

NOTES ON SIMULATION OF HAWKES PROCESSES:

Inhomogeneous Poisson process $\lambda \rightarrow \lambda(t)$:

$$\text{PDF (inter-event time } \Delta t) = \lambda(t) e^{-\int_t^{t+\Delta t} \lambda(t') dt'}. \quad (\text{see Chapter 22 anonymous note}).$$

↳ In order to generate Δt :

$$\begin{aligned} \text{accum}(\Delta t) &= \int_0^{\Delta t} \text{PDF}(\Delta t') dt' = u \in [0, 1]. \\ \int_0^{\Delta t} \underbrace{\lambda(t) e^{-\int_t^{t+\Delta t'} \lambda(t') dt'}}_{\frac{d}{d\Delta t'} \left[e^{-\int_t^{t+\Delta t'} \lambda(t') dt'} \right]} d\Delta t' &= u. \end{aligned}$$

Barrow

$$\downarrow$$
$$e^{-\int_t^{t+\Delta t} \lambda(t') dt'} - 1 = u$$

$$\log \downarrow \int_t^{t+\Delta t} \lambda(t') dt' = -\log(1-u) = -\log(-\bar{u}).$$

↳ Generate $\bar{u} \sim U[0, 1]$

↳ Solve last eq. for Δt .

Poisson Process as a Markov Process (exp. kernels)

$$\lambda(t) = \mu + \sum_{t_i < t} \alpha e^{-\beta(t-t_i)}$$

$$= \mu + \sum_{\substack{t_i < t_k \\ (t_k \text{ last event})}} \alpha e^{-\beta(t-t_k + t_k - t_i)} =$$

$$= \mu + e^{-\beta(t-t_k)} \underbrace{\sum_{t_i < t_k} \alpha e^{-\beta(t_k - t_i)}}_{\substack{\text{\textcolor{red}{\(\sum\)} goes for } t_i < t_k \\ \text{so it is } \lambda(t_k^-)}} \quad \checkmark \quad -\beta(t_k - t_i)$$

$$= \mu + e^{-\beta(t-t_k)} \underbrace{(\lambda(t_k) - \mu + \alpha)}_{\substack{\lambda(t_k) = \mu + \sum_{t_i < t_k} \alpha e^{-\beta(t_k - t_i)} \\ \Downarrow \\ \sum_{t_i < t_k} \alpha e^{-\beta(t_k - t_i)} = \lambda(t_k) - \mu + \alpha}} \quad \text{\textcolor{red}{\(\lambda(t_k^-)\)}}$$

→ Note that after t_k , we just need:

- * $\lambda(t_k) (= \lambda(t_k^-))$
 - * time elapsed $(t - t_k)$
- } (it doesn't have memory after t_k).

Simulation method

Recall that in order to find next t after t_k :

$$-\log(1-u) = \int_{t_k}^t \lambda(t') dt'.$$

That combined with $\lambda(t) = \mu + e^{-\beta(t-t_k)} (\lambda(t_k) + \alpha - \mu)$:

$$u = 1 - e^{-\mu(t-t_k)} e^{-(\lambda(t_k) + \alpha - \mu) \int_{t_k}^t e^{-\beta(t'-t_k)} dt'}$$
$$u = 1 - e^{-\mu(t-t_k)} e^{-\frac{1}{\beta} [e^{-\beta(t-t_k)} - 1]}$$
$$u = 1 - e^{-\mu(t-t_k)} e^{-\frac{1}{\beta} [e^{-\beta(t-t_k)} - 1]}$$
$$P_1(t_{k+1}^{(1)} > t) \cdot P(t_{k+1}^{(2)} > t)$$

Then we apply the composition method.

If we take $t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)})$; then $t_{k+1} \sim P(t_{k+1} > t)$.

$$\text{Prob}(t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)}) \leq t) = 1 - \text{Prob}(\min(t_{k+1}^{(1)}, t_{k+1}^{(2)}) > t)$$

$$= 1 - \text{Prob}(t_{k+1}^{(1)} > t) \cdot \text{Prob}(t_{k+1}^{(2)} > t)$$

which is the expression above.

la prob. de que el más pequeño sea mayor que t es que cada uno por separado sea más grande que t . (porque ambos tienen que ser mayores).

