Testing of Two parameter shifted exponential distribution

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Introduction :

Let us consider a real life statistics problem. Suppose starting from now onward, we want to estimate the amount of time it takes for an earthquake to occur. Now naturally this occurrence of the earthquake is a random phenomenon; no one can say exactly when the earthquake will occur. So, we can say that the time of occurrence of an earthquake is a random quantity or rather it is a random variable having some probability distribution.

Now, let us consider another scenario. Suppose we want to estimate the time till which a car battery lasts. Again we can see that this ending of battery life is a random phenomenon. So the time of occurrence of the failure (malfunctioning/ending of life) of the car battery can also be considered as a random variable having its probability distribution.

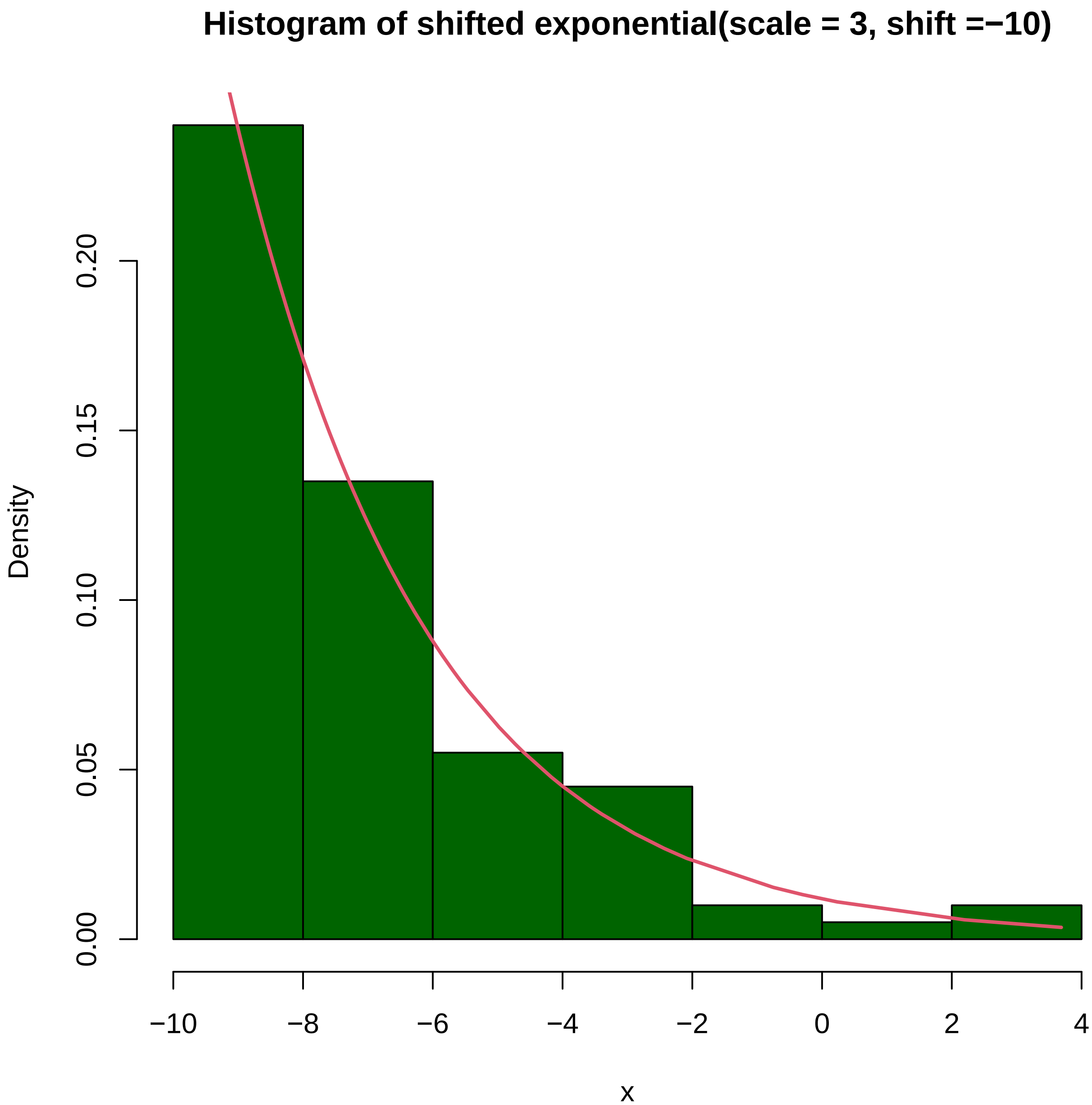
We can recall that this random variable is exactly what we call the **survival time** and its probability distribution is known as **survival or life-time distribution**. In many scenarios we can see that this survival distribution is nothing but an exponential distribution, or rather, more generally speaking, is a two-parameter exponential distribution.

The two-parameter exponential pdf is given by

$$f(x;\mu,\sigma) = \frac{1}{\sigma}e^{-\frac{x-\mu}{\sigma}} \quad 0 < x < \infty \quad x > \mu \,, \, \sigma > 0$$

where μ is the threshold parameter and σ is the scale parameter.

In this project, w**e take a deeper look at the two-parameter exponential distribution. We shall be constructing the hypothesis test of $H_0 : \mu = \mu_0(known)\,, \sigma = \sigma_0(known)$ against the all possible one-sided and two-sided alternatives theoretically calculating its power. Then we will see its practical implementations in R.**


Histogram of shifted exponential(scale = 3, shift =−10)
Density
0.00
0.05
0.10
0.15
0.20
−10
−8
−6
−4
−2
0
2
4
x

Testing procedure:

We consider 3 cases.Let us discuss one after another.

Case 1: σ **is known ,** μ **is unknown**

Let, $Y = X_{(1)}$ =min$\{X_1, X_2, ..., X_n\}$
Marginal pdf of Y is given by
$f_Y(y) = n(1 - F(y))^{n-1} f_\theta(y)$
$= n[\int_y^\infty \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}} dx]^{n-1} \times \frac{1}{\sigma} e^{-\frac{(y-\mu)}{\sigma}}$
$= \frac{n}{\sigma} e^{-\frac{n(y-\mu)}{\sigma}}, y \geq \mu$
Now let , $Z = (Y - \mu) \sim exp(n/\sigma), \sigma > 0$
Under $H_{01} : \mu = \mu_0$,test statistic $Z_{H_{01}} = (X_{(1)} - \mu_0)$
This test statistic will provide the above test.

