



## 1 UNIFORM RANDOM NUMBERS

Let  $U$  be a uniform random variable between 0 and 1.

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat .

**Solution:** Download the following files and execute the C program.

```
wget https://github.com/
Ritvik-Sai-C/Random_
numbers/blob/main/1.1/
exrand.c
wget https://github.com/
Ritvik-Sai-C/Random_
numbers/blob/main/1.1/
coeffs.h
```

Use the below command in the terminal to run the code

```
gcc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

**Solution:** The graph 0 is obtained by running the below code

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/1.2/uni_cdf.
py
```

Run the following command in the terminal to run the code

```
python3 uni_cdf.py
```

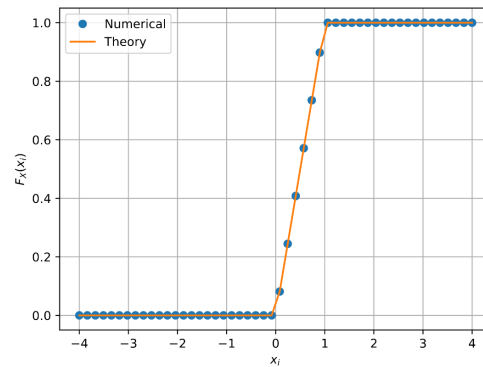


Fig. 0: The CDF of  $U$

- 1.3 Find a theoretical expression for  $F_U(x)$ . **Solution:** Since  $U$  is a uniform random variable distribution,  $P_U(x_i) = P_U(x_j) = k, \forall i, j$  CDF of  $P_U(x) = F_U(x)$

$$= \int P_U(x) dx \quad (1.2)$$

$$= \int k dx \quad (1.3)$$

$$\text{wkt } \int_0^1 k dx = 1 \quad (1.4)$$

$$\therefore k = 1 \quad (1.5)$$

$$\therefore F_U(x) = x \quad (1.6)$$

1.4 The mean of  $U$  is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.7)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.8)$$

Write a C program to find the mean and variance of  $U$ .

**Solution:**

```
wget https://github.com/
Ritvik-Sai-C/Random_
numbers/blob/main/1.4/
mean_var.c
```

Use below command to run file,

```
gcc mean_var.c -lm
./a.out
```

running the code gives us Mean =0.500007 , Variance =0.083301

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.9)$$

$$dF_U(x) = dx \quad (1.10)$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dx \quad (1.11)$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \quad (1.12)$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.13)$$

$$\therefore P_X(x) = 0, \forall x \in (1, \infty) \cap (-\infty, 0) \quad (1.14)$$

$$\text{Var}(X) = E[U^2] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.15)$$

## 2 CENTRAL LIMIT THEOREM

2.1 Generate  $10^6$  samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where  $U_i, i = 1, 2, \dots, 12$  are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

**Solution:**

```
wget https://github.com/
Ritvik-Sai-C/Random_
numbers/blob/main/1.1/
exrand.c
wget https://github.com/
Ritvik-Sai-C/Random_
numbers/blob/main/1.1/
coeffs.h
```

