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**PHYSICAL TECHNOLOGY:
RESEARCH AND APPLICATIONS**

MASTER'S THESIS

**Exploring the relationship between Hawkes processes
and self-organized criticality in living systems**

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Line of research: Modelling of complex systems and their interdisciplinary applications

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Abstract

Chapter 1

Introduction

1.1 Point processes

Within the large framework of complex systems, stochastic processes lend us a hand to decypher properties of living systems, bridging randomness with structured behaviour. This processes are used to model the dynamics of systems which evolve randomly in time. This is why they are ideal for describing natural phenomena such as the spread of diseases [1], social networks [2] or ecological systems [3]. Mathematically, a stochastic process is a collection of random variables [4], generally ordered in time $\{X_t\}_{t \in T}$, where t is the time and X_t is the system state at time t . T is the time index set, which can be discrete or continuous, in this work we will focus on the discrete case because we are interested in the study of point (Hawkes) processes for modeling neurons.

Point processes are a type of stochastic process that describe the occurrence of events in time or space. We will be interested in time point processes because we are going to model the spiking activity of neurons. For our purposes, they will be characterized by two parameters, the time of occurrence of the events t_k and the intensity or rate of occurrence of these events λ . This rate tell us how likely is that an event occurs at time t given the history of the process (probability density function, PDF) as pictured in Figure 1.1.

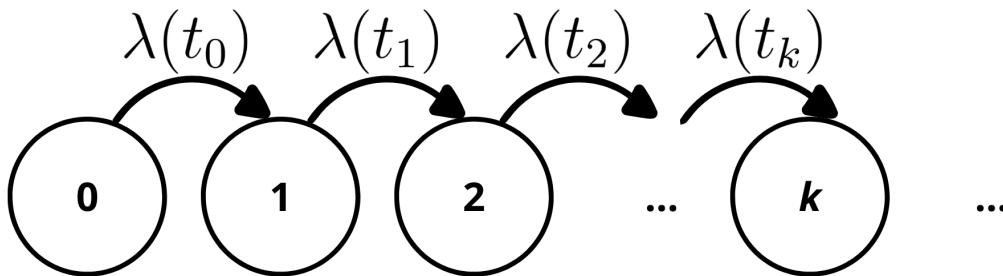


Figure 1.1. Representation of a point process. The intensity function $\lambda(t)$ is a time-dependent function.

In general, the rate is a function of the history of the process, which makes the process non-Markovian, but in our case, it will be a Markovian process, which means that the rate depends only on the last event that occurred as we will see. An example of a Markovian point process is the Poisson process, which is a simple and one of the most studied point processes because they are present in many everyday situations such as the arrival of customers at a store, occurrence of defects on a Production line. They are also present in some physics phenomena, for instance, the decay of radioactive particles or the arrival of photons at a detector. These processes are characterized by a rate of occurrence of events λ . The dynamics of these processes are described by the Poisson distribution which is the probability distribution of a random variable N such that the probability that $N = n$ is:

$$P(N = n) = \frac{\lambda^n}{n!} e^{-\lambda}. \quad (1.1)$$

Furthermore, the mean value and the variance of the distribution are also equal to λ . Poisson processes can be homogeneous or inhomogeneous, depending on whether the rate is constant or time-dependent. In Figure 1.2 we can see an example of a homogeneous Poisson process.

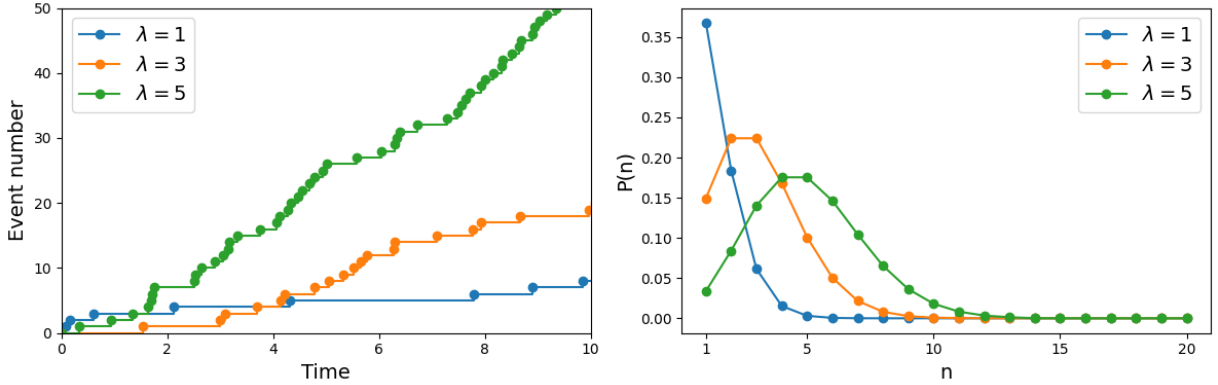


Figure 1.2. Left: event number in time for different rates. Right: Probability of having a certain number of events for different rates.

