Alternating Harmonic Series Grouped by Fibonacci and Triangular Numbers

Matthew Blake

INTRODUCTION

- The divergence of the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i} = \infty$ was proven in 1360 by Nicholas Oresme.
- However, the alternating harmonic series converges:

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$$

- Chen and Kennedy (2012) explored a variant of the harmonic series, and this was the inspiration for this project.

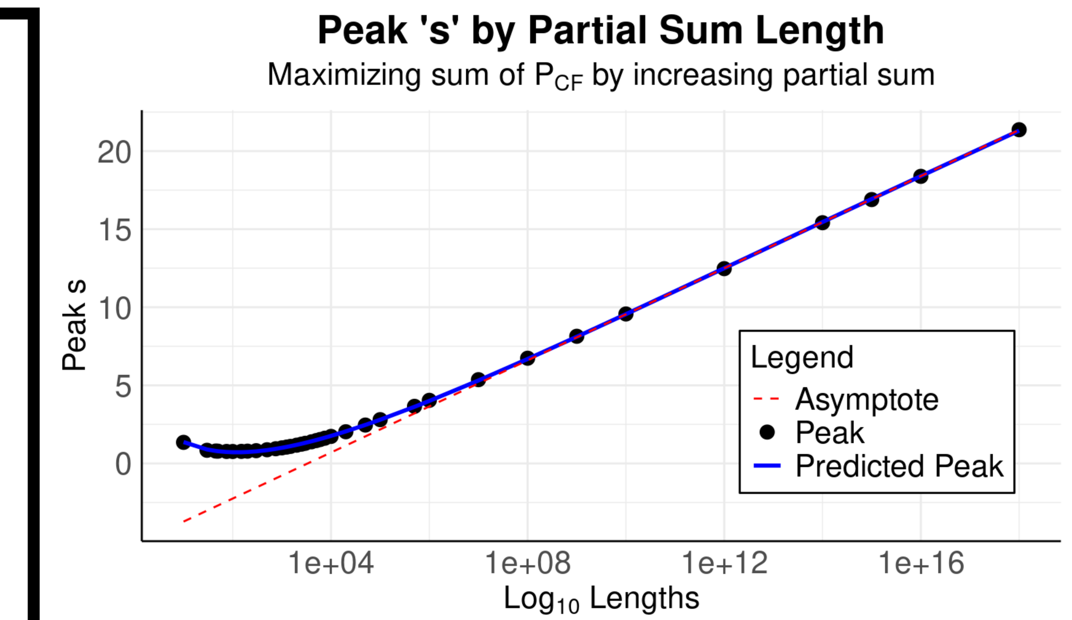
PRELIMINARIES

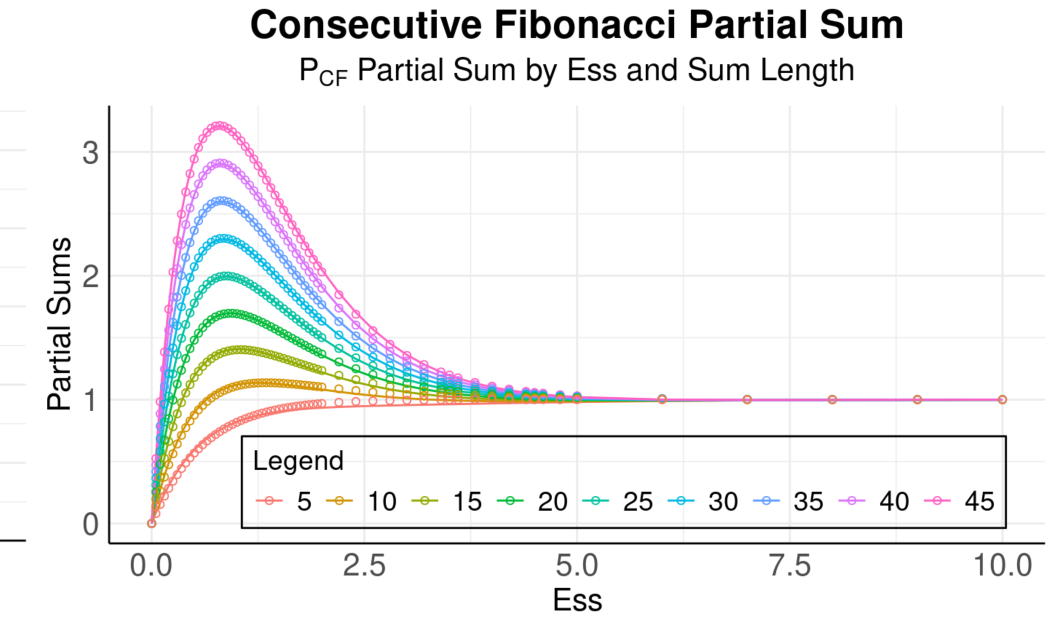
- F_i denotes the i^{th} number in the Fibonacci sequence
- $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi = \frac{\sqrt{5}-1}{2}$, the Golden Ratio, and $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.
- Young (1991) gives an inequality which yields, for m > 1,

