Alternating Harmonic Series Grouped by Fibonacci and Triangular Numbers

Matthew Blake

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Abstract

Chen and Kennedy (2012) explored reordering the Alternating Harmonic Series, rearranging the terms into blocks whose sizes are given by the Fibonacci numbers. In this paper, we modify their approach in two ways: by replacing the Fibonacci numbers with the triangular numbers, and by removing their parity restriction on the sign of each term; we thus create three new series. We analyze convergence and boundedness across all four series, finding that while the original diverges but is bounded, two of the new series converge, while the final series diverges without bound. Finally, inspired by the Riemann zeta function, we consider a new family of series by exponentiating every blocked term by a positive power s. We analyze properties of these generalized series, including convergence and boundedness.

Introduction

The Harmonic Series,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}=\infty,$$

is one of the oldest series studied. Its divergence was first proven in 1360 by Nicholas Oresme (1961).

A related series is the Alternating Harmonic Series,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}=\ln 2,$$

whose sum converges to $\ln 2$.
Chen and Kennedy (2012) modified this series by grouping terms into parenthetical groups, called "blocks", whose sizes increase in accordance with the

Fibonacci sequence. Despite the change of ordering, the pattern of adding odd inverses and subtracting even inverses is preserved from the Alternating Harmonic series. Therefore, we will call this the Parity Fibonacci Blocked Series, or S_{PF}:

$$\begin{aligned}
S_{PF} := &\overbrace{(1) - \left(\frac{1}{2}\right)}^{F_0=1} + \overbrace{\left(\frac{1}{3}\right) - \left(\frac{1}{4}\right)}^{F_1=1} + \overbrace{\left(\frac{1}{5}+\frac{1}{7}\right) - \left(\frac{1}{6}+\frac{1}{8}\right)}^{F_2=2} \\
&+ \overbrace{\left(\frac{1}{9}+\frac{1}{11}+\frac{1}{13}\right) - \left(\frac{1}{10}+\frac{1}{12}+\frac{1}{14}\right)}^{F_3=3} \\
&+ \overbrace{\left(\frac{1}{15}+\cdots+\frac{1}{23}\right) - \left(\frac{1}{16}+\cdots+\frac{1}{24}\right)}^{F_4=5} + \cdots \\
= &\sum_{i=0}^{\infty}\left(\sum_{m=0}^{F_i-1}\frac{1}{2F_{i+1}-1+2m} - \sum_{n=0}^{F_i-1}\frac{1}{2F_{i+1}+2n}\right)
\end{aligned}$$

where F_i is the ith Fibonacci number.

We want to examine S_{PF}, as well as three other related series. The second series we will look at will maintain the block sizes of S_{PF}, but rearrange the fractions such that the inverses are in increasing order. We will call this series the Consecutive Fibonacci Blocked Series, or S_{CF}:

$$\begin{aligned}
S_{CF} := &(1) - \left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) - \left(\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}\right) - \left(\frac{1}{7}+\frac{1}{8}\right) \\
&+ \left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}\right) - \left(\frac{1}{12}+\frac{1}{13}+\frac{1}{14}\right) + \cdots \\
= &\sum_{i=1}^{\infty}\left(\sum_{m=2\cdot F_i-1}^{F_{i+2}-2}\frac{1}{m} - \sum_{n=F_{i+2}-1}^{2\cdot F_{i+1}-2}\frac{1}{n}\right).
\end{aligned} \tag{1}$$

In addition to these, we will later examine two more series, the Alternating and Parity Triangular-Blocked series, for which block sizes are determined not by Fibonacci numbers, but by triangular numbers $T_i = \frac{i(i+1)}{2}$.

Throughout this paper, we will use the phrases "block" and "term" interchangeably to refer to a sum of F_n or T_n unit fractions (positive integer reciprocals), enclosed in parentheses. A "term pair" represents pair of terms (the first added, and the second subtracted) with the same length (F_n or T_n, depending on the series). The "size" of a term or block refers to the number of unit fractions summed within the parentheses of the block; the "size" of a term pair refers to the size of each of the terms contained within it. Finally, an "odd" term of a sequence or series $\{a_n\}$ is any a_i such that i is odd; an "even" term is defined similarly. As all of these series are alternating, any odd term is "positive" or added, and any even term is "negative" or subtracted.

For each series, we want to consider the following questions: First, does the series diverge or converge? If it converges, what value does it converge to? If it diverges, is the series still bounded (oscillating between two limits), or does it diverge without bound (going to positive or negative infinity)?

Finally, taking inspiration from the Riemann-Zeta function, we will modify each series by raising each block to a positive power s. We want to examine whether this affects convergence or boundedness, and how the sums are affected. We will model partial sums of these series with approximating formulae, and study their behavior.

Preliminaries

In this paper, we will use several useful properties of the Fibonacci sequence and of harmonic numbers.

Fibonacci Sequence Properties

The Fibonacci sequence is defined recursively by $F_0 = F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 2$. There are two properties we will use later in this paper:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi, \tag{2}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1, \tag{3}$$

which can be proven by basic induction.

Harmonic Number approximation

Young (1991) gives us the inequality

$$\ln(n) + \frac{1}{2(n+1)} < H_n - \gamma < \ln(n) + \frac{1}{2n}.$$

Rearranging and simplifying, this yields, (for $m > 1$)

$$\ln\left(\frac{n}{m-1}\right) - \frac{n+1-m}{2(n+1)m} < \sum_{i=m}^{n} \frac{1}{i} < \ln\left(\frac{n}{m-1}\right) - \frac{n-m+1}{2n(m-1)}, \tag{4}$$

which provides us bounds for the difference of two harmonic numbers, $H_n - H_{m-1}$.