Running the above codes generates uni.dat and gau.dat file. Use the command

```
gcc exrand.c -lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of  $X$  using the samples in gau.dat. What properties does a CDF have?

**Solution:** The CDF of  $X$  is plotted in 0, Properties of the CDF:

- $\Phi(x) = \frac{P(Z \leq x)}{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{u^2}{2}\right\} du} =$
- $\lim_{x \rightarrow \infty} \Phi(x) = 1, \lim_{x \rightarrow -\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

2.3 Load gau.dat in python and plot the empirical PDF of  $X$  using the samples

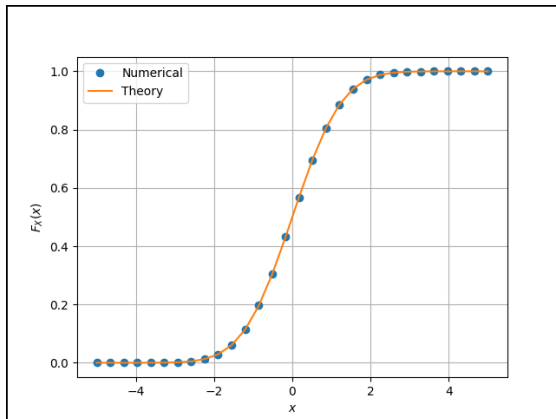


Fig. 0: The CDF of  $X$

in `gau.dat`. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

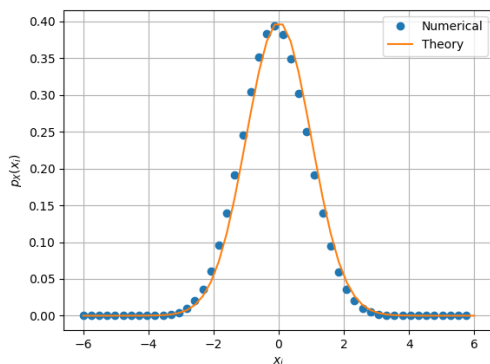


Fig. 0: The PDF of  $X$

**Solution:** The PDF of  $X$  is plotted in 0 using the code below

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
```

```
blob/main/2.3/gauss_
pdf.py
```

Use the below command to run the code:

```
python3 pdf.py
```

Properties of PDF:

- PDF is symmetric about  $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of  $X$  by writing a C program.

**Solution:** Running the below code gives Mean = 0.000294 Variance= 0.999560

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/2.4/mean_var
(gau).c
```

Command used:

```
gcc mean_var(gau).c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Given,  $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.4)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} dx \quad (2.5)$$

$$\because x e^{-\frac{x^2}{2}} \text{ is a odd function,} \quad (2.6)$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.7)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x (x e^{-\frac{x^2}{2}}) dx \quad (2.8)$$

Using integration by parts:

$$= x \int x e^{-\frac{x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{x^2}{2}} dx \quad (2.9)$$

$$I = \int x e^{-\frac{x^2}{2}} dx \quad (2.10)$$

$$\text{Let } \frac{x^2}{2} = t \quad (2.11)$$

$$\Rightarrow x dx = dt \quad (2.12)$$

$$\Rightarrow \int e^{-t} dt = -e^{-t} + c \quad (2.13)$$

$$\therefore \int x e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c \quad (2.14)$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \quad (2.15)$$

$$\text{Also, } \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \quad (2.16)$$

$$\therefore \text{ substituting limits we get, } E[x^2] = 1 \quad (2.17)$$

$$\text{Var}(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.18)$$

### 3 FROM UNIFORM TO OTHER

#### 3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

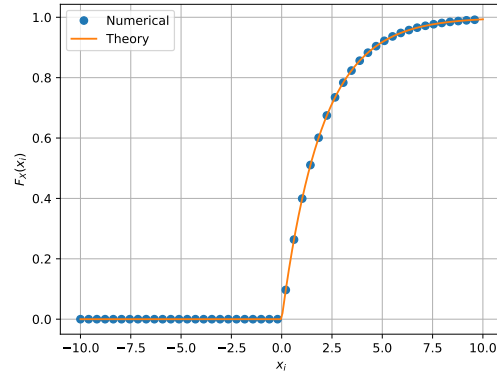


Fig. 0: CDF for (3)

**Solution:** Running the below code generates samples of V from file uni.dat(U).

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/3.1/V.py
```

Use the below command in the terminal to run the code:

```
python3 V.py
```

Now these samples are used to plot (0) by running the below code,

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/3.1/V_cdf.py
```

Use the below command to run the code:

```
python3 V_cdf.py
```

3.2 Find a theoretical expression for  $F_V(x)$ .

$$F_V(x) = P(V \leq x) \quad (3.2)$$

$$= P(-2\ln(1 - U) \leq x) \quad (3.3)$$

$$= P(1 - e^{\frac{-x}{2}} \geq U) \quad (3.4)$$

$$P(U < x) = \int_0^x dx = x \quad (3.5)$$

$$\therefore P(1 - e^{\frac{-x}{2}} \geq U) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \quad (3.6)$$

#### 4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

**Solution:** Run the below code to generate T.dat

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/4.1/T_gen_
dat.c
```

Run the command below in the terminal

