Algorithmic Game Theory

LECTURE 5

Main Topics Covered:

- Revenue Maximization is harder than Welfare Maximization
- Virtual Welfare Maximizers
- Case Study: Yahoo!

Revisiting Welfare-Maximization

- Why we focussed on Welfare-Maximization till now:
  - It's relevant to real-world scenarios [e.g., Govt. auction to sell wireless spectrum]
  - It's pedagogical – Welfare maximization is special !
- Example for illustration :- One item-One bidder Auction
  - Only 1 bidder → private valuation 'v'
  - Sellers posts a price 'r' → take-it-or-leave-it offer
  - Revenue = r if $v \geq r$ | Revenue = 0 if $v < r$
  - For maximizing welfare, simply set r=0 [i.e. give the item for free]
  - For maximizing revenue,
    - If 'v' is known beforehand, set r=v
    - If 'v' not known beforehand (which is generally the case) ?

      There's no definitive answer !

Conclusion: Welfare-maximization is special because→
In every single parameter environment, there is a DSIC mechanism for maximizing welfare *ex post* i.e. as if private information was known beforehand !

Bayesian Analysis

- The key ingredients are:
  - A single-parameter environment: We assume there is a constant M such that $x_i \le M$ for every i and feasible solution $(x_1,\dots,x_n) \in X$
  - Independent distributions $F_1,\dots,F_n$ with positive and continuous density functions $f_1,\dots,f_n$: We assume that the private valuation v_i of participant i is drawn from F_i →

    [$F_i(z)$ = probability that a random variable with distribution F_i has value at most z]
  - The support of every distribution F_i belongs to $[0, v_{max}]$ where $v_{max} < \infty$
  - We assume that the mechanism designer knows the distributions $F_1,\dots,F_n$

    [These distributions are derived from data such as bids in previous auctions]
  - The valuations are $v_1,\dots,v_n$ private
  - Agents do not need to know $F_1,\dots,F_n$ [since we focus on DSIC auctions where agents have a dominant strategy]
- In Bayesian environment, the optimal mechanism is the one among all DSIC mechanisms that has the highest expected revenue !

One-Item One-bidder Revisited

- We can now apply our knowledge of Bayesian analysis to the one item-one bidder problem
- The expected revenue of a posted price ‘r’ is given by → $r.(1 - F(r))$
- Given a distribution F, we need to find the best posted price ‘r’
  - An optimal posted price is called the *monopoly price* of distribution F
  - Posting a monopoly price is a revenue maximizing auction for DSIC mechanisms
- Example:
  - If F is the uniform distribution on [0,1] so that F(x) = x on [0,1], then
    - Monopoly price = ${}^{1}/_{2}$
    - ∴ Expected Revenue = ${}^{1}/_{2}.(1 - {}^{1}/_{2})$ = ${}^{1}/_{4}$

Characterization of Optimal DSIC Mechanisms

Goal: To understand optimal (expected revenue maximizing) DSIC mechanisms for every single-parameter environment and distribution $F_1,\ldots,F_n$

Preliminaries

- By Revelation principle [Lecture 4], every DSIC mechanism is equivalent to- and thus has the same expected revenue- as a direct revelation DSIC mechanism (**x**,**p**)
- ∴ We focus only on such mechanisms where **b**=**v** [truthful bidding]
- Expected revenue of a DSIC mechanism (**x**,**p**) is

$$E_{v\sim F}[\sum_{i=1}^{n} p_i(v)]$$

  where expectation is w.r.t the distribution **F**= F_1 x F_2 x … x F_n over agents' valuations

- Unfortunately, it is not known how to maximize this formula
- Therefore, we look for an alternative formula that
  - References only the allocation rule, not the payment rule
  - Therefore, is easier to maximize

Virtual Valuations

- For agent i with valuation distribution F_i and valuation v_i,

$$Virtual\ Valuation = \varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$$

- Virtual valuation depends on:
  - Agent i's valuation and not on other agents' valuation
  - Can be negative
- Interpretation:

$$\varphi_i(v_i) = \underbrace{v_i}_{\text{What you'd like to charge}} - \underbrace{\frac{1-F_i(v_i)}{f_i(v_i)}}_{\text{Information rent earned by agent}}$$

- v_i → maximum revenue obtainable from agent i
- $\frac{1-F_i(v_i)}{f_i(v_i)}$ → Information rent = the inevitable revenue loss due to not knowing v_i beforehand

Expected Revenue equals Expected Virtual Welfare (1/2)

- **LEMMA:**

For every single-parameter environment with valuation distributions $F_1,\ldots,F_n$, every DSIC mechanism ($\boldsymbol{x}$,$\boldsymbol{p}$), every agent i and value v_{-i} of the valuations of other agents,

$$E_{v_i \sim F_i}[p_i(v)] = E_{v_i \sim F_i}[\varphi_i(v_i).\, x_i(v)] \qquad \ldots(1)$$

- Interpretation: expected payment of an agent = expected virtual value earned by agent

- **THEOREM (Expected Revenue equals Expected Virtual Welfare)**

For every single-parameter environment with valuation distributions $F_1,\ldots,F_n$ and every DSIC mechanism ($\boldsymbol{x}$,$\boldsymbol{p}$),

$$\underbrace{\boldsymbol{E}_{v \sim F}\left[\sum_{i=1}^{n} p_i(v)\right]}_{\text{Expected revenue}} = \underbrace{\boldsymbol{E}_{v \sim F}\left[\sum_{i=1}^{n} \varphi_i(v_i).\, x_i(v)\right]}_{\text{Expected virtual welfare}}$$