For Right-sided test,

The best critical region of size α for testing $H_{01} : \mu = \mu_0$ vs $H_{01}' : \mu > \mu_0$ is given by
$w_1 = \{\underset{\sim}{x} : Z_{H_{01}} > c\}$ such that $P_{H_{01}}(w_1) = \alpha$
i.e. $c = \Gamma_{\alpha;1,n/\sigma}$=upper α level of $exp(n/\sigma)$
So,
$w_1 = \{\underset{\sim}{x} : Z_{H_{01}} > \Gamma_{\alpha;1,n/\sigma}\}$

Decision:

Hence at α level of significance,we reject H_{01} if obs $Z_{H_{01}} > \Gamma_{\alpha;1,n/\sigma}$ or ow we accept H_{01} at α level.

For Left-sided test,

The best critical region of size α for testing $H_{01} : \mu = \mu_0$ vs $H_{01}' : \mu < \mu_0$ is given by
$w_2 = \{\underset{\sim}{x} : Z_{H_{01}} < c\}$ such that $P_{H_{01}}(w_2) = \alpha$
i.e. $c = \Gamma_{1-\alpha;1,n/\sigma}$=upper $1 - \alpha$ level of $exp(n/\sigma)$
So,
$w_2 = \{\underset{\sim}{x} : Z_{H_{01}} < \Gamma_{\alpha;1,n/\sigma}\}$

Decision:

Hence at α level of significance,we reject H_{01} if obs $Z_{H_{01}} < \Gamma_{\alpha;1,n/\sigma}$ or ow we accept H_{01} at α level.

For Both-sided test,

The best critical region of size α for testing $H_{01} : \mu = \mu_0$ vs $H_{01}' : \mu \neq \mu_0$ is given by
$w_3 = \{\underset{\sim}{x} : Z_{H_{01}} > \Gamma_{\alpha;1,n/\sigma}$ or $Z_{H_{01}} < \Gamma_{1-\alpha;1,n/\sigma}\}$ such that
$P_{H_{01}}(w_3) = P_{H_{01}}(Z_{H_{01}} > \Gamma_{\alpha;1,n/\sigma}) + P_{H_{01}}(Z_{H_{01}} < \Gamma_{1-\alpha;1,n/\sigma})$
$= \alpha/2 + 1 - (1 - \alpha/2)$
$= \alpha$

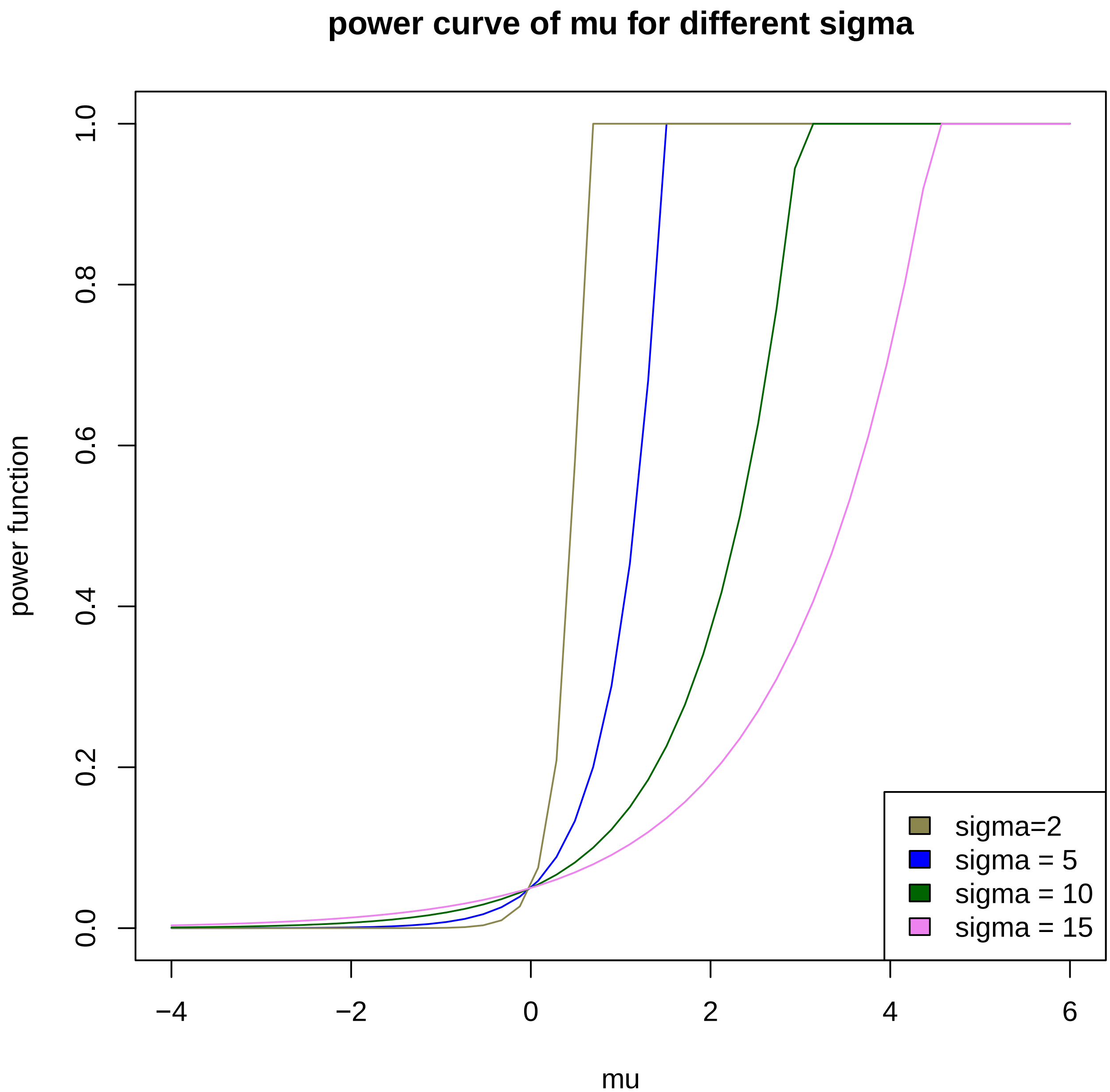
Decision:

Hence at α level of significance,we reject H_{01} if obs $Z_{H_{01}} > \Gamma_{\alpha;1,n/\sigma}$ or $Z_{H_{01}} < \Gamma_{1-\alpha;1,n/\sigma}$
ow we accept H_{01} at α level.