```
gcc T_gen_dat.c -lm
./a.out
```

4.2 Find the CDF of  $T$ .

$$F_T(t) = P(T < t) \quad (4.2)$$

$$= P(U_1 + U_2 < t) \quad (4.3)$$

we know that  $0 \leq U_1 \leq 1$  and  $0 \leq U_2 \leq 1$

$\therefore 0 \leq U_1 + U_2 \leq 2$ , so

$$\forall t > 2, P(U_1 + U_2 < t) = 1$$

$$\forall t < 0, P(U_1 + U_2 < t) = 0$$

for  $0 \leq t \leq 2$  let us split it into 2 cases, for  $0 \leq t \leq 1$  and  $1 < t \leq 2$

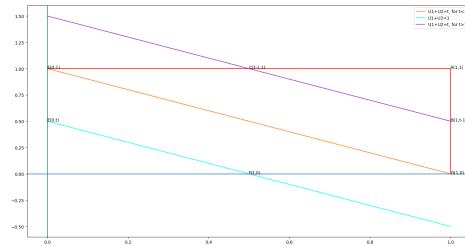


Fig. 0: Plot

The above figure is produced by the following code

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/4.2/T_plot.
py
```

Run the following command in the terminal to run the code

```
python3 T_plot.py
```

From Fig (0)

$$P(U_1 + U_2 < t, 0 \leq t \leq 1) = \frac{\Delta(EOF)}{\Delta(AEOD)} \quad (4.4)$$

$$= \frac{t^2}{2} \quad (4.5)$$

$$P(U_1 + U_2 < t, 1 \leq t \leq 2) = \frac{\Delta(ABC)}{\Delta(AEOD)} \quad (4.6)$$

$$= 1 - \frac{(2-t)^2}{2} \quad (4.7)$$

$$\therefore F_T(t) = P(U_1 + U_2 < t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.8)$$

4.3 Find the PDF of  $T$ .

**Solution:**

$$P_T(t) = \frac{d(F_T(t))}{dt} \quad (4.9)$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.10)$$

4.4 Find the theoretical expressions for the PDF and CDF of  $T$ .

**Solution:**

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.11)$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 1 - \frac{(2-t)^2}{2} & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.12)$$

4.5 Verify your results through a plot.  
**Solution:** Run the below code to get

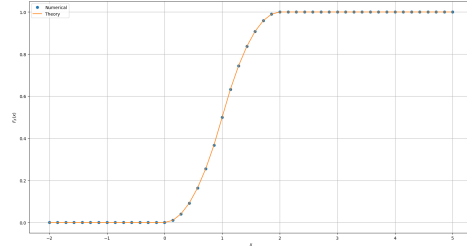


Fig. 0: CDF for (4)

the cdf

```
https://github.com/Ritvik-Sai-C/Random_numbers/blob/main/4.5/T_cdf.py
```

Use the following command in the terminal to run the code

```
python3 T_cdf.py
```

Run the below code to get the pdf

```
https://github.com/Ritvik-Sai-C/Random_numbers/blob/main/4.5/T_pdf.py
```

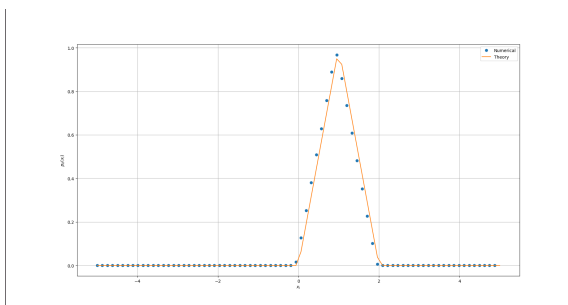


Fig. 0: PDF for (4)

Use the following command in the terminal to run the code

```
python3 T_pdf.py
```

## 5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable  $X \in \{1, -1\}$ .

**Solution:** Run the below code,

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/5.1/
bernoulli.c
```

Use the below command in the terminal to run the code

```
gcc bernoulli.c -lm
./a.out
```

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where  $A = 5$  dB, and  $N \sim \mathcal{N}(0, 1)$ .

**Solution:** Run the below code for generating samples of  $Y$ ,

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/5.2/Ygen.c
```

