Alternating Harmonic Series: Fibonacci and Triangular Number Blocks

Matthew Blake

St. Olaf College

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Outline

Introduction

- The Harmonic Series is one of the oldest sequences studied. Its divergence was first proven in 1360 by Nicholas Oresme.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \tag{1}$$

- A related series is the Alternating Harmonic Series, whose sum is known to converge to $\ln 2$.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} = \ln 2 \tag{2}$$

- The four series we shall examine are variations of the Alternating Harmonic series.

Series Name Definitions

- A "Fibonacci-blocked series" is a series whose signs alternate by Fibonacci-number length blocks, denoted with parentheses.
  - $F_n = 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \dots$
- Conversely, a "Triangular-blocked series" blocks terms by triangular numbers.
  - $T_n = \frac{n(n+1)}{2} = 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \dots$
- A "Consecutive" series maintains the *order* of the reciprocals when blocked,
  - ie. $\left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+k-1}\right) - \left(\frac{1}{n+k} + \frac{1}{n+k+1} + \frac{1}{n+k+2} + \cdots + \frac{1}{n+2k-1}\right)$
- whereas a "Parity" series adds odd reciprocals, and subtracts even reciprocals.
  - ie. $\left(\frac{1}{n} + \frac{1}{n+2} + \frac{1}{n+4} \cdots + \frac{1}{n+2k-2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+3} + \frac{1}{n+5} + \cdots + \frac{1}{n+2k-1}\right)$

Series Notations- Consecutive Fibonacci

These four criteria combine to give us our four series:

- The Consecutive Fibonacci series

$$\overbrace{1-\frac{1}{2}}^{1}+\overbrace{\frac{1}{3}-\frac{1}{4}}^{1}+\overbrace{\underbrace{\left(\frac{1}{5}+\frac{1}{6}\right)-\left(\frac{1}{7}+\frac{1}{8}\right)}_{Consecutive}}^{2}+\overbrace{\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}\right)}^{3}$$
$$\overbrace{-\left(\frac{1}{12}+\frac{1}{13}+\frac{1}{14}\right)}^{3}+\overbrace{\left(\frac{1}{15}+\cdots+\frac{1}{19}\right)-\left(\frac{1}{20}+\cdots+\frac{1}{24}\right)}^{5}+\cdots \tag{3}$$

$$=\sum_{i=1}^{\infty}\left(\sum_{n=2\cdot F_i-1}^{F_{i+2}-2}\frac{1}{n}-\sum_{n=F_{i+2}-1}^{2\cdot F_{i+1}-2}\frac{1}{n}\right)$$

Parity Fibonacci

The Parity Fibonacci series

$$\overbrace{1-\frac{1}{2}}^{1}+\overbrace{\frac{1}{3}-\frac{1}{4}}^{1}+\overbrace{\underbrace{\left(\frac{1}{5}+\frac{1}{7}\right)}_{Odd}-\underbrace{\left(\frac{1}{6}+\frac{1}{8}\right)}_{Even}}^{2}+\overbrace{\underbrace{\left(\frac{1}{9}+\frac{1}{11}+\frac{1}{13}\right)}_{Odd}}^{3}$$

$$\overbrace{-\underbrace{\left(\frac{1}{10}+\frac{1}{12}+\frac{1}{14}\right)}_{Even}}^{3}+\overbrace{\left(\frac{1}{15}+\cdots+\frac{1}{23}\right)-\left(\frac{1}{16}+\cdots+\frac{1}{24}\right)}^{5}+\cdots \tag{4}$$

$$=\sum_{i=0}^{\infty}\left(\sum_{m=0}^{F_i-1}\frac{1}{2F_{i+1}-1+2m}-\sum_{n=0}^{F_i-1}\frac{1}{2F_{i+1}+2n}\right)$$