Convergence State of Fibonacci Blocked Series

For the two Fibonacci blocked series, we will first examine whether each series converges. If these series converge, what do they converge to? If they diverge, do they have convergent subsequences, or do they increase without bound?

S_{PF} Divergence

First, we claim that the Parity Fibonacci Series S_{PF} diverges. We will show this by demonstrating the existence of an infinite subsequence that does not go to zero. Consider the subsequence containing just the positive (added) terms:

$$\{a_n^+\} := 1,\ \frac{1}{3},\ \left(\frac{1}{5}+\frac{1}{7}\right),\ \left(\frac{1}{9}+\frac{1}{11}+\frac{1}{13}\right),\ \left(\frac{1}{15}+\frac{1}{17}+\frac{1}{19}+\frac{1}{21}+\frac{1}{23}\right),\ \cdots .$$

We can create a smaller series by replacing every fraction a block with the smallest fraction in the block,

$$\{b_n^+\} := 1,\ \frac{1}{3},\ \frac{2}{7},\ \frac{3}{13},\ \frac{5}{23},\ \cdots , \tag{5}$$

so that $a_n \geq b_n$ for every term. We claim that the terms in b_n do not approach zero, and therefore that the corresponding (positive) terms in a_n also do not approach zero.

To justify this claim, first note that the numerator of b_n is simply F_n, the nth Fibonacci number, by design. The denominator, conversely, is one less than twice the sum of all the Fibonacci numbers to that point (the final odd reciprocal), yielding

$$b_n = \frac{F_n}{2\sum_{i=1}^{n} F_i - 1} = \frac{F_n}{2F_{n+2} - 3},$$

by (3). We now want to take the limit of this term as $n \to \infty$:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{F_n}{2\phi^2 F_n - 3} = \lim_{n\to\infty} \frac{F_n}{2\phi^2 F_n} = \frac{1}{2\phi^2} \neq 0.$$

Since we have that the terms in b_n do not go to zero, and the terms in a_n are positive and greater than b_n, we have that $a_n \not\to 0$, and so the S_{PF} must diverge, by the Test for Divergence.

S_{CF} Divergence

Now let us consider the Consecutive Fibonacci Blocked Series, S_{CF}. Taking the corresponding positive term subsequence

$$c_n := 1,\ \frac{1}{3},\ \left(\frac{1}{5}+\frac{1}{6}\right),\ \left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}\right),\ \left(\frac{1}{15}+\cdots+\frac{1}{19}\right),\ \cdots,$$

we note that each term c_i is greater than or equal to the corresponding term a_i in a_n. Therefore, we have that $c_i \geq a_i \geq b_i$, by the argument above, and hence the terms of c_i do not converge to zero. Because c_i is an infinite subsequence of S_{CF}, we have that the terms of S_{CF} do not converge to zero, and therefore that S_{CF} diverges, by the Test for Divergence.

Boundedness State of Fibonacci Blocked Series

Since these two series diverge, a natural next question is whether the sequences of *even* partial sums converge for these series. The even partial sums are the sums of an even number of terms in the series, always ending in a subtracted term. This accounts for all of the cancellation between positive and negative terms, leaving a monotonically increasing sequence, as each added term is greater than the subtracted term which follows it.

Because this is an alternating series, this is equivalent to asking if the series has two alternating convergent subsequences, or if the partial sums grow unbounded.

S_{CF} Unboundedness

We will show that the even partial sums of S_{CF},

$$\sum_{i=1}^{\infty}\left(\sum_{m=2F_i-1}^{F_{i+2}-2}\frac{1}{m}-\sum_{n=F_{i+2}-1}^{2F_{i+1}-2}\frac{1}{n}\right),$$

increase without bound:

First, we want to calculate the limit of the terms in this series, as $i \to \infty$. We can use (4) to make a limit from (1), yielding

$$\lim_{i\to\infty}\sum_{m=2F_i-1}^{F_{i+2}-2}\frac{1}{m}-\sum_{n=F_{i+2}-1}^{2F_{i+1}-2}\frac{1}{n} \geq \lim_{i\to\infty}\ln\frac{F_{i+2}-2}{2F_i-2}-\frac{F_{i-1}}{2(F_{i+2}-1)(2F_i-1)} - \ln\frac{2F_{i+1}-2}{F_{i+2}-2}+\frac{F_{i-1}}{2(2F_{i+1}-2)(F_{i+2}-2)}.$$

Simplifying, we get

$$\cdots = \lim_{i\to\infty} \ln \frac{(F_{i+2}-2)^2}{(2F_i-2)(2F_{i+1}-2)} + \frac{F_{i-1}}{2}\left(\frac{1}{2(2F_{i+1}-1)(F_{i+2}-2)} - \frac{1}{(F_{i+2}-1)(2F_i-1)}\right).$$