Use the below command in the terminal to run the code

```
gcc Ygen.c -lm
./a.out
```

5.3 Plot  $Y$  using a scatter plot.

**Solution:**

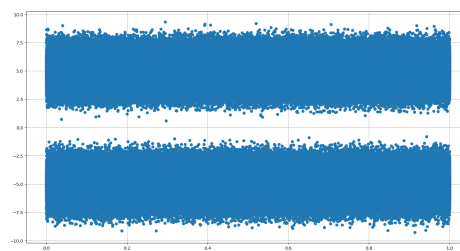


Fig. 0: plot for (5.3)

Run the following code to generate the scatter plot

```
https://github.com/Ritvik
-Sai-C/Random_numbers/
blob/main/5.3/Yplot.py
```

Use the below command to run the code,

```
python3 Yplot.py
```

5.4 Guess how to estimate  $X$  from  $Y$ .

**Solution:** if the received signal is greater than 0, then the receiver assumes  $s_1$  was transmitted.

if the received signal is less than or equal to 0, then the receiver assumes  $s_0$  was transmitted, where  $s_0$  and  $s_1$  are cases of  $X = 1$  and  $X = -1$  respectively where threshold 0 is taken to be

the decision boundary.

$$y > 0 \implies s_1 \quad (5.2)$$

$$y \leq 0 \implies s_0 \quad (5.3)$$

5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.4)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.5)$$

**Solution:** Here  $s_1$  and  $s_2$  are equally probable ie,  $p(s_1) = p(s_0) = \frac{1}{2}$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{x^2}{2}} dx \quad (5.6)$$

$$\begin{aligned} p(e|s_1) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{(y-A)^2}{2}} dy \\ &= Q(A) \end{aligned} \quad (5.7)$$

$$\begin{aligned} p(e|s_0) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{(y+A)^2}{2}} dy \\ &= Q(A) \end{aligned} \quad (5.8)$$

5.6 Find  $P_e$  assuming that  $X$  has equiprobable symbols.

**Solution:** Total probability of bit error:

$$P_e = p(s_1)p(e|s_1) + p(s_0)p(e|s_0) \quad (5.9)$$

$$= \frac{1}{2}[Q(A) + Q(A)] \quad (5.10)$$

$$\because p(s_1) = p(s_0) = \frac{1}{2}, X \text{ has equiprobable symbols} \quad (5.11)$$

$$= Q(A)$$

5.7 Verify by plotting the theoretical  $P_e$  with respect to  $A$  from 0 to 10 dB.

**Solution:**

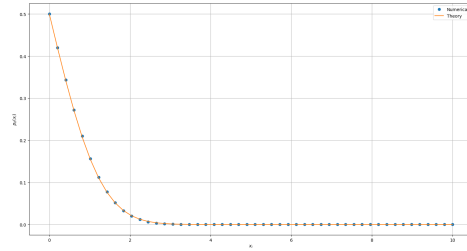


Fig. 0: plot for (5.7)

[https://github.com/Ritvik-Sai-C/Random\\_numbers/blob/main/5.7/Pplot.py](https://github.com/Ritvik-Sai-C/Random_numbers/blob/main/5.7/Pplot.py)

Use the below command to run the code,

```
python3 Pplot.py
```

5.8 Now, consider a threshold  $\delta$  while estimating  $X$  from  $Y$ . Find the value of  $\delta$  that minimizes the theoretical  $P_e$ .

**Solution:** Threshold= $\delta$ ,



$$\Rightarrow p_X(1) = 1 - p$$

$$y > \delta \Rightarrow s_1 \quad (5.12)$$

$$y \leq \delta \Rightarrow s_0 \quad (5.13)$$

$$p(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} e^{-\frac{(y-A)^2}{2}} dy \quad (5.14)$$

$$p(e|s_0) = \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} e^{-\frac{(y+A)^2}{2}} dy$$

$$P_e = \frac{1}{2\sqrt{2\pi}} \left( \int_{-\infty}^{\delta} e^{-\frac{(y-A)^2}{2}} dy + \int_{\delta}^{\infty} e^{-\frac{(y+A)^2}{2}} dy \right)$$