studied by Chen and Kennedy [2012],

Parity Triangular

- The Parity Triangular series

$$\overbrace{1-\frac{1}{2}}^{1}+\overbrace{\underbrace{\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)}_{Odd}-\underbrace{\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right)}_{Even}}^{3}+\overbrace{\underbrace{\left(\frac{1}{9}+\ldots+\frac{1}{19}\right)}_{Odd}}^{6} \tag{5}$$

$$\overbrace{\underbrace{-\left(\frac{1}{10}+\ldots+\frac{1}{20}\right)}_{Even}}^{6}+\overbrace{\left(\frac{1}{21}+\cdots+\frac{1}{39}\right)-\left(\frac{1}{22}+\cdots+\frac{1}{40}\right)}^{10}+\cdots=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\frac{1}{2}k(k+1)}\frac{1}{2n-1+\frac{1}{3}(k-1)k(k+1)}-\frac{1}{2n+\frac{1}{3}(k-1)k(k+1)}$$

Consecutive Triangular

- and the Consecutive Triangular series

$$\overbrace{1-\frac{1}{2}}^{1}+\underbrace{\overbrace{\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)-\left(\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}^{3}}_{Consecutive}+\overbrace{\left(\frac{1}{9}+\cdots+\frac{1}{14}\right)}^{6} \tag{6}$$

$$\overbrace{-\left(\frac{1}{15}+\cdots+\frac{1}{20}\right)}^{6}+\overbrace{\left(\frac{1}{21}+\cdots+\frac{1}{30}\right)-\left(\frac{1}{31}+\cdots+\frac{1}{40}\right)}^{10}+\cdots=$$

$$=\sum_{k=1}^{\infty}\sum_{n=1}^{\frac{1}{2}k(k+1)}\frac{1}{n+\frac{1}{3}(k-1)k(k+1)}-\frac{1}{n+\frac{1}{2}k(k+1)+\frac{1}{3}(k-1)k(k+1)}$$

Fibonacci Sequence

The Fibonacci Sequence, defined recursively by $F_0 = F_1 = 1$, and $F_n + F_{n+1} = F_{n+2}\ \forall n \in \mathbb{N}$, has two properties we will use later in this talk:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi \tag{7}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.
Additionally,

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1 \tag{8}$$

can be proven by induction.

Harmonic Series Approximation

Young [1991] gives us

$$\ln(n) + \gamma + \frac{1}{2(n+1)} < H_n < \ln(n) + \gamma + \frac{1}{2n}$$

Rearranging and simplifying, this yields, (for $m > 1$)

$$\ln\frac{n}{m-1} - \frac{n+1-m}{2(n+1)m} < \sum_{i=m}^{n}\frac{1}{i} < \ln\frac{n}{m-1} - \frac{n-m+1}{2n(m-1)} \qquad (9)$$

Parity Fibonacci Divergence

By the Test for Divergence, if we can find a(n infinite) subsequence of terms in the series which does not go to zero, the series diverges. We claim that the sequence of positive terms $\{a_n\}$ does not go to zero.

$$\{a_n\} := 1,\ \frac{1}{3},\ \left(\frac{1}{5} + \frac{1}{7}\right),\ \left(\frac{1}{9} + \frac{1}{11} + \frac{1}{13}\right), \left(\frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23}\right), \ldots$$

We can create a smaller series, b_n, by replacing each fraction in a_n with the smallest fraction in the parentheses:

$$\{b_n^+\} := 1,\ \frac{1}{3},\ \frac{2}{7},\ \frac{3}{13},\ \frac{5}{23},\ \ldots$$

so that

$$a_n \geq b_n \ \forall n \in \mathbb{N}$$

Parity Fibonacci Divergence

However,

$$b_n = \frac{F_n}{2\sum_{i=1}^{n} F_i - 1} = \frac{F_n}{2F_{n+2} - 3}$$

by (6).
Applying (5), we get that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{F_n}{2\phi^2 F_n - 3} = \lim_{n\to\infty} \frac{F_n}{2\phi^2 F_n} = \frac{1}{2\phi^2} \neq 0$$

Since $a_n \geq b_n \not\to 0$, we have that an infinite subsequence (namely a_n) of the Parity Fibonacci Series does not go to zero, meaning the series diverges by the Test for Divergence.