We evaluate the limit, using (2) to give us

$$\cdots = \lim_{i\to\infty} \ln \frac{(\phi^2 F_i-2)^2}{4(F_i-1)(\phi F_i-1)} + \frac{F_i}{2\phi}\left(\frac{1}{2(2\phi F_i-1)(\phi^2 F_i-1)} - \frac{1}{(\phi^2 F_i-2)(2F_i-1)}\right),$$

which simplifies to

$$\cdots = \lim_{i\to\infty} \ln \frac{\phi^4 F_i^2 - 4\phi^2 F_i + 4}{4(\phi F_i^2 - \phi^2 F_i + 1)} = \ln \frac{\phi^4 F_i^2}{4\phi F_i^2} = \ln \frac{\phi^3}{4} \approx 0.05734 > 0. \qquad (6)$$

Since the terms in (1) are greater than those of (6), whose terms go to a positive term, we conclude that (1) diverges by the Test for Divergence, and therefore so does the sequence of even partial sums of S_{CF}, by the Comparison Test. In every case, the fractions in the added part are greater than those in the subtracted part, and so the sequence of even partial terms is monotonically increasing.

Therefore, S_{CF}'s even partial sums also diverge, and we have unboundedness.

S_{PF} Boundedness

Conversely, Chen and Kennedy (2012) have shown that the even partial sums for S_{PF} converge to $\ln 2$, and the odd partial sums converge to $\ln 2 + \frac{1}{2}\ln\phi$. Thus, we have that S_{PF} is bounded.

Convergence State of Triangular Blocked Series

Next, we will examine the asymptotic behavior of S_{CT} and S_{PT}.

S_{CT} Convergence

We now consider the Consecutive Triangular Blocked Series, S_{CT}:

$$
\begin{aligned}
S_{CT} :=& \overbrace{(1) - \left(\frac{1}{2}\right)}^{T_1=1} + \overbrace{\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) - \left(\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}^{T_2=3} \\
&+ \overbrace{\left(\frac{1}{9}+\cdots+\frac{1}{14}\right) - \left(\frac{1}{15}+\cdots+\frac{1}{20}\right)}^{T_3=6}\cdots \\
=& \sum_{k=1}^{\infty}\sum_{n=1}^{\frac{1}{2}k(k+1)} \frac{1}{n+\frac{1}{3}(k-1)k(k+1)} - \frac{1}{n+\frac{1}{2}k(k+1)+\frac{1}{3}(k-1)k(k+1)}.
\end{aligned}
$$

Noting that

$$
\frac{1}{3}(k-1)k(k+1) + \frac{1}{2}k(k+1) = \frac{1}{3}k(k+0.5)(k+1),
$$

we rewrite the summation above as

$$
\begin{aligned}
S_{CT} &= \sum_{k=1}^{\infty}\left(\sum_{m=\frac{1}{3}(k-1)k(k+1)+1}^{\frac{1}{3}k(k+0.5)(k+1)} \frac{1}{m} - \sum_{n=\frac{1}{3}k(k+0.5)(k+1)+1}^{\frac{1}{3}k(k+1)(k+2)} \frac{1}{n}\right) \\
&= 1 - 0.5 + \sum_{k=2}^{\infty}\left(\sum_{m=\frac{1}{3}(k-1)k(k+1)+1}^{\frac{1}{3}k(k+0.5)(k+1)} \frac{1}{m} - \sum_{n=\frac{1}{3}k(k+0.5)(k+1)+1}^{\frac{1}{3}k(k+1)(k+2)} \frac{1}{n}\right)
\end{aligned}
$$

to avoid division by 0 errors in the next step.

By the Harmonic Number formula above (4), we have that

$$
\begin{aligned}
S_{CT} - 0.5 <& \sum_{k=2}^{\infty} \ln\left(\frac{\frac{1}{3}k(k+0.5)(k+1)}{\frac{1}{3}(k-1)k(k+1)}\right) - \frac{\frac{1}{3}k(k+0.5)(k+1) - \frac{1}{3}(k-1)k(k+1)}{2(\frac{1}{3}k(k+0.5)(k+1))(\frac{1}{3}(k-1)k(k+1))} \\
&- \left(\ln\left(\frac{\frac{1}{3}k(k+1)(k+2)}{\frac{1}{3}k(k+0.5)(k+1)}\right) - \frac{\frac{1}{3}k(k+0.5)(k+1) - \frac{1}{3}(k-1)k(k+1)}{2(\frac{1}{3}k(k+0.5)(k+1)+1)(\frac{1}{3}(k-1)k(k+1)+1)}\right) \\
=& \sum_{k=2}^{\infty} \ln\frac{(k+0.5)^2}{(k-1)(k+2)} \\
&+ \frac{9k(k+1)}{4(k(k+0.5)(k+1)+3)(k(k+1)(k+2)+3)} - \frac{9}{4(k-1)k(k+0.5)(k+1)} \\
<& \sum_{k=2}^{\infty} \ln\frac{(k+0.5)^2}{(k-1)(k+2)} + \frac{9}{4k(k+0.5)(k+1)}\left(\frac{1}{k+2} - \frac{1}{k-1}\right) \\
=& \sum_{k=2}^{\infty} \ln\frac{(k+0.5)^2}{(k-1)(k+2)} + \frac{-27}{4(k-1)k(k+0.5)(k+1)(k+2)}.
\end{aligned} \tag{7}
$$

However, we know that every pair of terms with the same number of reciprocals has a positive difference. Therefore, we have that

$$\forall k \in \mathbb{N}^+ \; S_{CT}(2k-1) - S_{CT}(2k) > 0.$$

This means that the sequence of even partial sums is monotonically increasing, and therefore that it either diverges to infinity or converges to a positive number. As any subsequence must converge to the same number as the original sequence, this implies that the sequence of partial sums of the sequence Sq_{CT} must also diverge to infinity or converge to a positive number, and therefore that the series S_{CT} must behave identically. In particular, this enables us to prove convergence of S_{CT} by establishing the convergence of a strictly larger series, such as (7).