2.1 Power function:

Here just we consider the case $H_{01} : \mu = 0$ vs $H_{01}^{'} : \mu > 0$

```
power_mu <- function(n=10,alpha=0.05,mu,mu0=0,sigma_known = 5)
{
exp_alpha = qgamma(1-alpha,shape=1,rate = n/sigma_known)
if(mu>(exp_alpha+mu0))
{
power_mu.vec=rep(1,length(mu>(exp_alpha+mu0)))
}
else
{
power_mu.vec = exp(-n*(exp_alpha+mu0-mu)/sigma_known)
 }
return(power_mu.vec)
}
supp=seq(-4,6,length=50)
power_mu.vec=NULL
power_mu.vec.1=NULL
power_mu.vec.2=NULL
power_mu.vec.3=NULL
for(i in 1:length(supp))
{
power_mu.vec[i]=power_mu(mu=supp[i])
}
for(i in 1:length(supp))
{
power_mu.vec.1[i]=power_mu(mu=supp[i],sigma_known = 2)
}
for(i in 1:length(supp))
{
power_mu.vec.2[i]=power_mu(mu=supp[i],sigma_known = 10)
}
for(i in 1:length(supp))
{
power_mu.vec.3[i]=power_mu(mu=supp[i],sigma_known = 15)
}
plot(supp,power_mu.vec,type="l",col="blue",,xlab="mu",ylab="power function",main="power cur
lines(supp,power_mu.vec.1,type="l",col="khaki4")
lines(supp,power_mu.vec.2,type="l",col="darkgreen")
lines(supp,power_mu.vec.3,type="l",col="violet")
legend("bottomright",c("sigma=2","sigma = 5","sigma = 10","sigma = 15"),fill=c("khaki4","bl
```



Case 2: μ is known , σ is unknown

$Z = (Y - \mu) \sim exp(n/\sigma), \beta > 0$

So, $\frac{2nZ}{\sigma} \sim \chi^2_2$

Under $H_{02} : \sigma = \sigma_0$,test statistic $T_{H_{02}} = \frac{2nZ}{\sigma_0}$

This test statistic will provide the above test.

For Right-sided test,

The best critical region of size α for testing $H_{02} : \sigma = \sigma_0$ vs $H_{02}' : \sigma > \sigma_0$ is given by

$w_1 = \{\underset{\sim}{x} : T_{H_{02}} > c\}$ such that $P_{H_{01}}(w_1) = \alpha$

i.e. $c = \chi^2_{2;\alpha}$=upper α level of χ^2_2 distribution.

So,

$w_1 = \{\underset{\sim}{x} : T_{H_{02}} > \chi^2_{2;\alpha}\}$

Decision:

Hence at α level of significance,we reject H_{02} if obs $T_{H_{02}} > \chi^2_{2;\alpha}$ or ow we accept H_{02} at α level.

For Left-sided test,

The best critical region of size α for testing $H_{02} : \sigma = \sigma_0$ vs $H^{'}_{02} : \sigma < \sigma_0$ is given by
$w_2 = \{\underset{\sim}{x} : T_{H_{02}} < c\}$ such that $P_{H_{02}}(w_2) = \alpha$
i.e. $c = \chi^2_{2;1-\alpha}$=upper $1 - \alpha$ level of χ^2_2 distribution.
So,
$w_2 = \{\underset{\sim}{x} : T_{H_{02}} < \chi^2_{2;1-\alpha}\}$

Decision:

Hence at α level of significance,we reject H_{02} if obs $T_{H_{02}} < \chi^2_{2;1-\alpha}$ or ow we accept H_{02} at α level.

For Both-sided test,

The best critical region of size α for testing $H_{02} : \sigma = \sigma_0$ vs $H^{'}_{02} : \sigma \neq \sigma_0$ is given by
$w_3 = \{\underset{\sim}{x} : T_{H_{02}} > \chi^2_{2;\alpha}$ or $T_{H_{02}} < \chi^2_{2;1-\alpha}\}$ such that
$P_{H_{02}}(w_3) = P_{H_{02}}(T_{H_{02}} > \chi^2_{2;\alpha}) + P_{H_{02}}(T_{H_{02}} < \chi^2_{2;1-\alpha})$
$= \alpha/2 + 1 - (1 - \alpha/2)$
$= \alpha$

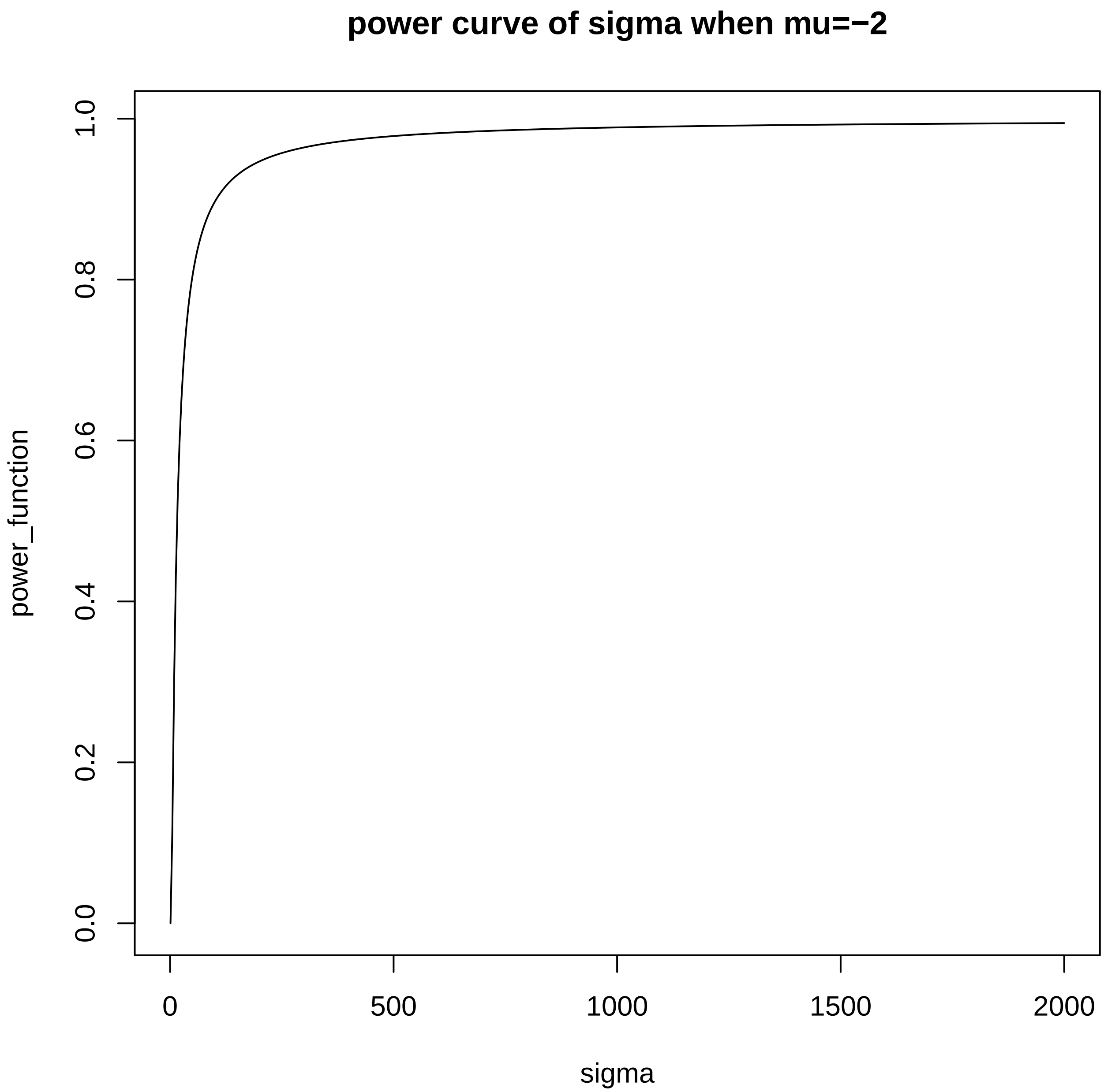
Decision:

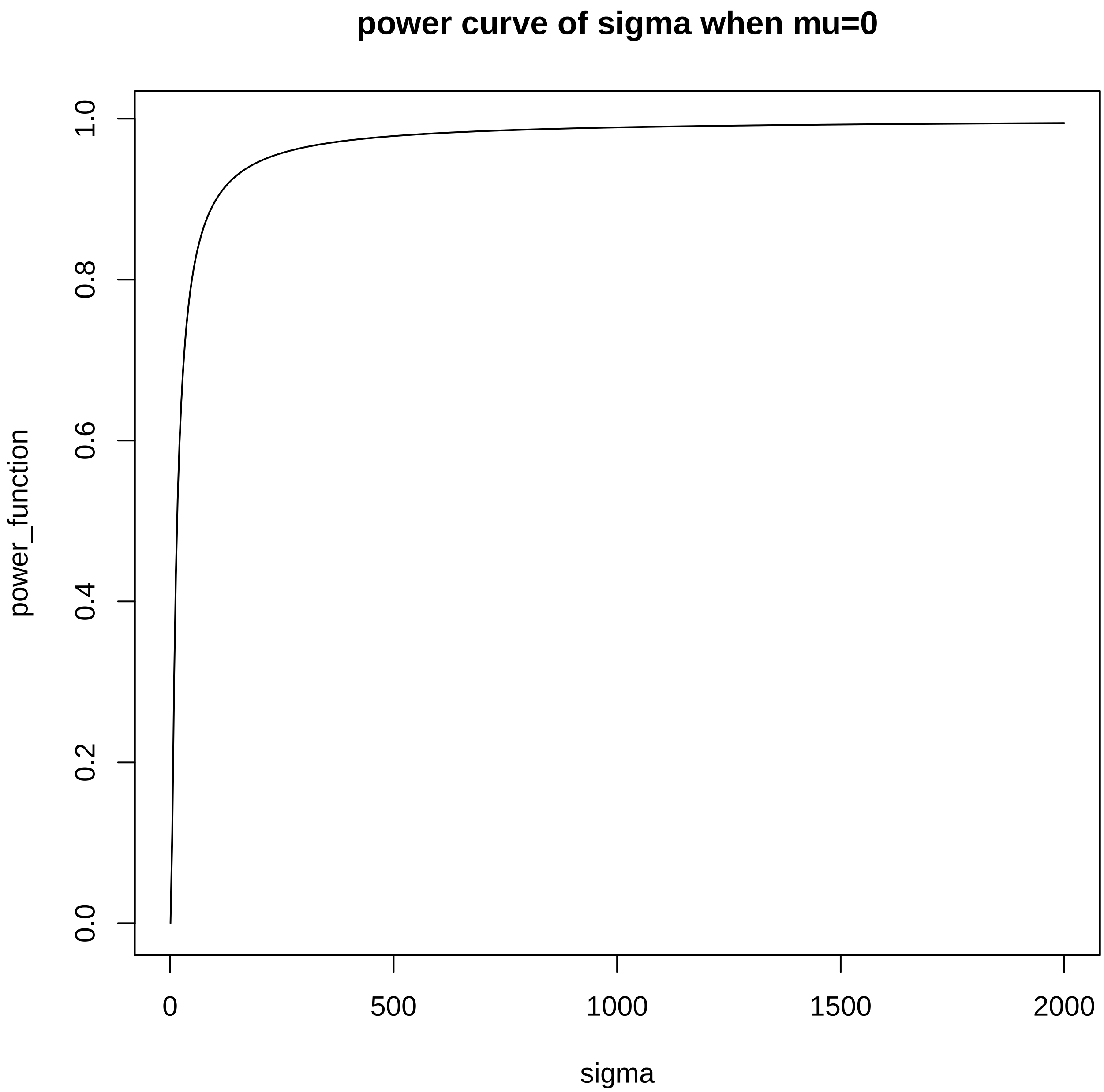
Hence at α level of significance,we reject H_{02} if obs $T_{H_{02}} > \chi^2_{2;\alpha}$ or $T_{H_{02}} < \chi^2_{2;1-\alpha}$
ow we accept H_{02} at α level.