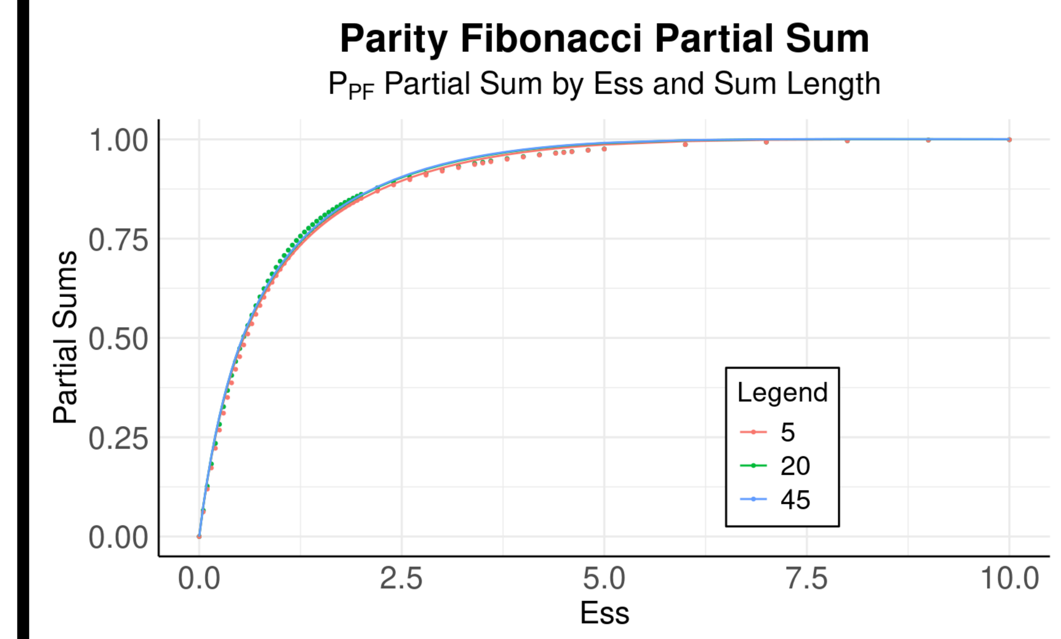
$$\ln\left(\frac{n}{m-1}\right) - \frac{n+1-m}{2(n+1)m} < \sum_{i=m}^{n} \frac{1}{i} < \ln\left(\frac{n}{m-1}\right) - \frac{n-m+1}{2n\,(m-1)}$$

