Algorithmic Game Theory

LECTURE 6

Main Topics Covered:

- Optimal Auctions Can Be Complex
- Prophet Inequality
- Simple Single-Item Auctions
- Prior-Independent Mechanisms

The Problem with Optimal Auctions

- In Lecture 5 we saw that for every single-parameter environment wherein bidders' valuations are drawn independently from regular distributions we obtain maximum expected revenue over all DSIC mechanisms by using the allocation rule:

  $x(v) = argmax_X \sum_{i=1}^{n} \varphi_i(v_i).x_i(v)$ where $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$

- We also saw, for optimal single-item auctions with i.i.d. bidders and a regular distribution the problem reduces to a simple case of second price auction with reserve price $\varphi^{-1}(0)$
- What if we have a complex problem where valuations are drawn from "different" regular distributions ?
  - Someone other than highest bidder might win
  - It becomes complicated to explain the winning price
  - The solution becomes COMPLEX !

The Prophet Inequality (1/4)

BACKGROUND:

Let us consider a ‘n’-stage game:

- In stage ‘i’ you are offered a non-negative prize π_i drawn from distribution G_i
- Distributions $G_1, \ldots, G_n$ are independent and are known in advance
- The prize π_i is revealed only at stage i
- You can either accept π_i at stage i and leave the game or you can reject π_i and advance to a later stage
- Difficulty of deciding:
  - If we accept π_i we might loose a bigger amount at a later stage
  - If we keep rejecting π_i we might have to end up with a lousy amount at the final stage

The Prophet Inequality proves useful here to offer a simple strategy almost like a clairvoyant prophet !

The Prophet Inequality (2/4)

Theorem:

For every sequence $G_1, \ldots, G_n$ of independent distributions, there is a strategy that guarantees expected reward at least $^1/_2\, E_{\pi\sim G}[max_i\, \pi_i]$. Moreover, there is such a threshold strategy, which accepts prize i if and only if π_i is at least some threshold t.

Proof:

- Let z^+ denote max{z,0}
- Consider a threshold strategy t
- Difficult to compare directly the expected payoff of this strategy with that of a prophet

  ∴ We will derive and compare the lower and upper bounds, respectively, of these two quantities
- Let q(t) = probability that threshold strategy accepts no prize at all
  - t increases ➔ risk q(t) increases ➔ average value of accepted prize goes up

The Prophet Inequality (3/4)

- What pay-off does t-threshold strategy obtain ?
  - If only 1 prize satisfies $\pi_i \geq t$ then we get an extra-credit = $\pi_i - t$
  - If more than 1 prizes satisfy $\pi_i \geq t$ then we credit only the baseline t to the payoff
- Formally, we have the following lower bound:

$$E[payoff\ \ of\ t - threshold\ strategy]$$

$$\geq \big(1 - q(t)\big).t + \sum_{i=1}^{n} E[\pi_i - t \mid \pi_i \geq t, \pi_j < t\ \forall\ j \neq i]\ \boldsymbol{Pr}[\pi_i \geq t]\ .\mathbf{Pr}[\pi_j < t\ \forall\ j \neq i]$$

$$= \big(1 - q(t)\big).t + \sum_{i=1}^{n} \underbrace{E[\pi_i - t\ |\pi_i \geq t]\ \mathbf{Pr}[\pi_i \geq t]}_{= E[(\pi_i - t)^+]}\ \ .\ \ \underbrace{\mathbf{Pr}[\pi_j < t\ \forall j \neq i]}_{\geq q(t)}$$

$$\geq \big(1 - q(t)\big).t + q(t) \sum_{i=1}^{n} E[(\pi_i - t)^+] \qquad \text{...(1)}$$

The Prophet Inequality (4/4)

Now we produce an upper bound on the "prophet's" expected pay-off, which is easy to compare with the lower bound of the t-threshold strategy's payoff

$$
\begin{aligned}
E_\pi[max_{i=1}^{n}\ \pi_i] &= \ E_\pi[t + \ max_{i=1}^{n}(\pi_i - t)] \\
&\leq t + \ E_\pi[max_{i=1}^{n}(\pi_i - t)^{+}] \\
&\leq t + \ \textstyle\sum_{i=1}^{n} E_\pi[(\pi_i - t)^{+}] \qquad \text{...(2)}
\end{aligned}
$$

Comparing (1) and (2), we can set t so that $q(t) = \frac{1}{2}$, with a 50/50 chance of accepting the prize.