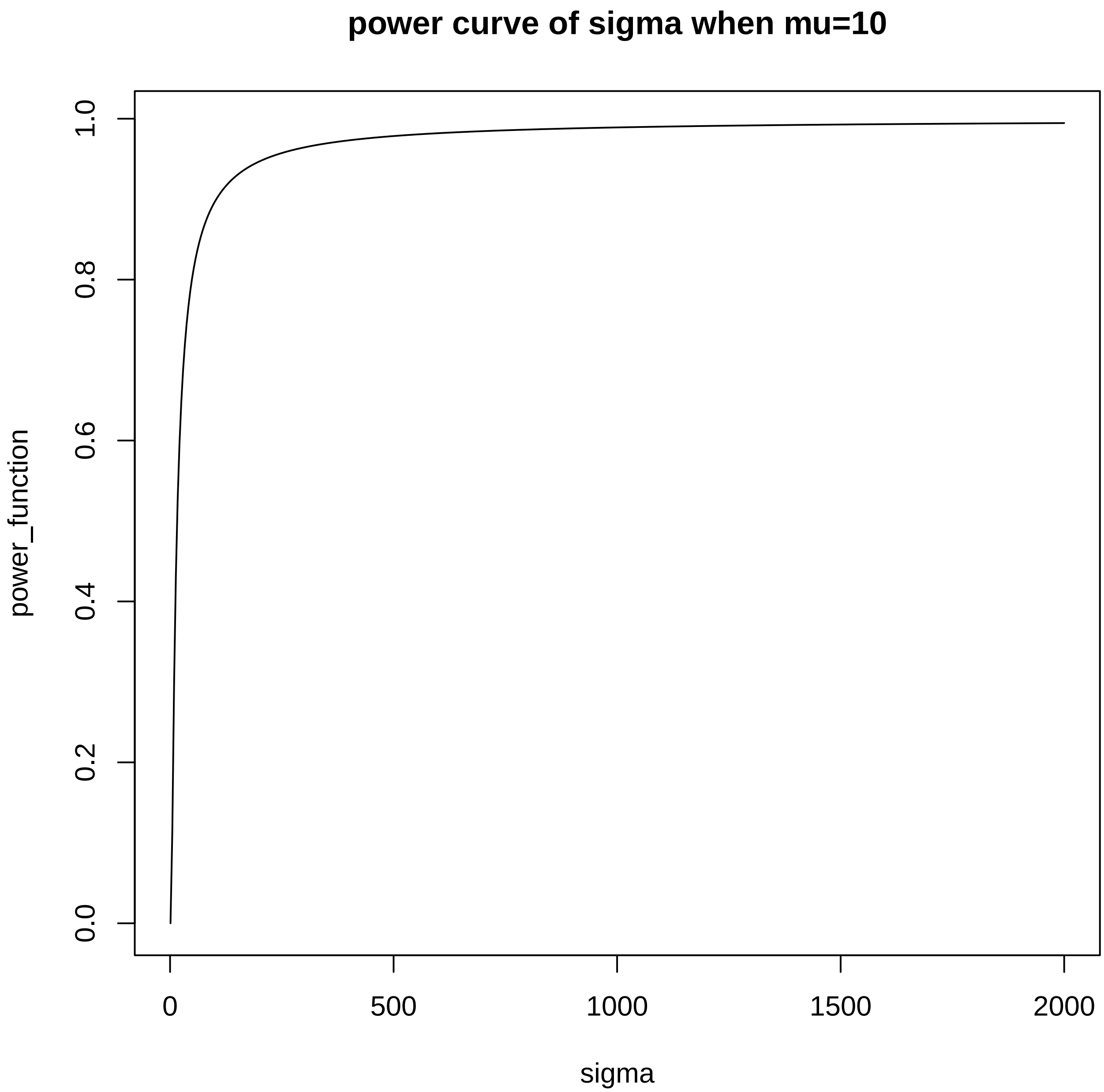
2.2 Power function:

Here just we consider the case $H_{02} : \sigma = 1$ vs $H^{'}_{02} : \sigma > 1$

```
power_sigma <- function(n=10,alpha=0.05,sigma,sigma0=1,mu_known = -10)
{
chi_alpha = qchisq(1-alpha,n,2)
power_mu.vec = exp(-(sigma0*chi_alpha)/(2*sigma))
return(power_mu.vec)
}
supp=seq(1,2000,length=500)
plot(supp,power_sigma(sigma=supp),type="l",xlab="sigma",ylab="power_function",main="power o
```


power curve of sigma when mu=−2
power_function
0.0
0.2
0.4
0.6
0.8
1.0
0
500
1000
1500
2000
sigma


power curve of sigma when mu=0
power_function
0.0
0.2
0.4
0.6
0.8
1.0
0
500
1000
1500
2000
sigma


power curve of sigma when mu=10
power_function
0.0
0.2
0.4
0.6
0.8
1.0
0
500
1000
1500
2000
sigma

Case 3: Both μ , σ unknown

Let,$\theta = (\mu, \sigma)$

Marginal pdf of X_i is given by,
$f_\theta(x_i) = \frac{1}{\sigma} e^{-\frac{(x_i-\mu)}{\sigma}}, i = 1(1)n$
Now let ,$U_i = (X_i - \mu) \sim exp(1/\sigma), \sigma > 0$
Joint pdf of $U_{(1)}, U_{(2)}, ..., U_{(n)}$ is given by
$f_\theta(.) = n! \prod_{i=1}^n f_\theta(u_i)$
$= n! \left(\frac{1}{\sigma}\right)^n e^{-\frac{\sum_{i=1}^n u_i}{\sigma}}$

Let us consider the transformation $(U_{(1)}, U_{(2)}, ..., U_{(n)}) \to (V_1, V_2, ..., V_n)$ such that
$V_1 = U_{(1)}$
$V_2 = U_{(2)} - U_{(1)}$
.
.
.
.
$V_n = U_{(n)} - U_{(n-1)}$
So,
$U_{(1)} = V_1, U_{(2)} = V_1 + V_2, ..., U_{(n)} = \sum_{i=1}^n V_i$
Jacobian of the transformation,
$|j| = 1$
So joint pdf of $(V_1, V_2, ..., V_n)$ is given by,
$f_\theta(\underset{\sim}{v}) = |j| \times f_\theta(\underset{\sim}{u})|_{u_1=v_1, u_2=v_2, ..., u_n=v_n}$
$= 1 \times n! \left(\frac{1}{\sigma}\right)^n e^{-\frac{v_1+(v_1+v_2)+...........+\sum_{i=1}^n v_i}{\sigma}}$
$= \frac{n!}{\sigma^n} \times e^{-\frac{nv_1+(n-1)v_2+................+v_n}{\sigma}}$
$= \frac{n}{\sigma} e - \frac{nv_1}{\sigma} \times \frac{(n-1)}{\sigma} e - \frac{(n-1)v_2}{\sigma} \times .... \times \frac{1}{\sigma} e - \frac{v_n}{\sigma}$
Hence,$V_i's$ are all independent.
$V_i \sim exp(mean = \frac{\sigma}{n-i+1}), i = 1(1)n$

Let us,$T = \sum_{i=1}^n (X_i - X_{(1)})$
$= \sum_{i=1}^n (X_i - \alpha) - n(X_{(1)} - \alpha)$
$= \sum_{i=1}^n U_i - nU_{(1)}$
$= U_{(2)} + U_{(3)} + ... + U_{(n)}$
$= (n-1)V_2 + (n-2)V_3................ + V_n$
So,$T \sim gamma(n-1, mean = \sigma)$ $[V_i \sim exp(mean = \frac{\sigma}{n-i+1}) => (n-i+1)V_i \sim exp(mean = \sigma)]$
Hence, $\frac{2T}{\sigma} \sim \chi^2_{2(n-1)}$..(*)

3.1

Suppose we want to test $H_{03} : \mu = \mu_0, \sigma = \sigma_0$ **vs** $H_{03}' : \mu > \mu_0, \sigma < \sigma_0$