From this step, I claim that the above series (7) is convergent, because it is the sum of two convergent series. In particular, if I can show that

$$\sum_{k=2}^{\infty} \ln \frac{(k+0.5)^2}{(k-1)(k+2)}$$

and

$$\sum_{k=2}^{\infty} \frac{-27}{4(k-1)k(k+0.5)(k+1)(k+2)}$$

are convergent series, then it follows that their sum is convergent. As this sum is strictly greater than the monotonically increasing S_{CT}, showing convergence of (7) is sufficient to show convergence of S_{CT}. Therefore, it remains to show that $\sum_{k=1}^{\infty} \ln \frac{(k+0.5)^2}{(k-1)(k+2)}$ and $\sum_{k=1}^{\infty} \frac{-27}{4(k-1)k(k+0.5)(k+1)(k+2)}$ are convergent.

To show convergence of the first series, first note that, for small values x, $\ln(1+x) \approx x$, and therefore, for large k, $\ln(1+\frac{1}{k}) \approx \frac{1}{k}$. Next, observe that

$$\sum_{k=2}^{\infty} \ln \frac{(k+0.5)^2}{(k-1)(k+2)} = \sum_{k=2}^{\infty} \ln \left(1 + \frac{2.25}{k^2+k-2}\right).$$

Finally, we will use the Limit Comparison Test to prove convergence of the above series. Let $a_n = \ln\left(1 + \frac{2.25}{n^2+n-2}\right)$ and $b_n = \frac{1}{n^2}$. We want to show that $\sum_{n=2}^{\infty} a_n$ is convergent. To do this, note that $\sum_{n=2}^{\infty}$ is a convergent p-series, and that $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2.25}{n^2+n-2}\right)}{x^2} = \frac{x^2}{x^2} = 1$, which is a nonzero constant. Therefore, since $\sum_{n=2}^{\infty} b_n$ converges, so must $\sum_{n=2}^{\infty} a_n$, as desired.

Second, we must show convergence of

$$\sum_{k=2}^{\infty} \frac{-27}{4(k-1)k(k+0.5)(k+1)(k+2)}.$$

To show this, we can apply the Limit Comparison Test, with

$$a_n = \frac{-27}{4(n-1)n(n+0.5)(n+1)(n+2)}$$

and using the convergent p-series

$$b_n = \frac{1}{n^5}$$

to yield

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{-27}{n^5}}{\frac{1}{n^5}} = -27,$$

a nonzero constant. Therefore, since b_n converges, so too does our series a_n.

As we have shown that

$$\sum_{k=2}^{\infty} \ln \frac{(k+0.5)^2}{(k-1)(k+2)}$$

and

$$\sum_{k=2}^{\infty} \frac{-27}{4(k-1)k(k+0.5)(k+1)(k+2)}$$

are convergent, so too is their sum

$$\sum_{k=2}^{\infty} \ln \frac{(k+0.5)^2}{(k-1)(k+2)} + \frac{-27}{4(k-1)k(k+0.5)(k+1)(k+2)}.$$

Because this sum is convergent and strictly greater than our monotonically increasing series S_{CT}, we know that S_{CT} is bounded from above, namely by

$$\sum_{k=2}^{\infty} \ln \frac{(k+0.5)^2}{(k-1)(k+2)} + \frac{-27}{4(k-1)k(k+0.5)(k+1)(k+2)}.$$

As S_{CT} is bound from above, there exists a supremum of S_{CT}, and therefore the series converges to this supremum, by the Monotone Convergence Theorem.

S_{PT} Blocked Series Convergence

The Parity Triangular Blocked series S_{PT} converges; we can show this with the Alternating Series test. By observing S_{PT},

$$S_{PT} := \overbrace{(1) - \left(\frac{1}{2}\right)}^{T_1=1} + \overbrace{\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right) - \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right)}^{T_2=3}$$
$$+ \overbrace{\left(\frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19}\right) - \left(\frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} + \frac{1}{18} + \frac{1}{20}\right)}^{T_3=6} + \cdots,$$

we see that for any even k, the kth partial sum will equal $\bar{H}_n$, the nth partial sum of the alternating harmonic series, where

$$n = 2\sum_{i=1}^{k} \frac{i(i+1)}{2} = \sum_{i=1}^{k} i^2 + i = \frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{3}.$$

Since the alternating harmonic series is known to converge to $\ln 2$, the even partial sums also converge to $\ln 2$. To show the entire series converges to $\ln 2$, it remains to show that the odd terms go to zero.

However, note that, each positive term in S_{PT} is less than or equal to the corresponding (positive) term in S_{CT}. Since we have established that the latter goes to zero, it follows that the former series does as well, by the Squeeze Theorem.

Since we have that the even partial sums converge to $\ln 2$, and the odd terms decrease to 0, we have that S_{PT} converges to $\ln 2$.

Series Generalization: Term Exponentiation

The second part of this paper will examine these series when each block (term) is raised to a positive power s. We focus on positive powers, because a power of 0 reduces to Grandi's series ($\sum_{n=1}^{\infty}(-1)^n$), and a negative exponent diverges exponentially in every case. This will enable us to derive equations to estimate the convergent values or partial sums of these series. We can extrapolate these equations to make further estimates of partial sums beyond what we can compute directly in R.