Consecutive Fibonacci Divergence

Note that, taking the analogous sub-sequence to the Parity Series'

$$\{a_n\} = 1, \frac{1}{3},\ \left(\frac{1}{5} + \frac{1}{7}\right), \left(\frac{1}{9} + \frac{1}{11} + \frac{1}{13}\right), \left(\frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23}\right), \dots$$

we can construct the positive subsequence

$$\{c_n\} := 1, \frac{1}{3}, \left(\frac{1}{5} + \frac{1}{6}\right), \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11}\right), \left(\frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{19}\right), \dots$$

and see that $c_n \geq a_n\ \forall n \in \mathbb{N}$. So clearly $c_n \geq b_n$, and the same argument as above shows divergence for the Parity Fibonacci Series.

Consecutive Triangular Convergence

We will show the convergence of this series by the Alternating Series Test.
Consider the subsequence of positive terms $\{a_k\} :=$

$$\sum_{n=1}^{\frac{1}{2}k(k+1)} \frac{1}{n+\frac{1}{3}(k-1)k(k+1)} = \sum_{n=\frac{1}{3}(k-1)k(k+1)+1}^{\frac{1}{3}k(k+0.5)(k+1)} \frac{1}{n}$$

via the identity

$$\frac{1}{3}(k-1)k(k+1) + \frac{1}{2}k(k+1) = \frac{1}{3}k(k+1)(k+2) - \frac{1}{2}k(k+1)$$

Consecutive Triangular Convergence

By (7) above, we have that

$$\lim_{k\to\infty} \sum_{n=\frac{1}{3}(k-1)k(k+1)+1}^{\frac{1}{3}k(k+0.5)(k+1)} \frac{1}{n} \leq \lim_{k\to\infty} \ln \frac{\frac{1}{3}k(k+0.5)(k+1)}{\frac{1}{3}(k-1)k(k+1)+1-1} - \frac{\frac{1}{2}k(k+1)}{\frac{2}{9}(k-1)k^2(k+0.5)(k+1)^2} = \lim_{k\to\infty} \ln \frac{k+0.5}{k-1} - \frac{9}{4(k-1)k(k+0.5)(k+1)} = \ln 1 = 0$$

- Because the positive terms are bounded above by a value which goes to zero, the positive terms go to zero.
- Furthermore, since each positive term is greater in absolute value than the negative term which follows it, the negative terms go to zero as well.
- Therefore, we have convergence by the Alternating Series Test.

Parity Triangular Convergence

- By the Alternating Series Test, if we can show that the terms of an alternating series go to zero, then we have convergence.
- First, note that, because any even partial sum is a simple rearrangement of the Alternating Harmonic Series it will equal an alternating harmonic series partial sum $\bar{H}_n$, where

$$n = 2\sum_{i=1}^{k} \frac{i(i+1)}{2} = \sum_{i=1}^{k} i^2 + i = \frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{3}$$

- It is known that the Alternating Harmonic Series converges to $\ln 2$, so the even partial sums must converge to $\ln 2$.
- It remains to show that the odd terms go to zero.
- Yet each odd term (beyond the first) is smaller than the corresponding term in the Consecutive series, whose odd terms have been shown to go to zero.
- Therefore, each odd term in the Parity Triangular Series goes to zero.
- This gives us convergence, by the Alternating Series Test.
- Moreover, we know that the series converges to $\ln 2$

Consecutive Fibonacci Unboundedness

- While we have established that this series is divergent, we can also show that it is unbounded: rather than oscillating between two values, the partial sums continue to infinity.
- We will do this by repeating the earlier proof, using the Divergence Test, on the *even* partial sums.
- Since every odd term is larger than the even term following it, each pair of terms is a positive difference, and therefore
- the sequence of even partial sums is monotonically increasing.
- Thus, to show that this series is unbounded, it suffices to show that the even partial sums do not go to zero.