Then, $\Theta_{03} = \{(\mu, \sigma) = (\mu_0, \sigma_0)\}, \Theta_{03'} = \{(\mu, \sigma) : \mu_0 < \mu, 0 < \sigma < \sigma_0)\}$
So the whole parametric space $\Omega = \Theta_{03} \cup \Theta_{03'} = \{(\mu, \sigma) : \mu_0 \leq \mu, 0 < \sigma \leq \sigma_0)\}$
The likelihood function of (μ, σ) given $\underset{\sim}{X} = (X_1, X_2, ..., X_n)$ is given by
$L_{\underset{\sim}{x}}(\mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{\sum(x_i-\mu)}{\sigma}} I(x_{(1)}, \mu)$
Where,

$$I(x_{(1)},\mu)=\begin{cases}1 & x_{(1)}>\mu\\ 0 & x_{(1)}<\mu\end{cases}$$

Under H_{03},$\text{Sup}_{\Theta_{03}} L_{\underset{\sim}{x}}(\mu,\sigma)=\left(\frac{1}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}} I(x_{(1)},\mu_0)$

Under **unrestricted case** $(\mu,\sigma)\in(-\infty,\infty)\times(0,\infty)$,

Mle of $\mu,\widehat{\mu}=x_{(1)}$,Mle of $\sigma,\widehat{\sigma}=\frac{1}{n}\sum_{i=1}^n(x_i-x_{(1)})$

So under **restricted case** $\Omega=\Theta_{03}\cup\Theta_{03'}$

Mle of μ,

$$\widehat{\widehat{\mu}}=\begin{cases}x_{(1)} & x_{(1)}\geq\mu_0\\ \mu_0 & x_{(1)}<\mu_0\end{cases}$$

Mle of σ,

$$\widehat{\widehat{\sigma}}=\begin{cases}\frac{1}{n}\sum_{i=1}^n(x_i-\widehat{\widehat{\mu}}) & \frac{1}{n}\sum_{i=1}^n(x_i-\widehat{\widehat{\mu}})\leq\sigma_0\\ \sigma_0 & \frac{1}{n}\sum_{i=1}^n(x_i-\widehat{\widehat{\mu}})>\sigma_0\end{cases}$$

$$=\begin{cases}\frac{1}{n}\sum_{i=1}^n(x_i-x_{(1)}) & \frac{1}{n}\sum_{i=1}^n(x_i-x_{(1)})\leq\sigma_0,x_{(1)}\geq\mu_0\\ \frac{1}{n}\sum_{i=1}^n(x_i-\mu_0) & \frac{1}{n}\sum_{i=1}^n(x_i-\mu_0)\leq\sigma_0,x_{(1)}<\mu_0\\ \sigma_0 & \frac{1}{n}\sum_{i=1}^n(x_i-\widehat{\widehat{\mu}})>\sigma_0\end{cases}$$

$$=\begin{cases}\widehat{\sigma} & \widehat{\sigma}\leq\sigma_0,x_{(1)}\geq\mu_0\\ \frac{1}{n}\sum_{i=1}^n(x_i-\mu_0) & \frac{1}{n}\sum_{i=1}^n(x_i-\mu_0)\leq\sigma_0,x_{(1)}<\mu_0\\ \sigma_0 & \frac{1}{n}\sum_{i=1}^n(x_i-\widehat{\widehat{\mu}})>\sigma_0\end{cases}$$

If $x_{(1)}<\mu_0, I(x_{(1)},\mu_0)=0$,Hence we ignore the case if $x_{(1)}<\mu_0$ for whatever the range of σ

So the LR criterian will be

$$\lambda(\underset{\sim}{x})=\frac{Sup_{\Theta_{03}}L_{\underset{\sim}{x}}(\mu,\sigma)}{Sup_{\Omega}L_{\underset{\sim}{x}}(\mu,\sigma)}=\begin{cases}\frac{\left(\frac{1}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}}}{\left(\frac{1}{\widehat{\sigma}}\right)^n e^{-\frac{\sum(x_i-x_{(1)})}{\widehat{\sigma}}}} & \widehat{\sigma}\leq\sigma_0,x_{(1)}\geq\mu_0\\ \frac{\left(\frac{1}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}}}{\left(\frac{1}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-x_{(1)})}{\sigma_0}}} & \widehat{\sigma}>\sigma_0,x_{(1)}\geq\mu_0\\ 0 & x_{(1)}<\mu_0\end{cases}$$

$$=\begin{cases}\left(\frac{\widehat{\sigma}}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}+\frac{\sum(x_i-x_{(1)})}{\widehat{\sigma}}} & \widehat{\sigma}\leq\sigma_0,x_{(1)}\geq\mu_0\\ e^{-\frac{\sum(x_{(1)}-\mu_0)}{\sigma_0}} & \widehat{\sigma}>\sigma_0,x_{(1)}\geq\mu_0\\ 0 & x_{(1)}<\mu_0\end{cases}$$

$$=\begin{cases}\left(\frac{\widehat{\sigma}}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}+\frac{\sum(x_i-x_{(1)})}{\frac{1}{n}\sum_{i=1}^n(x_i-x_{(1)})}} & \widehat{\sigma}\leq\sigma_0,x_{(1)}\geq\mu_0\\ e^{-\frac{\sum(x_{(1)}-\mu_0)}{\sigma_0}} & \widehat{\sigma}>\sigma_0,x_{(1)}\geq\mu_0\\ 0 & x_{(1)}<\mu_0\end{cases}$$

$$=\begin{cases}\left(\frac{\widehat{\sigma}}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}+n} & \widehat{\sigma}\leq\sigma_0,x_{(1)}\geq\mu_0\\ e^{-\frac{n(x_{(1)}-\mu_0)}{\sigma_0}} & \widehat{\sigma}>\sigma_0,x_{(1)}\geq\mu_0\\ 0 & x_{(1)}<\mu_0\end{cases}$$

We reject H_{03} when $\lambda(\underset{\sim}{x})<c$

Under the case $\widehat{\sigma}>\sigma_0,x_{(1)}\geq\mu_0$,

$\lambda(\underset{\sim}{x})<c => e^{-\frac{n(x_{(1)}-\mu_0)}{\sigma_0}}<c$

$=> -\frac{n(x_{(1)}-\mu_0)}{\sigma_0} < c$
$=> \frac{n(x_{(1)}-\mu_0)}{\sigma_0} > c^{'}$
$=> x_{(1)} > \mu_0$

