

Assignment 4

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Question

- The random variables X_i are i.i.d with density $ce^{-cx}U(x)$. Show that, if $x = x_1 + \cdots + x_n$, then $f_x(x)$ is an Erlang density.

Solution I

Since, we know that,

$$\text{When } f(x) = \gamma x^{b-1} e^{-cx} U(x) \text{ and } \gamma = \frac{c^{b+1}}{\Gamma(b+1)} \quad (1)$$

$$\text{It follows that } \phi(s) = \gamma \int_0^\infty x^{b-1} e^{-(c-s)x} dx = \frac{\gamma \Gamma(b)}{(c-s)^b} = \frac{c^b}{(c-s)^b} \quad (2)$$

Differentiating with respect to s and setting $s = 0$, we obtain

$$\phi^n(0) = \frac{b(b+1) \dots (b+n-1)}{c^n} = E(x^n) \quad (3)$$

Solution II

With $n = 1$ and $n = 2$, this yields

$$E(x) = \frac{b}{c} \quad (4)$$

$$E(x^2) = \frac{b(b+1)}{c^2} \quad (5)$$

$$\sigma^2 = \frac{b}{c^2} \quad (6)$$

The exponential density is a special case obtained with $b = 1$, $C = \lambda$:

$$f(x) = \lambda e^{-\lambda x} U(x) \quad (7)$$

$$\phi(s) = \frac{\lambda}{\lambda - s} \quad (8)$$

where $E(x) = \frac{1}{\lambda}$ and $\lambda^2 = \frac{1}{\sigma^2}$

Solution III

Now, It is given that,

$$f_1(x) = ce^{-cx} U(x) \quad (9)$$

$$\text{then } \phi_1(s) = \frac{c}{c-s} \quad (10)$$

$$\phi(s) = \phi_1(s) + \phi_2(s) + \cdots + \phi_n(s) = \frac{c^n}{(c-s)^n} \quad (11)$$

$$\text{Hence, } f(x) = \frac{c^n x^{n-1}}{(n-1)!} e^{-cx} U(x) \quad (12)$$