$$P_e = \frac{Q(\delta + A) + Q(A - \delta)}{2} \quad (5.15)$$

$$P_e = f(\delta) \quad (5.16)$$

$$\text{to minimize } P_e, \frac{d(f(\delta))}{d\delta} = 0 \text{ and } f''(\delta) > 0 \quad (5.17)$$

$$e^{-\frac{(A-\delta)^2}{2}} - e^{-\frac{(A+\delta)^2}{2}} = 0 \quad (5.18)$$

$$\therefore A - \delta = A + \delta, \Rightarrow \delta = 0 \quad (5.19)$$

$$f''(\delta) = k((A - \delta)e^{-\frac{(A-\delta)^2}{2}} + (A + \delta)e^{-\frac{(A+\delta)^2}{2}}) > 0 \quad (5.20)$$

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.21)$$

**Solution:**  $p_X(0) = p$

$$P_e = pP(e|s_0) + (1 - p)P(e|s_1) \quad (5.22)$$

$$= pQ(A + \delta) + (1 - p)Q(A - \delta) \quad (5.23)$$

$$\frac{d(P_e)}{d(\delta)} = 0 \quad (5.24)$$

$$\Rightarrow e^{\frac{(A+\delta)^2 - (A-\delta)^2}{2}} = \frac{p}{1-p} \quad (5.25)$$

$$\therefore \delta = \frac{1}{2A} \log\left(\frac{p}{1-p}\right) \quad (5.26)$$

$$\frac{d(P_e)}{d(\delta)} \quad \text{at } \delta + \epsilon > 0 \quad (5.27)$$

$$\frac{d(P_e)}{d(\delta)} \quad \text{at } \delta - \epsilon < 0$$

$$\therefore \delta = \frac{1}{2A} \log\left(\frac{p}{1-p}\right) \rightarrow \text{minima}$$

$$A = 5 \Rightarrow \delta = \frac{1}{10} \log\left(\frac{p}{1-p}\right) \quad (5.28)$$

5.10 Repeat the above exercise using the MAP criterion.

**Solution:**

$$P_{X|Y}(x|y)\big|_{X=1} = \frac{P(Y = y|X = 1)P(X = 1)}{P(Y = y)} \quad (5.29)$$

$$P(Y = y) = P(Y = y|X = 1)P(X = 1) + P(Y = y|X = -1)P(X = -1) \quad (5.30)$$

$$P(Y = y|X = 1)P(X = 1) = pP(Y = A + N) \quad (5.31)$$

$$= p \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}} \right) \quad (5.32)$$

$$\therefore P_{X|Y}(x|y)\big|_{X=1} = \frac{p \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}} \right)}{P(Y = y)} \quad (5.33)$$

$$P_{X|Y}(x|y)\big|_{X=-1} = \frac{P(Y = y|X = -1)P(X = -1)}{P(Y = y)} \quad (5.34)$$

$$P(Y = y|X = -1)P(X = -1) = (1 - p)P(Y = -A + N) \\ = (1 - p) \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} \right) \quad (5.35)$$

$$\therefore P_{X|Y}(x|y)\big|_{X=-1} = \frac{(1 - p) \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} \right)}{P(Y = y)} \quad (5.36)$$

Now comparing  $a = P_{X|Y}(x|y)\big|_{X=-1}$  and  $b = P_{X|Y}(x|y)\big|_{X=1}$ , if  $a > b$ ,  $X = -1$  is more likely,  $a < b$ ,  $X = 1$  is more likely.

$$pe^{-\frac{(y-A)^2}{2}} \geq (1-p)e^{-\frac{(y+A)^2}{2}} \\ \implies e^{2Ay} \geq \frac{1-p}{p}$$

$$\implies y \geq \frac{1}{2A} \log \left( \frac{1-p}{p} \right)$$

$$\delta = \frac{1}{2A} \log \left( \frac{1-p}{p} \right)$$

$$y > \delta \implies X=1 \text{ is more likely}$$

$$y < \delta \implies X=-1 \text{ is more likely}$$