Under the case $\widehat{\sigma} \leq \sigma_0, x_{(1)} \geq \mu_0,$
$\lambda(\underset{\sim}{x}) < c => \left(\frac{\widehat{\sigma}}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}+n} < c$
$=> nlog\left(\frac{\widehat{\sigma}}{\sigma_0}\right) - \frac{\sum(x_i-\mu_0)}{\sigma_0} + n < c_1$
$=> nlog\left(\frac{\widehat{\sigma}}{\sigma_0}\right) - \frac{\sum(x_i-x_{(1)})+n(x_{(1)}-\mu_0)}{\sigma_0} < c_1$
$=> nlog\left(\frac{\widehat{\sigma}}{\sigma_0}\right) - n\left(\frac{\widehat{\sigma}}{\sigma_0}\right) - \frac{n(x_{(1)}-\mu_0)}{\sigma_0} < c_1$
$=> nlog\left(\frac{t}{n\sigma_0}\right) - \left(\frac{t}{\sigma_0}\right) - \frac{\widehat{y}}{\sigma_0} < c_1$ $[n(x_{(1)}-\mu_0) = \widehat{y}, \widehat{\sigma} = t/n]$
or $nlog\left(\frac{t}{\sigma_0}\right) - \left(\frac{t}{\sigma_0}\right) - \frac{\widehat{y}}{\sigma_0} < c_2$
Let, $nlog\left(\frac{t}{\sigma_0}\right) - \left(\frac{t}{\sigma_0}\right) - \frac{\widehat{y}}{\sigma_0} = \Psi(t,\widehat{y})$
$\frac{\delta\Psi(t,\widehat{y})}{\delta t} = 0 => \frac{n}{t} - \frac{1}{\sigma_0} = 0 => t = n\sigma_0$
and $\frac{\delta^2\Psi(t,\widehat{y})}{\delta t^2}|_{t=n\sigma_0} = -\frac{n}{t^2}|_{t=n\sigma_0} = -\frac{1}{n\sigma_0^2} < 0$
So $\Psi(t,\widehat{y})$ attains its maximum at $t = n\sigma_0 => \widehat{\sigma} = \sigma_0$

$\frac{\delta\Psi(t,\widehat{y})}{\delta\widehat{y}} = -\frac{1}{\sigma_0}$
So, $\Psi(t,\widehat{y}) \downarrow \widehat{y}$
Max$\{\Psi(t,\widehat{y})\} = nlog(n) - n$
At $\widehat{y} = 0, nlog\left(\frac{t}{\sigma_0}\right) - \left(\frac{t}{\sigma_0}\right) = c_2$
$=> nlog(\frac{n\widehat{\sigma}}{\sigma_0}) - \frac{n\widehat{\sigma}}{\sigma_0} = c_2$
$=> nlog(\frac{\widehat{\sigma}}{\sigma_0}) - \frac{n\widehat{\sigma}}{\sigma_0} = c_3$
$=> nlog(u) - nu = c_3$ $[u = \frac{\widehat{\sigma}}{\sigma_0}]$
Let $, nlog(u) - nu = \Psi(u)$
$\frac{\delta\Psi(u)}{\delta u} = 0 => \frac{n}{u} - n = 0 => u = 1$
and $\frac{\delta^2\Psi(u)}{\delta u^2}|_{u=1} = -n < 0$
So,$\Psi(u)$ attains its maximum at $u = 1$
Hence, $\Psi(u) < c_3 => u < c_3 => \frac{\widehat{\sigma}}{\sigma_0} < c_3$
$=> \widehat{\sigma} < k$
Hence $\lambda(\underset{\sim}{x}) < c => \{x_{(1)} \geq \mu_0, \widehat{\sigma} < k\}$

Hence the CR by LRT will be :
$w = \{\lambda(\underset{\sim}{x}) = 0 \text{ or } \lambda(\underset{\sim}{x}) < c\}$
$= \{x_{(1)} < \mu_0 \text{ or } x_{(1)} \geq \mu_0, \widehat{\sigma} < k\}$
Where k is such that $P_{H_{03}}(w) = \alpha => P_{H_{03}}\{x_{(1)} < \mu_0 \text{ or } x_{(1)} \geq \mu_0, \widehat{\sigma} < k\} = \alpha$
$=> P_{H_{03}}(x_{(1)} < \mu_0) + P_{H_{03}}(x_{(1)} \geq \mu_0, \widehat{\sigma} < k) = \alpha$
$=> P_{H_{03}}(\widehat{\sigma} < k) = \alpha$ $[P_{H_{03}}(x_{(1)} < \mu_0) = 0, \{x_{(1)} \geq \mu_0\}$ =sure event$\}$
$=> P_{H_{03}}(\widehat{\sigma} < k) = \alpha$
$=> P_{H_{03}}(\frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)}) < k) = \alpha$
$=> P_{H_{03}}(T < nk) = \alpha$
$=> P_{H_{03}}(\frac{2T}{\sigma_0} < \frac{2nk}{\sigma_0}) = \alpha$
$=> \frac{2nk}{\sigma_0} = \chi^2_{1-\alpha,2(n-1)}$ $[From\ (*)\ \frac{2T}{\sigma} \sim \chi^2_{2(n-1)}]$

$=> k = \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n}$

So The CR by LRT for testing $H_{03} : \mu = \mu_0, \sigma = \sigma_0$ **vs** $H^{'}_{03} : \mu > \mu_0, \sigma < \sigma_0$is given by $w = \{x_{(1)} < \mu_0$ or $x_{(1)} \geq \mu_0, \widehat{\sigma} < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n}\}$

Power of the test $= P_{H^{'}_{03}}(w) = P_{H^{'}_{03}}(x_{(1)} < \mu_0$ or $x_{(1)} \geq \mu_0, \widehat{\sigma} < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n})$
$= P_{H^{'}_{03}}(x_{(1)} < \mu_0) + P_{H^{'}_{03}}(x_{(1)} \geq \mu_0, \widehat{\sigma} < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n})$
Under $H^{'}_{03} : \mu = \mu_1 (> \mu_0), \sigma = \sigma_1 (< \sigma_0)$, $x_{(1)} \geq \mu_1 > \mu_0$ is sure event
So,
$P_{H^{'}_{03}}(w)$
$= 0 + P_{H^{'}_{03}}(\frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)}) < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n})$
$= 0 + P_{H^{'}_{03}}(T < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2})$
$= 0 + P_{H^{'}_{03}}(\frac{2T}{\sigma_1} < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{\sigma_1})$ $[Under\ H^{'}_{03} : \mu = \mu_1 (> \mu_0), \sigma = \sigma_1 (< \sigma_0)]$
$= \int_0^{\frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{\sigma_1}} k e^{-x/2 \cdot} x^{(n-2)} dx$
$\geq \int_0^{\chi^2_{1-\alpha,2(n-1)}} k e^{-x/2 \cdot} x^{(n-2)} dx$ $[\sigma_1 < \sigma_0 => \frac{\sigma_0}{\sigma_1} > 1]$
$\geq 1 - P(X > \chi^2_{1-\alpha,2(n-1)})$
$\geq 1 - (1 - \alpha)$ =Size of the test
Hence **the test is unbiased**.

