Algorithmic Game Theory

LECTURE 3

Main points covered:

- Single-Parameter Environments
- Allocation and Payment Rules
- Some Useful Definitions
- Myerson's Lemma
- Proof of Myerson's Lemma
- Applying the payment formula

Single-Parameter Environments (1/2)

- A generalization of the mechanism design problems introduced in Lecture 2
- Setup:
  - ‘n’ agents (bidders)
  - Each agent *i* has a private valuation v_i, her value “per unit of stuff” that she requires
  - There’s a feasible set X

    Each element of X is a non-negative n-vector $(x_1, x_2, \dots, x_n)$ where x_i denotes the “amount of stuff” given to agent i

AUCTION		MECHANISM
bidder	⟺	agent
bid	⟺	report
valuation	⟺	valuation

Single-Parameter Environments (2/2)

Some Examples to make things clear ➔

- Single-item auction → Here X is the set of 0-1 vectors that have at most one 1 i.e., $\sum_{i=1}^{n} x_i \leq 1$

- k-Unit Auctions → There are k identical items and each bidder can get at most one. ∴ X is the set of 0-1 vectors such that $\sum_{i=1}^{n} x_i \leq \mathrm{k}$

- Sponsored search auctions → Here, X is the set of n-vectors corresponding to the assignment of bidders to slots i.e., if bidder i is assigned to slot j then x_i is equal to α_j (CTR of slot j)

Allocation and Payment Rules

RECAP: In a sealed-bid auction, we need to make two decisions viz. 1)who wins and 2) who pays what

- We can formalize these as *allocation* and *payment rules* in 3 steps:
  - Collect bids b = (b_1, b_2,…, b_n) | b = bid vector/ bid profile
  - [allocation rule] Choose a feasible allocation $x(b) \in X \subseteq \mathbb{R}^n$ as a function of the bids
  - [payment rule] Choose payments $\mathrm{p}(b) \in \mathbb{R}^n$ as a function of the bids

Direct-revelation mechanism (agents directly reveal their private valuations)

Our new quasilinear utility model ➔ Agent i receives utility $u_i(b) = v_i . x_i(b) - p_i(b)$ when the bid profile is b.

Note: We must have

1. $p_i(b) \geq 0$ [so that seller does not have to pay to the agents]
2. $p_i(b) \leq b_i.x_i(b)$ [so that a truthful agent has non-negative utility]

Therefore, $p_i(b) \in [0, b_i . x_i(b)]$

Some Useful Definitions (1/2)

Implementable Allocation Rule:

*An allocation rule **x** for a single-parameter environment is "implementable" if there is a payment rule **p** such that the direct-revelation mechanism (**x**,**p**) is DSIC.*

Example →

- In a single-item auction, we award the item to the highest bidder.
- Is this allocation rule implementable ?

  YES !
- The second-price payment rule is the answer as it renders the mechanism DSIC

❑ An Observation:
If we had wanted an allocation rule that awarded the item to 2^{nd} highest bidder, then we cannot call it implementable !
There is no payment rule possible for such an allocation !

Some Useful Definitions (2/2)

Monotone Allocation Rule:

An allocation rule $\mathbf{x}$ *for a single-parameter environment is monotone if for every agent i and bids* b_{-i} *by the other agents, the allocation* $x_i(z, b_{-i})$ *is non-decreasing in her bid z.*

- In simple words, in a monotone allocation rule, bidding higher can only get you more stuff

Examples →

- In a single-item auction, allocating the item to highest bidder is "monotone"
  - If the winner(highest bidder) keeps raising her bid, she still remains winner !
- In single-item auction, allocating the item to 2^{nd} highest bidder is "non-monotone"
  - If the winner(2^{nd} highest bidder) raises her bid significantly, she may lose !
- The welfare maximizing allocation rule for sponsored search auctions where i^{th} highest bidder gets the i^{th} highest slot is "monotone"
  - On raising her bid, the bidder's position can only increase !

Myerson's Lemma

Fix a single-parameter environment:

a) An allocation rule **x** is implementable if and only if it is monotone

b) If **x** is monotone, then there is a unique payment rule for which the direct-revelation mechanism (**x**,**p**) is DSIC and $p_i(b) = 0$ whenever $b_i = 0$

c) The payment rule in (b) is given by an explicit formula

$$p_i(b_i, \boldsymbol{b}_{-i}) = \sum_{j=1}^{l} z_j \cdot [jump\ in\ x_i(\cdot, b_{-i})\ at\ z_j]$$

where, z_1, z_2,…, z_l are the breakpoints of the allocation function $x_i(\cdot, b_{-i})$ in the range $[0, b_i]$

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University of Chicago

Proof of Myerson's Lemma (1/3)

Let us fix a single-parameter environment and consider **x** to be an allocation rule that may or may not be monotone.