For example, P_{CF}, the generalized S_{CF}, is defined as

$$\begin{aligned} P_{CF}(s, L) :=& \sum_{i=2}^{L} \left(\left(\sum_{m=2F_i-1}^{F_{i+2}-2} \frac{1}{m} \right)^s - \left(\sum_{n=F_{i+2}-1}^{2F_{i+1}-2} \frac{1}{n} \right)^s \right) \\ =& (1)^s - \left(\frac{1}{2}\right)^s + \left(\frac{1}{3}\right)^s - \left(\frac{1}{4}\right)^s + \left(\frac{1}{5} + \frac{1}{6}\right)^s - \left(\frac{1}{7} + \frac{1}{8}\right)^s \\ & + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11}\right)^s - \left(\frac{1}{12} + \frac{1}{13} + \frac{1}{14}\right)^s + \cdots, \end{aligned}$$

S_{PF} is generalized to

$$\begin{aligned} P_{PF}(s) :=& (1)^s - \left(\frac{1}{2}\right)^s + \left(\frac{1}{3}\right)^s - \left(\frac{1}{4}\right)^s + \left(\frac{1}{5} + \frac{1}{7}\right)^s - \left(\frac{1}{6} + \frac{1}{8}\right)^s \\ & + \left(\frac{1}{9} + \frac{1}{11} + \frac{1}{13}\right)^s - \left(\frac{1}{10} + \frac{1}{12} + \frac{1}{14}\right)^s + \cdots, \end{aligned}$$

and so on. Note that the original series S is the special case of the generalized series P (for Power), at $s = 1$. For divergent Fibonacci series, P may be a function of L, the number of term pairs calculated in the partial sum, as well as of s.

Generalized Series Convergence and Boundedness State

The convergence states of these series, for positive exponents $s \neq 1$ can be shown from the proofs above:

Convergence State

- The Fibonacci blocked series' *terms* converge to a value t bounded from below by $t > \frac{1}{2\phi^2} > 0$. Raising every term to a real power s will cause the terms to converge to t^s, which will always be positive. Therefore, the Test for Divergence will still apply, and the modified series will diverge.

- To show that the generalized Triangular blocked series P_{CT} and P_{PT} diverge, we can modify the approach used to prove the convergence of S_{CT}. We will start with P_{CT}. Using monotonicity of (positive) exponentiation we have

$$P_{CT} - 0.5 < \sum_{k=2}^{\infty} \left(\ln\left(\frac{\frac{1}{3}k(k+0.5)(k+1)}{\frac{1}{3}(k-1)k(k+1)} \right) - \frac{\frac{1}{3}k(k+0.5)(k+1) - \frac{1}{3}(k-1)k(k+1)}{2(\frac{1}{3}k(k+0.5)(k+1))(\frac{1}{3}(k-1)k(k+1))} \right)^s - \left(\ln\left(\frac{\frac{1}{3}k(k+1)(k+2)}{\frac{1}{3}k(k+0.5)(k+1)} \right) - \frac{\frac{1}{3}k(k+0.5)(k+1) - \frac{1}{3}(k-1)k(k+1)}{2(\frac{1}{3}k(k+0.5)(k+1)+1)(\frac{1}{3}(k-1)k(k+1)+1)} \right)^s. \tag{8}$$

Note that, with the substitutions

$$(a_k) = \ln\left(\frac{\frac{1}{3}k(k+0.5)(k+1)}{\frac{1}{3}(k-1)k(k+1)} \right) - \frac{\frac{1}{3}k(k+0.5)(k+1) - \frac{1}{3}(k-1)k(k+1)}{2(\frac{1}{3}k(k+0.5)(k+1))(\frac{1}{3}(k-1)k(k+1))}$$

$$(b_k) = \ln\left(\frac{\frac{1}{3}k(k+1)(k+2)}{\frac{1}{3}k(k+0.5)(k+1)} \right) - \frac{\frac{1}{3}k(k+0.5)(k+1) - \frac{1}{3}(k-1)k(k+1)}{2(\frac{1}{3}k(k+0.5)(k+1)+1)(\frac{1}{3}(k-1)k(k+1)+1)},$$

we can rewrite (8) as

$$P_{CT} - 0.5 < \sum_{k=2}^{\infty} a_k^s - b_k^s, \tag{9}$$

where

$$\sum_{k=2}^{\infty} a_k - b_k$$

was the upper bound for $S_{CT} - 0.5$ in (7), which was shown to be convergent.

I claim that, given two sequences $a_n > b_n > 0 \ \forall k$, where $a_n - b_n$ is a convergent series, $a_n^s - b_n^s$ is convergent for all $s > 0$. Clearly, if we can prove this general statement, its application to (9) would suffice to show convergence of P_{CT}. This is because the monotonically positive nature $a_k - b_k$ is unaffected by the monotonic positive exponentiation, and this would provide an upper bound for the sum of the monotonic series P_{CT}, which would give it a supremum and therefore ensure convergence by the Monotone Convergence Theorem as above.

I will now prove the statement above: Let a_n, b_n be two sequences such that $a_n > b_n > 0 \ \forall n$, and $a_n - b_n$ is a convergent series. Let $s > 0$.