(Hence proved!)

Simple Single-Item Auction (1/2)

- Setup:
  - 1 item, n bidders with valuations drawn independently from n regular distributions that are not necessarily identical
  - We will use the Prophet Inequality to design a simple, near-optimal auction
- Idea:
  - Consider i'th prize = virtual valuation $\varphi_i(v_i)^+$ of bidder
  - Then the corresponding F_i is same as G_i
- Connection to Prophet Inequality:
  - Expected revenue of optimal auction = Expected value obtained by prophet

$$E_{v\sim F}[\sum_{i=1}^{n} \varphi_i(v_i)x_i(v)] = E_{v\sim F}[max_{i=1}^{n}\varphi_i(v_i)^+]$$

- Let us consider an allocation rule (Virtual Threshold Allocation Rule) with the following form:
  - Choose t such that $\Pr[max_i\ \varphi_i(v_i)^+ \ \geq t] = \frac{1}{2}$
  - Give the item to bidder i with $\varphi_i(v_i) \geq t$, if any, breaking ties among multiple candidate winners arbitrarily

Simple Single-Item Auction (2/2)

- Virtual Threshold Rules are Near-Optimal

  If x is a virtual threshold allocation rule, then $E_v[\sum_{i=1}^{n} \varphi_i(v_i)^+ x_i(v)] \geq \frac{1}{2} E_v[max_{i=1}^{n} \varphi_i(v_i)^+]$

- A specific virtual threshold allocation rule:

  Second-Price with Bidder Specific Reserves

  - Set a reserve price $r_i = {\varphi_i}^{-1}(t)$ for each bidder i with t defined as for virtual threshold allocation rules
  - Give the item to the highest bidder that meets her reserve, if any

  This allocation rule is monotone → Can be extended to DSIC auction using Myerson's Lemma

  Here, winner's payment = MAX{her reserve price, highest bid by another bidder who meets reserve}

  This auction approximately maximizes expected revenue over all DSIC auctions.

- Simple vs Optimal Auctions:
  - The expected revenue of a 2nd price auction with suitable reserve prices is at least 50% of that of optimal auction

Prior-Independent Mechanisms (1/2)

- Till now we assumed that the valuation distributions were known in advance to the mechanism designer
- This is reasonable in cases where we have lots of data and bidders' preference does not change too rapidly
- What if these valuation distributions were not known in advance ?
- In that case, we use distributions in the analysis of mechanism but NOT in their design
- Goal: To create a good "*Prior-Independent*" mechanism i.e., whose description makes no reference to a valuation distribution

  Example: Second-price Single-item Auctions
- The Bulow-Klemperer Theorem:

Let F be a regular distribution and n a positive integer. Let p and p^* denote the payment rules of the second price auction with n+1 bidders and the optimal auction (for F) with n bidders, respectively. Then,

$$E_{v\sim F^{n+1}}\left[\sum_{i=1}^{n+1} p_i(v)\right] \geq E_{v\sim F^{n}}\left[\sum_{i=1}^{n} p_i^*(v)\right]$$

Prior-Independent Mechanisms (2/2)

- Interpretation of Bulow-Klemperer Theorem:
  - The expected revenue of an optimal auction is at most that of a second price auction (with no reserve) with one extra bidder
  - It is beneficial to invest more resources to recruit more serious participants than to know more about their preferences alone
  - Extra competition is more important than getting the auction format just right
- Proof of Bulow-Klemperer Theorem:
  - We use an indirect approach: proof by a hypothetical auction's example as follows →
    1. Simulate an optimal n-bidder auction for F on the first n bidders 1,2,…,n
    2. If item is not awarded in 1^{st} step, give the item to the n+1 bidder for free
  - Here we see, the expected revenue equals that of an optimal auction with n bidders
  - Also, the item is always allocated
  - Therefore, the expected revenue of a second price auction with n+1 bidders is at least that of our hypothetical auction (Hence Proved!)

References

- Twenty Lectures on Algorithmic Game Theory by Tim Roughgarden, 2016, Cambridge University Press

- https://www.youtube.com/playlist?list=PLEGCF-WLh2RJBqmxvZ0_ie-mleCFhi2N4

THANK YOU