n+1) \times 2(n+1)$ complex matrix A such that

$$
\overline{W(T(a,b,c))} = W(A)
$$

where the matrix A is chosen so that it satisfies

$$F_A(t,x,y) = \Big(G_n(t,x,y) - |\alpha_n x + \gamma_n y|^2 H_n(t,x,y)\Big)^2 - 4\prod_{j=0}^{n} \left|\alpha_j x + \gamma_j y\right|^2,$$

where the polynomials G_n and H_n are as in Proposition 2.2.

Proof. By Theorem 2.6, there exists a $2(n+1)\times 2(n+1)$ matrix A such that $P(t,x,y) = F_A(t,x,y)$, where P is the homogeneous polynomial in Proposition 2.4. But also, by the same proposition,

$$\begin{aligned}\sup\Big\{\mathrm{Re}(e^{-i\theta}z)\colon z\in W(T(a,b,c))\Big\} &= \max\{t\in\mathbb{R}\colon F_A(t,-\cos\theta,-\sin\theta)=0\}\\ &= \max\Big\{\mathrm{Re}(e^{-i\theta}z)\colon z\in W(A)\Big\}\end{aligned}$$

for each $\theta\in[0,2\pi)$, and hence the closure of the numerical range of $T(a,b,c)$ equals the numerical range of A. □

It is clear that given the operator T, one can compute the polynomial P which, by the Plaumann-Vinzant Theorem, is the Kippenhahn polynomial of some matrix A. The question arises on whether the matrix A can be explicitly computed. The paper [20] shows a method for constructing such a matrix A (see also [12, 19]).

In some cases, the matrix A can be found explicitly, as the next proposition shows. The reader should compare our next result to Theorem 4.1 in [1], where an alternative method for computing the numerical range of the tridiagonal operator $T(a,b,c)$ is obtained, when a, b and c are real 2-periodic sequences.

Proposition 2.8. Let a and c be real 2-periodic sequences and let b the constant 0 sequence. If

$$S = \begin{pmatrix} \alpha_0+\alpha_1 & -\gamma_0 & -\gamma_1 & 0\\ -\gamma_0 & -\alpha_0+\alpha_1 & 0 & -\gamma_1\\ -\gamma_1 & 0 & \alpha_0-\alpha_1 & -\gamma_0\\ 0 & -\gamma_1 & -\gamma_0 & -\alpha_0-\alpha_1\end{pmatrix}$$

then $\overline{W(T(a,b,c))} = W(S)$.

Proof. It is a straightforward computation that the polynomial P in Proposition 2.4 equals

$$P(t,x,y) = \Big(t^2 - |\alpha_0 x+\gamma_0 y|^2 - |\alpha_1 x + \gamma_1 y|^2\Big)^2 - 4|\alpha_0 x+\gamma_0 y|^2 |\alpha_1 x+\gamma_1 y|^2$$

But a computation also shows that $F_S(t,x,y) = P(t,x,y)$ and hence, by Theorem 2.7, we have $\overline{W(T(a,b,c))} = W(S)$. □

We illustrate the above proposition with some examples.

Example 2.9. Let a be the 2-periodic sequence with period word 1 3, let b be the constant 0 sequence and let c be the 2-periodic sequence with period word 4 8. Then, by Proposition 2.8, if

$$S = \begin{pmatrix} 8 & \frac{1}{2}i & -\frac{7}{2}i & 0\\ \frac{1}{2}i & 1 & 0 & -\frac{7}{2}i\\ -\frac{7}{2}i & 0 & -1 & \frac{1}{2}i\\ 0 & -\frac{7}{2}i & \frac{1}{2}i & -8\end{pmatrix},$$

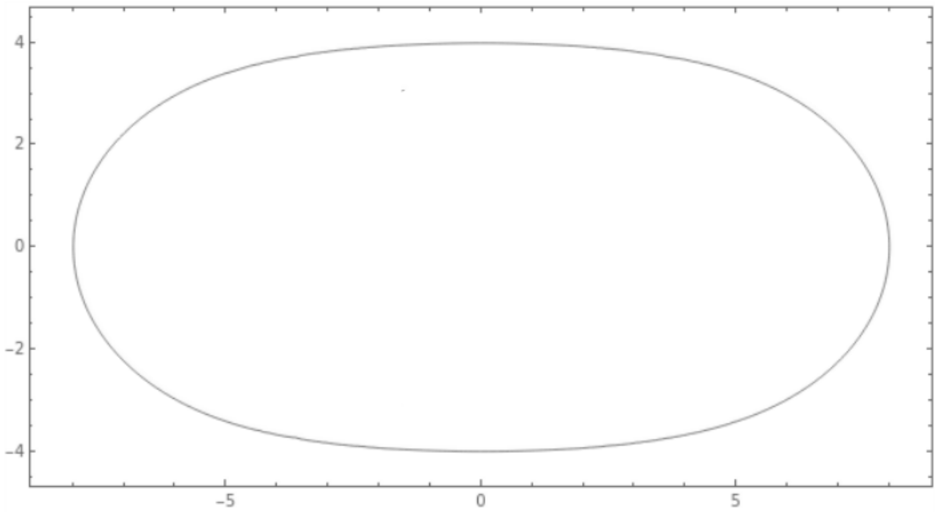

FIGURE 1. Boundary of the numerical range of S

then $\overline{W(T(a,b,c))} = W(S)$. The boundary of the numerical range of S is shown in Figure 1. The Kippenhahn polynomial of S equals

$$P(t,x,y) = t^4 - 65t^2x^2 - 25t^2y^2 + 64x^4 + 192x^2y^2 + 144y^4.$$

The quartic curve $P(t,x,y) = 0$ in the complex projective plane has a pair of ordinary singular points of multiplicity 2 at $(t,x,y) = (0,1,\pm i\sqrt{2/3})$ and there is no other singular point. So the algebraic curve theory tell us that the homogeneous polynomial $P(t,x,y)$ is irreducible in the polynomial ring.