Note that, because $a_n - b_n$ converges, $b_n \to a_n$ as $n \to \infty$.

Therefore, let $f(x) = x^s$. As exponentiation is a differentiable function, we can apply Taylor's Theorem to yield the equation

$$f(a_n) = f(b_n) + f'(b_n)(a_n - b_n) + O((a_n - b_n)^2)$$

or

$$a_n^s = b_n^s + s b_n^{s-1}(a_n - b_n) + O((a_n - b_n)^2).$$

Rearranging and applying to our example (replacing a_n, b_n with a_k, b_k), we get

$$a_k^s - b_k^s = s b_k^{s-1}(a_k - b_k) + O((a_k - b_k)^2).$$

Furthermore, b_k is bounded, as $b_k > 0$ and

$$\begin{aligned}
\lim_{k\to\infty} b_k = \lim_{k\to\infty} \ln\left(\frac{\frac{1}{3}k(k+1)(k+2)}{\frac{1}{3}k(k+0.5)(k+1)}\right) - \frac{\frac{1}{3}k(k+0.5)(k+1) - \frac{1}{3}(k-1)k(k+1)}{2(\frac{1}{3}k(k+0.5)(k+1)+1)(\frac{1}{3}(k-1)k(k+1)+1)} \\
= \lim_{k\to\infty} \ln\left(\frac{k+2}{k+0.5}\right) - \frac{9k(k+1)}{4(k(k+0.5)(k+1)+3)(k-1)k(k+1)+3)} \\
< \lim_{k\to\infty} \ln\left(\frac{4k+2}{k+0.5}\right) - \frac{9k(k+1)}{4(k(k+0.5)(k+1)+3(k^2+k))(k-1)k(k+1)+3(k^2+k))} \\
= \lim_{k\to\infty} \ln 4 - \frac{9}{4(k+3.5)(k+2)} = \ln 4.
\end{aligned}$$

Therefore, we have that $a_k^s - b_k^s$ can be written as the sum of $s b_k^{s-1}(a_k - b_k)$ and the remainder $O((a_n - b_n)^2)$. The first series clearly converges as it is bounded from above by a product of a convergent series, and from below by zero, and remains monotonically increasing. The second also converges, as $a_k - b_k$'s convergence means that there is a k for which this difference is maximal, and so it is bounded above by the product of this maximal $a_k - b_k$ and a $O(a_k - b_k)$ term, which converges as above.

Because P_{CT} is a series with a monotonically increasing subsequence, which is bounded above by $a_k^s - b_k^s$ which is in turn bound from above by the sum of two convergent sequences, we know that P_{CT} is bound from above, and therefore that the subsequence converges to a supremum; as the terms in P_{CT}, as exponents of the terms in S_{CT}, also go to zero, this means that the entire series converges to this supremum, and is therefore convergent, as desired.

As S_{PT} has partial sums strictly less than S_{CT}, any partial sum of P_{PT} will be less than P_{CT} for a given value of c. It follows that P_{PT} is also bounded, that its terms also go to zero, and that P_{PT} is also convergent.

Unboundedness of P_{CF}

The boundedness states for the divergent series are less straightforward. However, we can show that neither the unboundedness of P_{CF} nor the boundedness of P_{PF} are changed by modifying the parameter s:

Let A_n be the n^{th} term of P_{CF}, and let $B_n = A_{2n-1},\ C_n = A_{2n}$. To show that the generalized P_{CF} diverges without bound, we must show that the n^{th} term pair of P_{CF}, namely $B_n^s - C_n^s$, does not go to zero as $n \to \infty$. Because each $B_n > C_n$, each term pair is positive, and so to demonstrate unboundedness, it suffices to show that their difference does not go to zero.

First, we want to establish an upper bound for the terms in S_{CF}. Recalling the process in (5), we start by taking the positive subsequence

$$\{a_n^+\} = 1,\ \frac{1}{3},\ \left(\frac{1}{5}+\frac{1}{6}\right),\ \left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}\right),\ \left(\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}+\frac{1}{19}\right),\ \cdots,$$

and we instead replace every fraction with the *largest* fraction in the same parentheses, giving us

$$\{d_n^+\} := 1,\ \frac{1}{3},\ \frac{2}{5},\ \frac{3}{9},\ \frac{5}{15},\ \frac{8}{25},\cdots.$$

Note that

$$\{d_n^+\} := \frac{f_n}{2 \cdot f_{n+1} - 1},$$

so taking the limit

$$\lim_{n\to\infty} d_n^+ = \frac{1}{2\phi}$$

gives us $\frac{1}{2\phi}$ as an upper bound for the value of an odd term. Using (6), we get an even term upper bound of $\frac{1}{2\phi} - \ln\frac{\phi^3}{4}$.

With this bound, we can establish that, for all n past a certain N,

$$\frac{1}{2\phi^2} < B_n < \frac{1}{2\phi},$$

and, from (6),

$$\frac{1}{2\phi^2} < C_n < B_n - \ln\frac{\phi^3}{4}.$$

This gives us

$$\begin{aligned} B_n^s - C_n^s >& B_n^s - \left(B_n - \ln\frac{\phi^3}{4}\right)^s > B_n^s - \left(B_n - 2\phi B_n \ln\frac{\phi^3}{4}\right)^s \\ =& B_n^s\left(1 - \left(1 - 2\phi\ln\frac{\phi^3}{4}\right)^s\right) \end{aligned}$$

Since the expression $1-2\phi\ln\frac{\phi^3}{4}$ is less than one, any positive power s of the expression will be within $(0,1)$. In particular, let $r = 1-2\phi\ln\frac{\phi^3}{4} \in (0,1)$. Then

$$B_n^s\left(1 - \left(1 - 2\phi\ln\frac{\phi^3}{4}\right)^s\right) = B_n^s(1-r^s) > \frac{1-r^s}{(2\phi^2)^s} > 0$$

Since all but finitely many pairs of terms are bounded from below by this positive value, which does not depend on the term n, we have divergence without bound.