3.2 Power function:

Power function of the above test$= P_{(\mu,\sigma)}(w) = P(x_{(1)} < \mu_0$ or $x_{(1)} \geq \mu_0, \widehat{\sigma} < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n})$
$= P_{(\mu,\sigma)}(x_{(1)} < \mu_0) + P_{(\mu,\sigma)}(x_{(1)} \geq \mu_0, \widehat{\sigma} < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n})$
$= e^{-n\frac{(\mu_0 - \mu)}{\sigma}} + P_{(\mu,\sigma)}(x_{(1)} \geq \mu_0, \widehat{\sigma} < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n})$
$= e^{-n\frac{(\mu_0 - \mu)}{\sigma}} + P_{(\mu,\sigma)}(x_{(1)} \geq \mu_0, \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)}) < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2n})$
$= e^{-n\frac{(\mu_0 - \mu)}{\sigma}} + P_{(\mu,\sigma)}(x_{(1)} \geq \mu_0, \frac{2T}{\sigma} < \frac{2\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{2\sigma})$
$= e^{-n\frac{(\mu_0 - \mu)}{\sigma}} + P_{(\mu,\sigma)}(x_{(1)} \geq \mu_0, \frac{2T}{\sigma} < \frac{\sigma_0 \chi^2_{1-\alpha,2(n-1)}}{\sigma})$ $[\frac{2T}{\sigma} \sim \chi^2_{2(n-1)}]$

3.3 Alternative testing procedure

To test, $H_{03} : \mu = \mu_0, \sigma = \sigma_0$ **vs** $H^{'}_{03} : \mu = \mu_1 (> \mu_0), \sigma = \sigma_1 (< \sigma_0)$

$$\frac{f_{H^{'}_{03}}(\underset{\sim}{x},\mu,\sigma)}{f_{H_{03}}(\underset{\sim}{x},\mu,\sigma)} = \frac{\left(\frac{1}{\sigma_1}\right)^n e^{-\frac{\sum(x_i-\mu_1)}{\sigma_1}} I(x_{(1)},\mu_1)}{\left(\frac{1}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}} I(x_{(1)},\mu_0)} = \begin{cases} finite & x_{(1)} < \mu_1 \\ 0 & x_{(1)} > \mu_1 \end{cases}$$

Because,

$$I(x_{(1)}, \mu_1) = \begin{cases} 1 & x_{(1)} > \mu_1 \\ 0 & x_{(1)} < \mu_1 \end{cases}$$

$$\frac{f_{H^{'}_{03}}(\underset{\sim}{x},\mu,\sigma)}{f_{H_{03}}(\underset{\sim}{x},\mu,\sigma)} = \frac{\left(\frac{1}{\sigma_1}\right)^n e^{-\frac{\sum(x_i-\mu_1)}{\sigma_1}} I(x_{(1)},\mu_1)}{\left(\frac{1}{\sigma_0}\right)^n e^{-\frac{\sum(x_i-\mu_0)}{\sigma_0}} I(x_{(1)},\mu_0)}$$
$$= \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{(\frac{1}{\sigma_0} - \frac{1}{\sigma_1})\sum x_i + (\frac{\mu_1}{\sigma_1} - \frac{\mu_0}{\sigma_0})} \frac{I(x_{(1)},\mu_1)}{I(x_{(0)},\mu_0)}$$

So the CR by MP size α test for testing $H_{03}:\mu=\mu_0,\sigma=\sigma_0$ **vs** $H_{03}^{'}:\mu=\mu_1(>\mu_0),\sigma=\sigma_1(<\sigma_0)$ is given by:

$U=\{x_{(1)}<\mu_1 \text{or } \frac{f_{H_{03}^{'}}(\underset{\sim}{x},\mu,\sigma)}{f_{H_{03}}(\underset{\sim}{x},\mu,\sigma)}>k\}$

$=\{x_{(1)}<\mu_1 \text{ or } \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{(\frac{1}{\sigma_0}-\frac{1}{\sigma_1})\sum x_i+(\frac{\mu_1}{\sigma_1}-\frac{\mu_0}{\sigma_0})}\frac{I(x_{(1)},\mu_1)}{I(x_{(0)},\mu_0)}>k\}$

$=>\{x_{(1)}<\mu_1 \text{ or } \sum x_i<c\}$ $[\sigma_1<\sigma_0=>\frac{1}{\sigma_0}-\frac{1}{\sigma_1}<0$, so $\frac{f_{H_{03}^{'}}(\underset{\sim}{x},\mu,\sigma)}{f_{H_{03}}(\underset{\sim}{x},\mu,\sigma)}>k=>\sum x_i<c]$

Where c is such that $P_{H_{03}}(U)=\alpha=>P_{H_{03}}\{x_{(1)}<\mu_1 \text{ or } \sum x_i<c\}=\alpha$

$=>P_{H_{03}}(x_{(1)}<\mu_1)+P_{H_{03}}(\sum x_i<c)=\alpha$

$=>P_{H_{03}}(x_{(1)}<\mu_1)+P_{H_{03}}(\sum_{i=1}^n(x_i-\mu_0)<c-n\mu_0)=\alpha$

$=>e^{-n\frac{(\mu_1-\mu_0)}{\sigma_0}}+P_{H_{03}}(\sum_{i=1}^n U_i<c-n\mu_0)=\alpha$

$=>e^{-n\frac{(\mu_1-\mu_0)}{\sigma_0}}+P_{H_{03}}(\frac{2\sum_{i=1}^n U_i}{\sigma_0}<\frac{2(c-n\mu_0)}{\sigma_0})=\alpha$

$=>\frac{2(c-n\mu_0)}{\sigma_0}=(\alpha-e^{-n\frac{(\mu_1-\mu_0)}{\sigma_0}})\chi^2_{2n;1-\alpha}$ $[\sum_{i=1}^n U_i\sim gamma(n,\frac{1}{\sigma})=>\frac{2\sum_{i=1}^n U_i}{\sigma_0}\sim\chi^2_{2n}]$

$=>c=n\mu_0+\frac{\sigma_0}{2}(\alpha-e^{-n\frac{(\mu_1-\mu_0)}{\sigma_0}})\chi^2_{2n;1-\alpha}$

So The CR by MP size α test for testing $H_{03}:\mu=\mu_0,\sigma=\sigma_0$ **vs** $H_{03}^{'}:\mu>\mu_0,\sigma<\sigma_0$ is given by

$U=\{x_{(1)}<\mu_1 \text{ or } \sum x_i<n\mu_0+\frac{\sigma_0}{2}(\alpha-e^{-n\frac{(\mu_1-\mu_0)}{\sigma_0}})\chi^2_{2n;1-\alpha}\}$

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