Hence, using for example Proposition 2.3 in [8], there cannot be a matrix R of size $m\times m$, with $1 \le m < 4$ with $W(R) = W(S)$. Incidentally, this shows that the size of the matrix A in Theorem 2.7 is optimal.

Example 2.10. Let a and c be real 2-periodic sequences with period words a_0a_1 and c_0c_1 respectively, and let b be the constant sequence 0. If $a_0 = c_1$, then $\gamma_1 = 0$ and then, by Proposition 2.8, $\overline{W(T(a,b,c))} = W(S)$, where

$$S = \begin{pmatrix} \alpha_0+\alpha_1 & -\gamma_0 & 0 & 0 \\ -\gamma_0 & -\alpha_0+\alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_0-\alpha_1 & -\gamma_0 \\ 0 & 0 & -\gamma_0 & -\alpha_0-\alpha_1 \end{pmatrix}.$$

But this implies that

$$\overline{W(T(a,b,c))} = \operatorname{conv}\left(W(A+\alpha_1 I) \cup W(A-\alpha_1 I)\right),$$

where

$$A = \begin{pmatrix} \alpha_0 & -\gamma_0 \\ -\gamma_0 & -\alpha_0 \end{pmatrix}.$$

That is, $\overline{W(T(a,b,c))}$ is the convex hull of two ellipses (possibly degenerate), each one a translation of a single elllipse (possibly degenarate) centered at the origin.

Example 2.11. Let a be the 2-periodic sequence with period word $1,-1$, let b be the constant 0 sequence and let c be the constant 1 sequence. Then, by Example 2.10, we have that $\overline{W(T(a,b,c))} = \operatorname{conv}(W(A+I) \cup W(A-I))$, where

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

But it is easy to see that $W(A)$ is the closed line segment joining $-i$ and i. Hence, $\overline{W(T(a,b,c))}$ equals the convex hull of the closed line segment joining $-1-i$ and $-1+i$ and the closed line segment joining $1-i$ and $1+i$; i.e., the square with vertices $-1-i$, $-1+i$, $1-i$ and $1+i$, recovering (most of) Theorem 9 in [5].

Example 2.12. Let a and c be real 2-periodic sequences with period words a_0a_1 and c_0c_1 respectively, and let b be the constant sequence 0. If $c_0 = a_1$, then $\gamma_0 = 0$ and then, by Proposition 2.8, $\overline{W(T(a,b,c))} = W(S)$, where

$$S = \begin{pmatrix} \alpha_0+\alpha_1 & 0 & -\gamma_1 & 0 \\ 0 & -\alpha_0+\alpha_1 & 0 & -\gamma_1 \\ -\gamma_1 & 0 & \alpha_0-\alpha_1 & 0 \\ 0 & -\gamma_1 & 0 & -\alpha_0-\alpha_1 \end{pmatrix}.$$

But if

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$U^*SU = \begin{pmatrix} \alpha_0+\alpha_1 & -\gamma_1 & 0 & 0 \\ -\gamma_1 & \alpha_0-\alpha_1 & 0 & 0 \\ 0 & 0 & -\alpha_0+\alpha_1 & -\gamma_1 \\ 0 & 0 & -\gamma_1 & -\alpha_0-\alpha_1 \end{pmatrix}.$$

But this implies that

$$\overline{W(T(a,b,c))} = \operatorname{conv}(W(A+\alpha_0 I) \cup W(A-\alpha_0 I)),$$

where

$$A = \begin{pmatrix} \alpha_1 & -\gamma_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix}.$$

That is, $\overline{W(T(a,b,c))}$ is the convex hull of two ellipses (possibly degenerate), each one a translation of a single elllipse (possibly degenarate) centered at the origin.

Example 2.13. Let a be the 2-periodic sequence with period word 01, let b be the constant 0 sequence and let c be the constant 1 sequence. Then, by Example 2.12, we have that $\overline{W(T(a,b,c))} = \operatorname{conv}(W(A+I) \cup W(A-I))$, where

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i \\ -\frac{1}{2}i & -\frac{1}{2} \end{pmatrix}.$$

But, if

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \end{pmatrix},$$

then U is unitary and U^*AU equals

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $\overline{W(T(a,b,c))} = \operatorname{conv}(W(B+I) \cup W(B-I))$, recovering the result in [14, Theorem 3.6].

In the paper [15] we explore some sufficient conditions under which the matrix A can be explicitly found, namely if $b = 0$ and there is some symmetry in the periodic sequences a and c, then the polynomial P can be factored as the product of the Kippenhahn polynomials of two computable matrices, which generalizes the previous four examples.

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