Expected Revenue equals Expected Virtual Welfare (2/2)

Proof of Theorem:

Taking the expectation w.r.t $v_{-i} \sim F_{-i}$, of both sides of (1), we get

$$E_{v\sim F}[p_i(v)] = E_{v\sim F}[\varphi_i(v_i).x_i(v)]$$

Applying the linearity of expectation (twice) then gives,

$$E_{v\sim F}\left[\sum_{i=1}^{n} p_i(v)\right] = \sum_{i=1}^{n} E_{v\sim F}[p_i(v)]$$

$$= \sum_{i=1}^{n} E_{v\sim F}[\varphi_i(v_i).x_i(v)]$$

$$= E_{v\sim F}[\sum_{i=1}^{n} \varphi_i(v_i).x_i(v)],$$

as desired !

Maximizing Expected Virtual Welfare

- We know, maximizing expected virtual welfare will maximize the expected revenue
- The Expected Virtual Welfare : $E_{v\sim F}[\sum_{i=1}^{n} \varphi_i(v_i).x_i(v)]$
- We have no control over the input distribution **F** and the virtual valuation $\varphi_i(v_i)$
- How should we choose the allocation rule **x** to maximize ?
  - Possible Approach : Maximize Pointwise separately for each value of v
  - For each value of v, choose x(v) that maximizes expected virtual welfare subject to the feasibility of the allocation rule
  - Such an x(v) is called the Virtual Welfare-Maximizing Allocation Rule
- Interpretation:

In a single-item auction where the feasibility constraint is $\sum_{i=1}^{n} x_i(v) \leq 1$, the item is awarded to the bidder with the highest virtual valuation.

Exception: Since, virtual valuation can be negative too, if every bidder has a negative valuation in the above example, then the virtual welfare is maximized by NOT awarding the item to anyone !

Regular Distributions

- An important question to be raised:

  Is the virtual welfare-maximizing allocation rule monotone ?

- It is monotone 'if' the agents' valuation is drawn from a Regular Distribution
- What is a regular distribution ?

A distribution F is regular if the corresponding virtual valuation $\left[v - \frac{1-F(v)}{f(v)}\right]$ *is non-decreasing*

- With regular valuation distributions, we can extend the (monotone) virtual welfare maximizing allocation rule to a DSIC mechanism using Myerson's Lemma
- This is an expected revenue maximizing DSIC mechanism and is called the 'Virtual Welfare Maximizer'

Virtual Welfare Maximizer

Assumption: The valuation distribution F_i of every agent is regular

- Transform the truthfully reported valuation v_i of agent i into the corresponding virtual valuation $\varphi_i(v_i)$.
- Choose the feasible allocation $(x_1, \ldots, x_n)$ that maximizes the virtual welfare $\sum_{i=1}^{n} \varphi_i(v_i).x_i$.
- Charge payments according to Myerson's payment formula.

NOTE 1: Theorem: Virtual Welfare Maximizers are Optimal.

NOTE 2: The mechanism maximizes revenue not only over DSIC mechanisms but more generally over "Bayesian Incentive Compatible" (BIC) mechanisms.

Optimal Single Item Auctions

- Assume all bidders have a common valuation distribution **F** and hence a common virtual valuation function φ
- Also assume F is regular → φ is strictly increasing !
- The virtual welfare maximizer mechanism awards the item to the bidder with highest non-negative virtual valuation, if any
- Since all bidders share same increasing virtual valuation function →

  Bidder with highest virtual valuation = Bidder with highest valuation
- This allocation rule is same as that of a second-price auction with a reserve price of $\varphi^{-1}(0)$

  (Reserve price 'r' means the bidder has to bid at least that much amount and winner is charged either r or the 2nd-highest bid)

Case Study: (1/2)
Reserve Prices in Sponsored Search

- This case study highlights the importance of Optimal Mechanism Design Theory
- In a sponsored search auction [Lecture 2] expected revenue can be maximized if →
  - We assume the bidders' valuations-per-click are drawn i.i.d (independently and identically distributed) where F is a regular distribution
  - Theoretically, the optimal auction chooses from the bidders who bid at least the reserve price $\varphi^{-1}(0)$
  - These bidders are ranked from best to worst
- The Background:
  - Till 2008, YAHOO! had been using relatively low reserve prices like $.01, $.05, $.10
  - They were using the same reserve price for all types of different keywords
  - What if they had separate reserve prices for different keywords ?

Case Study: (2/2)
Reserve Prices in Sponsored Search

- The Experiment:
  - A regular (lognormal) distribution function was fitted for approximately 500,000 different keywords
  - Theoretically optimal reserve prices were computed for each keyword
  - It was found that the reserve prices varied significantly and went as high as \$.30 and \$.40
  - YAHOO! set the new reserve prices to be the average of the old one (\$.10) and the new theoretically computed one
- The Result:
  - The change worked and the company's revenue increased significantly !

References

- Twenty Lectures on Algorithmic Game Theory by Tim Roughgarden, 2016, Cambridge University Press

- https://www.youtube.com/playlist?list=PLEGCF-WLh2RJBqmxvZ0_ie-mleCFhi2N4

THANK YOU