Consecutive Fibonacci Unboundedness

The Consecutive Fibonacci Series,

$$\sum_{i=1}^{\infty}\left(\sum_{m=2F_i-1}^{F_{i+2}-2}\frac{1}{m}-\sum_{n=F_{i+2}-1}^{2F_{i+1}-2}\frac{1}{n}\right)$$

can be approximated via (9) as

$$\lim_{i\to\infty}\sum_{m=2F_i-1}^{F_{i+2}-2}\frac{1}{m}-\sum_{n=F_{i+2}-1}^{2F_{i+1}-2}\frac{1}{n} \tag{10}$$

$$>\lim_{i\to\infty}\ln\frac{F_{i+2}-2}{2F_i-2}-\frac{F_{i-1}}{2(F_{i+2}-1)(2F_i-1)}-\ln\frac{2F_{i+1}-2}{F_{i+2}-2}+\frac{F_{i-1}}{2(2F_{i+1}-2)(F_{i+2}-2)}$$

Simplifying,

$$\ldots=\lim_{i\to\infty}\ln\frac{(F_{i+2}-2)^2}{(2F_i-2)(2F_{i+1}-2)}+\frac{F_{i-1}}{2}\left(\frac{1}{2(2F_{i+1}-1)(F_{i+2}-2)}-\frac{1}{(F_{i+2}-1)(2F_i-1)}\right)$$

Consecutive Fibonacci Unboundedness

We evaluate the limit, remembering that, as $i \to \infty$, $F_{i+1} \to \phi \cdot F_i$, giving us

$$\ldots = \lim_{i\to\infty} \ln \frac{(\phi^2 F_i - 2)^2}{4(F_i - 1)(\phi F_i - 1)} + \frac{F_i}{2\phi}\left(\frac{1}{2(2\phi F_i - 1)(\phi^2 F_i - 1)} - \frac{1}{(\phi^2 F_i - 2)(2F_i - 1)}\right)$$

which simplifies to

$$\ldots = \lim_{i\to\infty} \ln \frac{\phi^4 F_i^2 - 4\phi^2 F_i + 4}{4(\phi F_i^2 - \phi^2 F_i + 1)} = \ln \frac{\phi^4 F_i^2}{4\phi F_i^2} = \ln \frac{\phi^3}{4} \approx 0.05734 > 0 \tag{11}$$

The terms in (10) are strictly greater than the terms in (11), which converge greater than 0. Therefore, the (monotonic) sequence of even partial sums of the Consecutive Fibonacci series does not go to zero, and therefore the Consecutive Fibonacci series diverges to infinity.

Parity Fibonacci Boundedness and Optimization Intro

Conversely, the Parity Fibonacci series does not diverge; Chen and Kennedy [2012] show that the even partial sums of this series converge to $\ln 2$, while the odd partial sums converge to $\ln 2 + \frac{1}{2} \ln \phi$. This series is therefore bounded.

- The next part of this presentation involves calculating partial sums of the above series, with each term raised to a positive power s.
  - A power of 0 would yield 0, and negative powers would lead to exponential growth.
- Using R, we calculated these series partial sums.
  - However, we ran into computational limits, so we created approximating functions for the series.
  - Using these, we attempted to extrapolate to find larger partial sums

Power Optimization- Equation Examples

For example, the Consecutive Fibonacci series is modified as

$$P_{CF}(s,L) = \sum_{i=1}^{L} \left(\left(\sum_{m=2\cdot F_i-1}^{F_{i+\mathbf{2}}-2} \frac{1}{m} \right)^s - \left(\sum_{n=F_{i+\mathbf{2}}-1}^{2\cdot F_{i+\mathbf{1}}-2} \frac{1}{n} \right)^s \right)$$
$$= (1)^s - \left(\frac{1}{2}\right)^s + \left(\frac{1}{3}\right)^s - \left(\frac{1}{4}\right)^s + \left(\frac{1}{5}+\frac{1}{6}\right)^s - \left(\frac{1}{7}+\frac{1}{8}\right)^s$$
$$+ \left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}\right)^s - \left(\frac{1}{12}+\frac{1}{13}+\frac{1}{14}\right)^s + \cdots$$