Boundedness of P_{PF}

Finally, we must show that the modified S_{PF} remains bounded.

To this end, let A_n be the n^{th} term in S_{PF}, and $B_n = A_{2n-1}$, $C_n = C_{2n}$. From above, we have $\sum_{n=1}^{\infty} B_n - C_n = \ln 2$. We must show that $P_{PF} = \sum_{n=1}^{\infty} B_n^s - C_n^s < \infty$.

In particular, we wish to show that for any $s > 0$, there exists a constant k such that $B_n^s - C_n^s \leq k(B_n - C_n)$. If we can show this, then clearly

$$\sum_{n=1}^{\infty} B_n^s - C_n^s \leq k \sum_{n=1}^{\infty} B_n - C_n = k \ln 2 < \infty.$$

As a preliminary, we have shown above that the odd (added) terms of S_{CF} are greater than or equal to the odd terms of S_{PF}. Therefore, our odd term upper bound of $\frac{1}{2\phi}$ for S_{CF} also serves as an odd term upper bound of S_{PF}.

First, consider the case when $s \in \mathbb{Z}_+$. Then, we can factor

$$B_n^s - C_n^s = (B_n - C_n)(B_n^{s-1} + B_n^{s-2} C_n + \ldots + B_n C_n^{s-2} + C_n^{s-1}).$$

Because, from above, we have $C_n < B_n < \frac{1}{2\phi}$, we can create an upper bound by

$$B_n^s - C_n^s < (B_n - C_n)(s B_n^s) < \frac{s}{(2\phi)^s}(B_n - C_n),$$

giving an upper bound with $k = \frac{s}{(2\phi)^s}$, and thus a sum bounded from above by $\frac{s}{(2\phi)^s} \ln 2$.

Next, suppose the exponent is a reciprocal of $s \in \mathbb{Z}_+$:

$$B_n^{\frac{1}{s}} - C_n^{\frac{1}{s}} = (B_n - C_n)\left(B_n^{\frac{s-1}{s}} + B_n^{\frac{s-2}{s}} C_n^{\frac{1}{s}} + \ldots + B_n^{\frac{1}{s}} C_n^{\frac{s-2}{s}} + C_n^{\frac{s-1}{s}}\right)^{-1}.$$

Just as above, we can create an upper bound for this:

$$B_n^{\frac{1}{s}} - C_n^{\frac{1}{s}} < (B_n - C_n)(s B_n^{\frac{s-1}{s}})^{-1} < \frac{1}{s}(2\phi)^{\frac{s-1}{s}}(B_n - C_n),$$

from which we can bound the infinite sum from above by $\frac{1}{s}(2\phi)^{\frac{s-1}{s}} \ln 2$.

Through the composition of the above two processes, we can bound $B_n^s - C_n^s$ from above by a constant k times $B_n - C_n$, for any rational number s (as rational numbers can be expressed as a quotient of integers). Since each procedure above is a finite process, it will always produce a finite number k. It only remains to show that an upper bound can be established for irrational positive numbers.

However, we know that positive exponentiation is a continuous function over the positive reals. This means that, for any irrational number s_i, we can find another s_i^* such that $\sum_{n=1}^{\infty} B_n^{s_i^*} - C_n^{s_i^*}$ is arbitrarily close to the series $\sum_{n=1}^{\infty} B_n^{s_i} - C_n^{s_i}$, by taking s_i^* sufficiently close to s_i. However, there exist rational numbers s_i^* within any positive distance of s_i, by the density of rationals in the reals. Therefore, for any $\epsilon > 0$, we can find a rational s_i^* for which

$$\left| \sum_{n=1}^{\infty} \left(B_n^{s_i^*} - C_n^{s_i^*}\right) - \sum_{n=1}^{\infty} (B_n^{s_i} - C_n^{s_i}) \right| < \epsilon.$$

Because the former series is bounded from above by $k \ln 2$ for some positive k, $\sum_{n=1}^{\infty} (B_n^{s_i} - C_n^{s_i})$ is bounded from above by $k \ln 2 + \epsilon$. Therefore, the P_{PF} is bounded from above for any positive s.

Finally, we have that $\sum_{n=1}^{\infty} (B_n^{s} - C_n^{s})$ is convergent for any positive s, because the series is bounded and has monotonically increasing partial sums.

Consecutive Fibonacci Blocked Series

After computing various partial sums with various powers s, we graphed them in R, and found equations to approximate the points, which appeared to follow a fairly consistent curve.

We observed several key features: the start at (0,0), the steep, concave downwards increase to a peak shortly before 1, and the decrease, inflection point, and asymptote at $P(s) = 1$ as $s \to \infty$.

With these features in mind, we explored various functions in Desmos which had these features. We returned to R to perform linear regression to determine these parameter values.