To show: There exists a payment rule **p** such that the mechanism (**x**,**p**) is DSIC

We will use shorthand $x(z)$ for $x_i(z,b_{-i})$ and $p(z)$ for $p_i(z,b_{-i})$ when agent i bids 'z'

Now,

Suppose (**x**,**p**) is DSIC and consider any $0 \leq y \leq z$

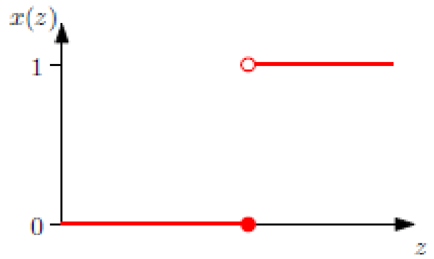
- Case 1: Agent i has private valuation z and submits false bid y. Then,

$$\underbrace{z.x(z) - p(z)}_{\text{Utility of bidding z}} \geq \underbrace{z.x(y) - p(y)}_{\text{Utility of bidding y}} \qquad \ldots(1)$$

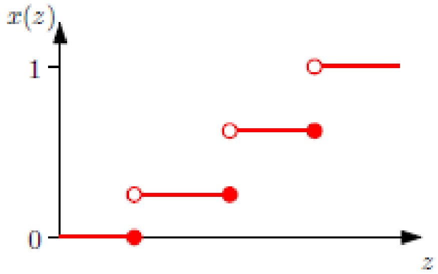
- Case 2: Agent i has private valuation y and submits false bid z. Then,

$$\underbrace{y.x(y) - p(y)}_{\text{Utility of bidding y}} \geq \underbrace{y.x(z) - p(z)}_{\text{Utility of bidding z}} \qquad \ldots(2)$$

Two examples of possible allocation curves



(a) 0-1 Monotone Curve



(b) Piecewise Constant Monotone Curve

Proof of Myerson's Lemma (2/3)

Rearranging (1) and (2) we get a "payment difference sandwich"

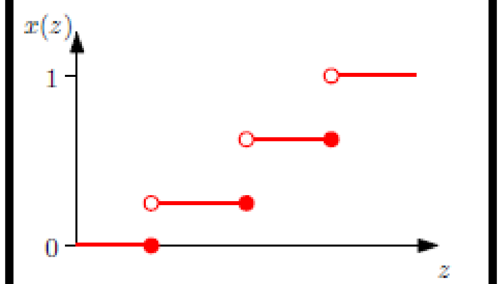
$z.[x(y) - x(z)] \leq p(y) - p(z) \leq y.[x(y) - x(z)]$ …(3)

The payment difference sandwich already implies that every implementable allocation rule is monotone

➔ We can say **x** is monotone !

Next,

We consider **x** to be piece-wise constant [Fig.(b)]



(b) Piecewise Constant Monotone Curve

We fix z and let y tend to z from above ($y \downarrow z$)

- Applying the limit in (3), the left and right sides become
  - 0 if there is no jump in x at z
  - Tending to z.h if there is a jump of magnitude h at z
  - Therefore, [jump in p at z] = z.[jump in x at z] …(4)
- Combining (4) with the initial condition p(0)=0, we get the required payment formula !

Proof of Myerson's Lemma (3/3)

Thus we have derived the payment formula for a piece-wise constant function

$$p_i(b_i, \boldsymbol{b}_{-i}) = \sum_{j=1}^{l} z_j \,.\, [jump\ in\ x_i(\cdot\,, b_{-i})\ at\ z_j]$$

Where, z_1, z_2,…, z_l are the breakpoints of the allocation function $x_i(\cdot, b_{-i})$ in the range $[0, b_i]$

- NOTE: The payment formula can be generalized to a case where **x** is not piece-wise constant. The formula would then take a form as follows:

$$p_i(b_i, \boldsymbol{b}_{-i}) = \int_0^{b_i} z . \frac{d}{dz} x_i(z, b_{-i})\, dz$$

for every agent i, bid b_i and bids b_{-i} by other agents.

Applying the Payment Formula (1/2)

- Single-item Auctions →

- Allocation Rule allocates item to the highest bidder
- Fixing a bidder i and bids b_{-i} by other agents, the function $x_i(z, b_{-i})$ is 0 upto $B=\max_{j!=i} b_j$ and 1 thereafter
- Therefore, it is piece-wise constant !
- We can apply the payment formula
  - If $b_i < B$, payment = 0
  - If $b_i > B$, there is a single breakpoint (jump of 1 at B), payment = B

Thus, Myerson's Lemma regenerates the second-price payment rule as a special case !

Applying the Payment Formula (2/2)

- Sponsored Search Auctions →
- There are k slots with CTRs $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$
- Allocation rule assigns j^{th} highest bidder to j^{th} best slot
- The rule is monotone and welfare-maximizing (assuming truthful bids)
- We can apply Myerson's payment formula
  - Re-index bids in the bid profile **b** as $b_1 \geq b_2 \geq \ldots \geq b_n$ first
  - Considering only first bidder, we can imagine bidder raising her bid from 0 to b_1 , holding other bids fixed
  - The allocation $x_i(z,b_{-i})$ ranges from 0 to α_1 as z ranges from 0 to b_1, with a jump of $\alpha_j - \alpha_{j+1}$ at the point where z becomes the j^{th} highest bid in the profile (z,b_{-i}), namely b_{j+1} !
- Thus in general,

$p_i(b) = \sum_{j=i}^{k} b_{j+1}(\alpha_j - \alpha_{j+1})$ for the i^{th} highest bidder (where α_{k+1} =0)

References

- Twenty Lectures on Algorithmic Game Theory by Tim Roughgarden, 2016, Cambridge University Press
- https://www.youtube.com/playlist?list=PLEGCF-WLh2RJBqmxvZ0_ie-mleCFhi2N4

THANK YOU