the Parity Fibonacci series as

$$P_{PF}(s) = (1)^s - \left(\frac{1}{2}\right)^s + \left(\frac{1}{3}\right)^s - \left(\frac{1}{4}\right)^s + \left(\frac{1}{5}+\frac{1}{7}\right)^s - \left(\frac{1}{6}+\frac{1}{8}\right)^s$$
$$+ \left(\frac{1}{9}+\frac{1}{11}+\frac{1}{13}\right)^s - \left(\frac{1}{10}+\frac{1}{12}+\frac{1}{14}\right)^s + \cdots$$

and so on.

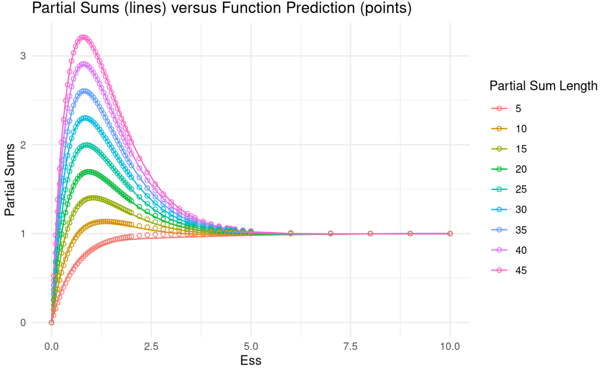
Convergence and Boundedness Consistency

- Fibonacci blocked series diverge for any (positive) power s.
  - Divergent term limit $w > \ln \frac{\phi^3}{4} > 0 \implies w^s > 0$.
- Triangular blocked series converge for any power s.
  - Convergent term limit $w = 0 \implies w^s = 0$.
- P_{CF} diverges without bound for any power s.
  - Established an upper bound for a term in S_{CF}, and from there create lower bound greater than zero for terms in P_{CF}.
- P_{PF} remains bounded for any power s.
  - Factor a rational s_r sufficiently close to s (using continuity), to bound term in P_{PF} from above by a constant times S_{PF}, whose sum is bounded.

Consecutive Fibonacci

For exponent s, the partial sum computed up to block size F_L is

$$P_{CF}(s, L) \approx \frac{-a + (dL + b)s + ae^{cs}}{L + ae^{cs} + 1} \qquad (12)$$



with parameters

$$a = 496.244$$
$$b = -555.268$$
$$c = 1.48991$$
$$d = 130.793$$

Figure 1:

This model has a RSE of approximately 0.0127.

Parity Fibonacci

The Parity Fibonacci series partial sum formula

$$P_{PF}(s, L) \approx \frac{-1 + as + e^{bs} + cs(1 - \frac{1}{L})}{1 + as + e^{bs}} \tag{13}$$

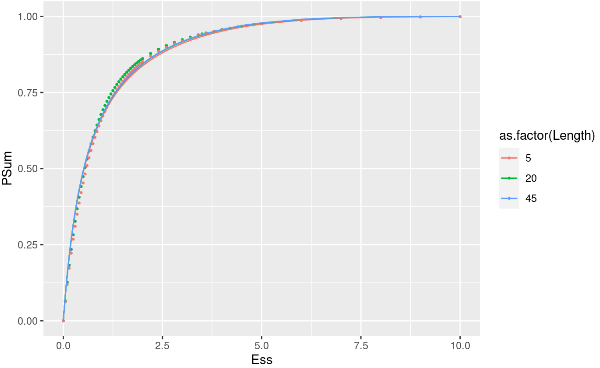


Figure 2:

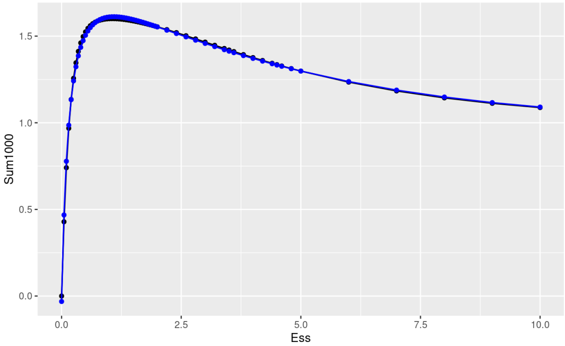
with parameters

$$a = 2.63985$$
$$b = 0.698774$$
$$c = 0.203249$$

This model has a RSE of approximately 0.0043065.

Consecutive Triangular

$$P_{CT}(s) \approx \frac{-a + \ln(ds) + s + ae^{bs}}{c + ae^{bs}} \tag{14}$$



with parameters

$$a = 5.12770$$
$$b = 0.329482$$
$$c = -4.83819$$
$$d = 0.990935$$

Black = Part. Sum

Blue = Funct. Est.

This model has a RSE of approximately 0.0128.
The series converges; the model was trained on the T_{1000} partial sum. This is a slight underestimation of the final distribution.

Peaks- Consecutive Fibonacci

From this formula (13), we want to calculate where the peaks are:
For a given partial sum length L, the exponent s which will produce the largest sum.
To do this, we will substitute

$$u = dL + b$$
$$v = L + 1$$

into (13) to yield

$$P_{CF}(s, L) \approx \frac{dLs - a + bs + ae^{cs}}{L + ae^{cs} + 1} = \frac{-a + us + ae^{cs}}{v + ae^{cs}} \tag{15}$$

Setting

$$\frac{\partial P_{CF}}{\partial s} = 0$$

Peaks- Consecutive Fibonacci

we get

$$\frac{\partial P_{CF}}{\partial s} = \frac{(u + cae^{cs})(v + ae^{cs}) - (-a + us + ae^{cs})(cae^{cs})}{(v + ae^{cs})^2} = 0$$

$$\implies uv + ae^{cs}(u + cv) + ca^2e^{2cs} = ca(-a + us)e^{cs} + ca^2e^{2s}$$

$$\implies uv + ae^{cs}(u + cv + ca) = cause^{cs}$$

$$\implies -cs = -\frac{v}{a}e^{-cs} - \frac{u + cv + ca}{u}$$

Via Mathematica:

$$\implies -cs = -\frac{u + cv + ca}{u} - W_0\left(\frac{v}{a}e^{-\frac{u+cv+ca}{cu}}\right)$$

where W_0 is the principal branch of the Lambert-W function.

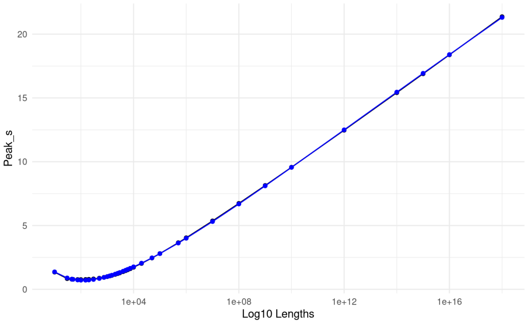
Peaks- Consecutive Fibonacci

Letting $k = \frac{u+cv+ca}{u} = \frac{dL+b+cL+c+ca}{dL+b}$, we get:

$$\implies s = \frac{k}{c} + \frac{1}{c} W_0 \left(\frac{L+1}{a} e^{-k} \right)$$

An approximating formula for the peak s is given by

$$F_{Peak}(L) = \frac{a + b \ln L + c \ln^2 L}{1 + d \ln L + f \ln^2 L}$$



with parameters

$$a = 4.18424$$
$$b = -1.31051$$
$$c = 0.150838$$
$$d = 0.188535$$
$$f = 0.000577577$$

Black = Part. Sum

Blue = Funct. Est.

with a RSE of approximately 0.03624.

References and Funding

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