Formula

We have computed that the partial sum P_{2L} of the series up to the blocks of size F_L, raised to the s power, is of the form

$$P_{CF}(s, L) \approx \frac{dLs - a + bs + ae^{cs}}{L + ae^{cs} + 1}, \tag{10}$$

with parameters computed to be approximately

$$\begin{aligned} a &= 496.245, \\ b &= -555.268, \\ c &= 1.48991, \\ d &= 130.793. \end{aligned}$$

This model has a RSE of 0.01272. This model was calculated with partial sums up to $L = 45$; that is, computed with term sizes up to the 45^{th} Fibonacci number. This model was trained with s values ranging from 0 to 10. Future work may involve re-tuning this model to include higher partial sums, or perhaps more s values.

Peak Calculation

One interesting feature of this curve is that it appears to have a "peak": for a given L, a value s for which the partial sum is maximal. We want to see if we can calculate this "peak" as a function of the length L, and if this will allow us to determine what value of s is maximal for the whole series, when $L \to \infty$:

To do this, we s by set

$$\frac{\partial P}{\partial s} = 0.$$

To do this, we will substitute

$$\begin{aligned} u &= dL + b \\ v &= L + 1 \end{aligned}$$

into (10) to yield

$$P_{CF}(s, L) \approx \frac{dLs - a + bs + ae^{cs}}{L + ae^{cs} + 1} = \frac{-a + us + ae^{cs}}{v + ae^{cs}}.$$

This gives us

$$\begin{aligned} \frac{\partial P}{\partial s} =& \frac{(u + cae^{cs})(v + ae^{cs}) - (-a + us + ae^{cs})(cae^{cs})}{(v + ae^{cs})^2} = 0 \\ \Longrightarrow& \; uv + ae^{cs}(u + cv) + ca^2 e^{2cs} = ca(-a + us)e^{cs} + ca^2 e^{2s} \\ \Longrightarrow& \; uv + ae^{cs}(u + cv + ca) = cause^{cs} \\ \Longrightarrow& \; -cs = -\frac{v}{a}e^{-cs} - \frac{u + cv + ca}{u}. \end{aligned}$$

Using Mathematica, we simplify this to

$$-cs = -\frac{u + cv + ca}{u} - W_0\left(\frac{v}{a}e^{-\frac{u+cv+ca}{cu}}\right),$$

where W_0 denotes the principal branch of the Lambert-W function.

Substituting $k = \frac{u+cv+ca}{u} = \frac{dL+b+cL+c+ca}{dL+b}$, we get

$$s = \frac{k}{c} + \frac{1}{c}W_0\left(\frac{L+1}{a}e^{-k}\right).$$

Taking the limit as $L \to \infty$, we get that $k \to \frac{d+c}{d}$, meaning that

$$\lim_{L\to\infty} s_{peak} = \frac{d+c}{dc} + \frac{1}{c}\lim_{L\to\infty} W_0\left(\frac{L+1}{a}e^{-\frac{d+c}{d}}\right) = \infty.$$

This shows that the peak does not converge to any particular s value, but diverges to ∞.

Peak Formula

An approximating formula for the peak s is given by

$$F_{Peak}(L) \approx \frac{a + b\ln L + c\ln^2 L}{1 + d\ln L + f\ln^2 L},$$

with parameters

$$\begin{aligned} a &= 4.18424 \\ b &= -1.31051 \\ c &= 0.150838 \\ d &= 0.188535 \\ f &= 5.775765 \times 10^{-4} \end{aligned}$$

and a RSE of approximately 0.03624.

Parity Fibonacci Blocked Series

Because this series' even partial sums converge, we computed the limiting function

$$P_{PF}(s) \approx \frac{-1 + as + e^{bs}}{as + e^{bs}},$$

where

$$\begin{aligned} a &= 0.73481 \\ b &= 0.87946 \end{aligned}$$

has a RSE of approximately 0.01098.

Additionally, the partial sum length formula

$$P(s, L) \approx \frac{-1 + as + e^{bs} + cs(1 - \frac{1}{L})}{1 + as + e^{bs}}$$

for

$$\begin{aligned} a &= 2.19506 \\ b &= 0.774792 \\ c &= 0.301244 \end{aligned}$$

generalizes the formula to account for partial sums. This model has a RSE of 0.01188. These models were likewise computed from s values ranging from 0 to 10, and partial sum sized computed up to $L = 45$, the term pair with size F_{45}.

Consecutive Triangular Blocked series

The model for S_{CT} is

$$P_{CT}(s) \approx \frac{-a + \ln(s+d) + s + ae^{bs}}{c + ae^{bs}}$$

for

$$\begin{aligned} a &= 5.38541 \\ b &= 0.321631 \\ c &= -5.11466 \\ d &= 0.999703, \end{aligned}$$

with a RSE of approximately 0.0128. Because the sums quickly converge, we computed the partial sum up to blocks of size T_{2000}. The s values ranged from 0 to 10.

Parity Triangular Blocked Series

For S_{PT}, we computed

$$P_{PT}(s) \approx \frac{-a + bs + ae^{cs}}{1 + ds + ae^{cs}}$$

for

$$\begin{aligned} a &= 2.27299 \times 10^6 \\ b &= 1.24110 \times 10^6 \\ c &= 0.552346 \\ d &= 2.62199 \times 10^5, \end{aligned}$$

which has a RSE of approximately 0.001137. As above, the model was calculated with s values between 0 and 10, and with terms of up to size T_{2000}.

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