Then:

* Generate $t_{k+1}^{(1)} \sim P(t_{k+1}^{(1)} > t) = e^{-\mu(t-t_k)}$.
 ($P(t_{k+1}^{(1)} \leq t) = 1 - e^{-\mu(t-t_k)} = u_1 \in [0,1]$.) *equivalent!*
 $\bar{u}_1 \in [0,1] = 1 - \bar{u}_1$

This is done with:
 $u_1 = e^{-\mu(t_{k+1}^{(1)} - t_k)}$

$\log u_1 = -\mu(t_{k+1}^{(1)} - t_k) \Rightarrow t_{k+1}^{(1)} = t_k - \frac{1}{\mu} \log u_1$

* Generate $t_{k+1}^{(2)} \sim P(t_{k+1}^{(2)} > t) = e^{-(\lambda(t_k^-) + \alpha - \mu)\beta^{-1} (1 - e^{-\beta(t_{k+1}^{(2)} - t_k)})}$

$-\log u_2 = [\lambda(t_k^-) + \alpha - \mu] \beta^{-1} (1 - e^{-\beta(t_{k+1}^{(2)} - t_k)})$

$1 + \frac{\beta \cdot \log u_2}{\lambda(t_k^-) + \alpha - \mu} = e^{-\beta(t_{k+1}^{(2)} - t_k)}$

this should be a positive number!

$t_{k+1}^{(2)} = t_k - \beta^{-1} \log \left(1 + \frac{\beta \log u_2}{\lambda(t_k^-) + \alpha - \mu} \right)$

* Take $t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)})$.

* Calculate $\lambda(t_{k+1}) = \mu + e^{-\beta(t_{k+1} - t_k)} (\lambda(t_k^-) - \mu + \alpha)$

$= \mu + e^{-\beta(t_{k+1} - t_k)} (\lambda(t_k^+) - \mu)$

$\lambda(t_{k+1}^+) = \mu + \alpha + e^{-\beta(t_{k+1} - t_k)} (\lambda(t_k^+) - \mu)$

This is in complete agreement with what is done in the Notermanni paper.

extension to a BIVARIATE process:

* Markovian?

we could use β 's.

$$\lambda_1(t) = \mu_1 + \sum_{t_i^{(1)} < t} \alpha_{1 \rightarrow 1} e^{-\beta(t-t_i^{(1)})} + \sum_{t_i^{(2)} < t} \alpha_{2 \rightarrow 1} e^{-\beta(t-t_i^{(2)})}$$

$$\lambda_2(t) = \mu_2 + \sum_{t_i^{(1)} < t} \alpha_{1 \rightarrow 2} e^{-\beta(t-t_i^{(1)})} + \sum_{t_i^{(2)} < t} \alpha_{2 \rightarrow 2} e^{-\beta(t-t_i^{(2)})}$$

Last event was $t_k^{(e)}$

$$\lambda_2(t) = \mu_1 + \sum_{t_i^{(1)} < t} \alpha_{1 \rightarrow 1} e^{-\beta(t-t_k^{(e)} + t_k^{(e)} - t_i^{(1)})} + \sum_{t_i^{(2)} < t} \alpha_{2 \rightarrow 2} e^{-\beta(t-t_k^{(e)} + t_k^{(e)} - t_i^{(2)})}$$

$$= \mu_1 + e^{-\beta(t-t_k^{(e)})} \left\{ \sum_{t_i^{(1)} < t_k^{(e)}} \alpha_{1 \rightarrow 1} e^{-\beta(t_k^{(e)} - t_i^{(1)})} + \sum_{t_i^{(2)} < t_k^{(e)}} \alpha_{2 \rightarrow 1} e^{-\beta(t_k^{(e)} - t_i^{(2)})} \right\}$$

$$= \mu_1 + e^{-\beta(t-t_k^{(e)})} \left\{ \lambda_1(t_k^{(e)-}) - \mu_1 + \alpha_{e \rightarrow 1} \right\}$$

$$= \mu_1 + e^{-\beta(t-t_k^{(e)})} \left\{ \lambda_1(t_k^{(e)+}) - \mu_1 \right\}$$

... to be finished.

Extension to a BIVARIATE process:

*) According to Dassios & Zhao, $\Delta_{k+1} = t_{k+1} - t_k$ is taken as:

$$\Delta_{k+1} = \min \{ \Delta_{k+1}^{(1)}, \Delta_{k+1}^{(2)} \}, \text{ with } \Delta_{k+1}^{(j)} \text{ generated as usual:}$$

$$\Delta_{k+1}^{(j)} = \min \left(-\frac{1}{\mu_j} \log u_1^{(j)}, -\beta \log \left(1 + \underbrace{\frac{\beta \log u_2^{(j)}}{\lambda_j(t_k^+) - \mu_j}}_{g_j} \right) \right)$$

$$\left[\text{unless } g_j \leq 0 \text{ (then take first argument).} \right] \overset{g_j}{\swarrow}$$

And check what process (ℓ) corresponds to Δ_{k+1} ($= \Delta_{k+1}^{(\ell)}$).

*) Update time for next event: $t_k \rightarrow t_{k+1} = t_k + \Delta_{k+1}^{(\ell)}$.

*) Update $\lambda_j(t_{k+1}^+) = (\lambda_j(t_k^+) - \mu_j) e^{-\beta(t_{k+1} - t_k)} + \mu_j + d_{\ell \rightarrow j}$.

with $j = 1, 2$.