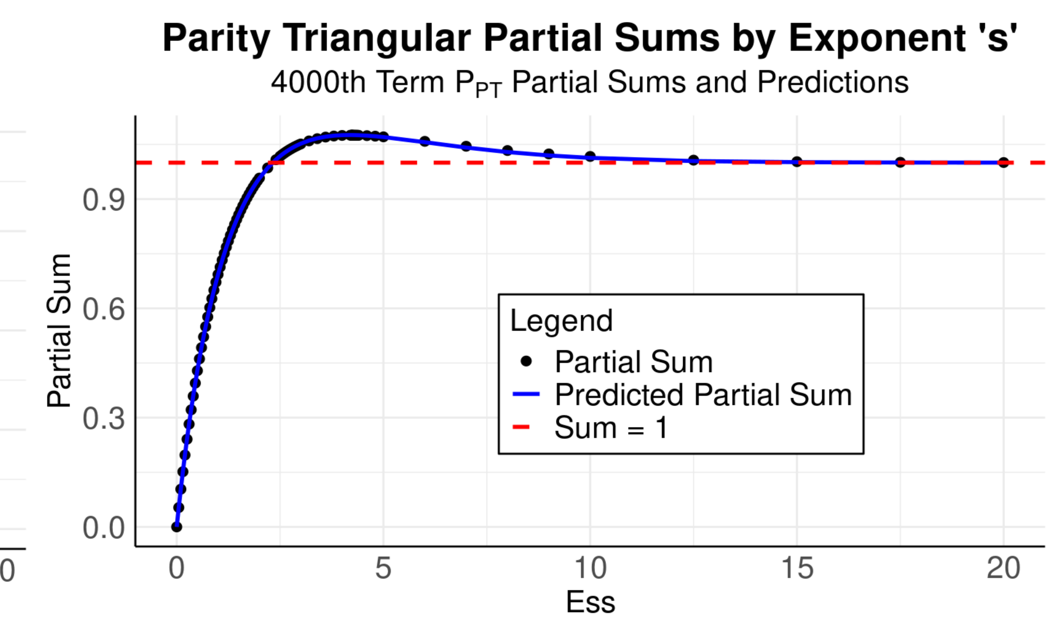
PROOFS- CONVERGENCE, BOUNDS

- S_{PF} is divergent, because the value of terms in the odd subsequence (terms with odd reciprocals) is bounded below by sequence $1, \frac{1}{3}, \frac{2}{7}, \frac{3}{13}, \frac{5}{23}$, which goes to $\frac{1}{2\phi^2} > 0$.
  Thus, S_{PF} **diverges** by Test for Divergence.
- Every odd term in S_{CF} is greater than corresponding term in S_{PF}, so similarly bounded from below, and we have **divergence**.
- S_{CT} converges because the sequence of even partial sums is a monotonically increasing sequence bounded from above by the partial sums of a convergent sequence. So S_{CT} has a supremum, and is thus **convergent** by the Monotone Convergence Theorem.
- S_{PT} converges because the even partial sums are rearrangements of the Alternating Harmonic Series. So even partial sums converge to ln 2, and odd terms go to zero, and so we have **convergence to ln 2** by Alternating Series Test.
- S_{CF} diverges without bound because we can use the Harmonic inequality to bound even partial sums from above, by a value which approaches $\ln\frac{\phi^3}{4} \approx 0.05734 > 0$. Since pairs of terms are positive, the series of partial sums, and thus the series, **diverges to infinity** by Test for Divergence.
- The boundedness of S_{PF} was shown by Chen and Kennedy, namely that the even partial sums converge to ln 2, and the odd partial sums converge to $\ln 2 + \frac{1}{2}\ln\phi$.









PROOFS- GENERALIZED SERIES

- Divergence is **unchanged**: if the original series S goes to some value $x \neq 0$, then the generalized series' terms will go to x^s, which will be nonzero. Thus, P_{CF} and P_{PF} diverge.
- Convergence is **unchanged**: This can be shown by bounding the generalized series from above by a multiple of the original series, by using linear approximation. Thus, P_{CT} and P_{PT} converge.
- Boundedness is **unchanged:** P_{CF} is unbounded, and P_{PF} is bounded:
  -The unboundedness of P_{CF} is shown by creating bounds for even and odd terms, and using them to create a (positive) lower bound for the difference between the $(2k\text{-}1)^{\text{st}}$ and $(2k)^{\text{th}}$ terms, for $k \in \mathbb{N}$. As these differences are bounded below by a positive number, their infinite sum diverges to infinity.
  - The boundedness of P_{PF} is shown by establishing boundedness for (positive) integers s, by factoring. We can similarly show integer inverses by factoring, and by composition, any rational s. As the rationals are dense in the reals, we extend to $\mathbb{R}^+$ by continuity of exponentiation.

SERIES INTRODUCTIONS

- We will call Chen and Kennedy's series the Parity Fibonacci Series:
- $S_{PF} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \left(\frac{1}{5} + \frac{1}{7}\right) - \left(\frac{1}{6} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{11} + \frac{1}{13}\right) - \left(\frac{1}{10} + \frac{1}{12} + \frac{1}{14}\right) +$

$$\ldots = \sum_{i=1}^{\infty} \left(\sum_{m=0}^{F_i - 1} \frac{1}{2F_{i+1} - 1 + 2m} - \sum_{n=0}^{F_i - 1} \frac{1}{2F_{i+1} + 2n} \right)$$

- We can change the groupings in the parentheses to form the Consecutive Fibonacci Series:
- $S_{CF} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \left(\frac{1}{5} + \frac{1}{6}\right) - \left(\frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11}\right) - \left(\frac{1}{12} + \frac{1}{13} + \frac{1}{14}\right) +$

$$\ldots = \sum_{i=1}^{\infty} \left(\sum_{m=2F_i - 1}^{F_{i+2} - 2} \frac{1}{m} - \sum_{n=F_{i+2} - 1}^{2F_{i+1} - 2} \frac{1}{n} \right)$$

- Note that for both S_{PF} and S_{CF}, the number of terms in each pair of parentheses increases with the Fibonacci Sequence.
- By changing this underlying sequence to the sequence of Triangular Numbers, we can create two more series:
- The Consecutive Triangular Series is given by
- $S_{CT} = 1 - \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) - \left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{14}\right) - \left(\frac{1}{15} + \ldots + \frac{1}{20}\right) +$

$$\ldots = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\frac{1}{2}k(k+1)} \frac{1}{n + \frac{1}{3}(k-1)k(k+1)} - \frac{1}{n + \frac{1}{2}k(k+1) + \frac{1}{3}(k-1)k(k+1)} \right)$$

- The Parity Triangular Series is given by
- $S_{PT} = 1 - \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right) - \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{19}\right) - \left(\frac{1}{10} + \ldots + \frac{1}{20}\right) +$

$$\ldots = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\frac{1}{2}k(k+1)} \frac{1}{2n - 1 + \frac{1}{3}(k-1)k(k+1)} - \frac{1}{2n + \frac{1}{3}(k-1)k(k+1)} \right)$$

RESULTS

- S_{CF} diverges without bound
- S_{PF} diverges, but is bounded.
- S_{CT} and S_{PT} converge.
- Exponential generalization does not affect convergence or boundedness.

Note that convergent series are trivially bounded.

SERIES GENERALIZATIONS

For each series S, we can create a corresponding series P by raising every term to a positive power s.

$$P_{CF} = 1^s - \left(\frac{1}{2}\right)^s + \left(\frac{1}{3}\right)^s - \left(\frac{1}{4}\right)^s + \left(\frac{1}{5} + \frac{1}{6}\right)^s - \left(\frac{1}{7} + \frac{1}{8}\right)^s + \cdots = \sum_{i=1}^{\infty} \left(\sum_{m=2F_i - 1}^{F_{i+2} - 2} \frac{1}{m} - \sum_{n=F_{i+2} - 1}^{2F_{i+1} - 2} \frac{1}{n} \right)^s$$

$$P_{PF} = 1^s - \left(\frac{1}{2}\right)^s + \left(\frac{1}{3}\right)^s - \left(\frac{1}{4}\right)^s + \left(\frac{1}{5} + \frac{1}{7}\right)^s - \left(\frac{1}{6} + \frac{1}{8}\right)^s + \ldots = \sum_{i=1}^{\infty} \left(\sum_{m=0}^{F_i - 1} \left(\frac{1}{2F_{i+1} - 1 + 2m}\right)^s - \sum_{n=0}^{F_i - 1} \left(\frac{1}{2F_{i+1} + 2n}\right)^s \right)$$

$$P_{CT} = 1 - \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) - \left(\frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{14}\right) - \left(\frac{1}{15} + \ldots + \frac{1}{20}\right) + \ldots$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\frac{1}{2}k(k+1)} \frac{1}{n + \frac{1}{3}(k-1)k(k+1)} \right)^s - \left(\sum_{n=1}^{\frac{1}{2}k(k+1)} \frac{1}{n + \frac{1}{2}k(k+1) + \frac{1}{3}(k-1)k(k+1)} \right)^s$$

$$P_{PT} = 1^s - \left(\frac{1}{2}\right)^s + \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right)^s - \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right)^s + \left(\frac{1}{9} + \ldots + \frac{1}{19}\right)^s - \left(\frac{1}{10} + \ldots + \frac{1}{20}\right)^s + \cdots$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\frac{1}{2}k(k+1)} \frac{1}{2n - 1 + \frac{1}{3}(k-1)k(k+1)} \right)^s - \left(\sum_{n=1}^{\frac{1}{2}k(k+1)} \frac{1}{2n + \frac{1}{3}(k-1)k(k+1)} \right)^s$$

REFERENCES

Chen, H. and Kennedy, C. (2012). *Harmonic Series Meets Fibonacci Sequence*. The College Mathematics Journal, 43(3):237–243.

Oresme, N. (1961). *Questiones Super Geometriam Euclidis*. E. J. Brill. Question 5.27, Conclusion C.

Young, R. M. (1991). 75.9 Euler's Constant. The Mathematical Gazette, 75(472):187–190.

ACKNOWLEDGEMENTS

- I would like to thank Drs. Sooie-Hoe Loke and Dominic Klyve for their mentorship throughout this project.
- This work is part of the Central Convergence REU, funded by the NSF (DMS-2050692).
- Travel to JMM was funded by Pi Mu Epsilon.