1.2 Hawkes processes

On the other hand if we consider a non-homogeneous Poisson process, the rate is a function of time, $\lambda(t)$, which is the case of the Hawkes process. The rate can be written in several ways [5, 6, 7, 8]. We will use the expression from [5]:

$$\lambda(t|t_1, \dots, t_k) = \mu + n \sum_{i=1}^k \phi(t - t_i), \quad (1.2)$$

where μ is the background rate of a homogeneous Poisson process, n is a parameter that controls the strength the self-excitation, and $\phi(t)$ is the kernel function that describes the influence of the past events on the rate of occurrence of the events. The kernel function is a non-negative and monotonically non-increasing function that integrates to 1. Typical choices for the kernel function are the exponential or the power-law functions. In this work we will focus on the exponential kernel. From Eq 1.2 we can see that the rate depends on the history of the process, making it non-Markovian in general, but

with an exponential kernel, the process becomes Markovian. The kernel function can be written as: $\phi(t) = \sum_{t_i < t} \alpha e^{-\beta(t-t_i)}$ so the rate becomes:

$$\begin{aligned}
\lambda(t) &= \mu + \sum_{t_i < t} \alpha e^{-\beta(t-t_i)} \\
&= \mu + \sum_{\substack{t_i < t_k \\ t_k: \text{last event}}} \alpha e^{-\beta(t-t_k+t_k-t_i)} \\
&= \mu + e^{-\beta(t-t_k)} \underbrace{\sum_{t_i < t_k} \alpha e^{-\beta(t_k-t_i)}}_{\lambda(t_k)} \\
&= \mu + e^{-\beta(t-t_k)} (\lambda(t_k) - \mu + n).
\end{aligned} \tag{1.3}$$

Where we have used the following expression for the rate of the Hawkes process at time t_k :

$$\lambda(t_k) = \mu + \sum_{t_i < t_k} \alpha e^{-\beta(t_k-t_i)} \Rightarrow \sum_{t_i < t_k} \alpha e^{-\beta(t_k-t_i)} = \lambda(t_k) - \mu + \alpha \tag{1.4}$$

Despite being a Markovian process, it is still an inhomogeneous Poisson process because the rate is not constant. In addition, it is a self-exciting process, which means that the occurrence of an event increases the probability of the occurrence of another event. This is why it is used to model the spiking activity of neurons, where the occurrence of a spike increases the probability of the occurrence of another spike. This self-excitation will enable the appearance of bursts of activity that we will measure. The parameters chosen for the kernel function will be $\alpha = \beta = 1$ and we will vary the background rate μ from values much smaller than 1 to values greater than 1. In Figures 1.3 and 1.4 we can see typical diagrams of Hawkes processes with these parameters.

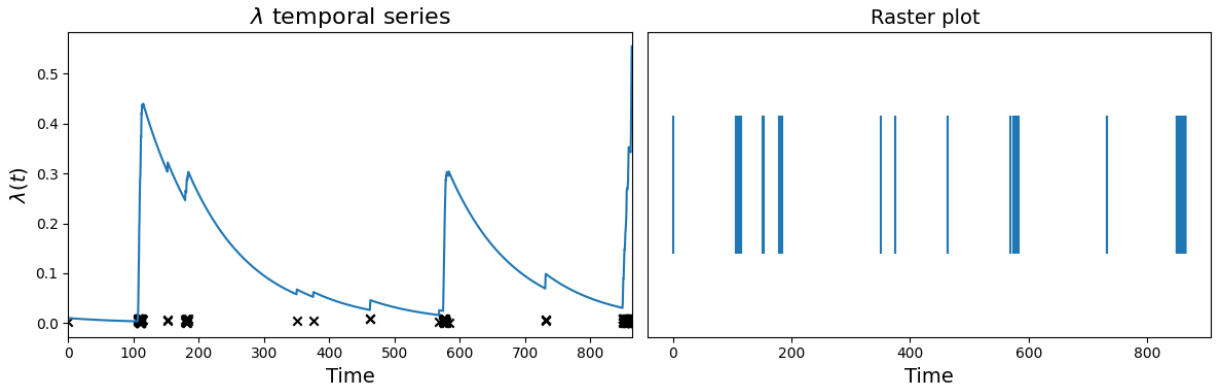


Figure 1.3. On the left, a temporal series of $K = 150$ events of a Hawkes process with $\mu = 0.01$, on the right, a raster plot of the same process.

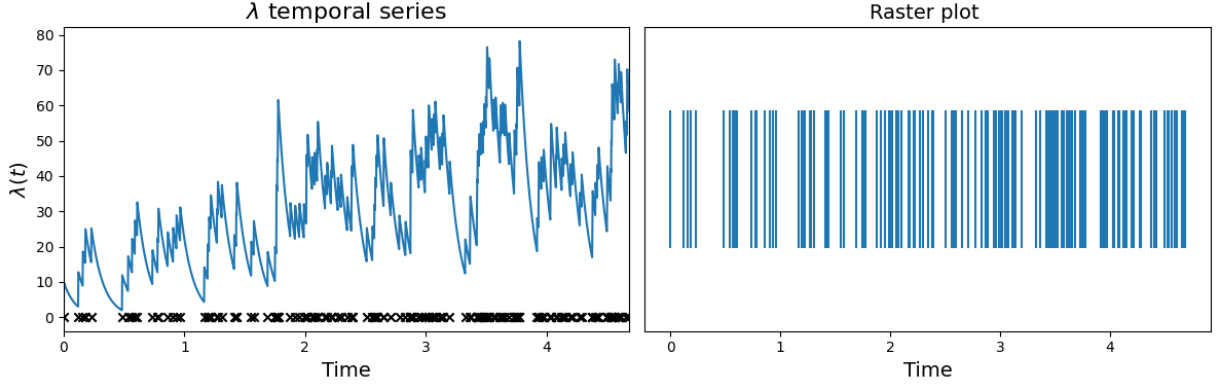


Figure 1.4. On the left, a temporal series of $K = 150$ events of a Hawkes process with $\mu = 0.01$, on the right, a raster plot of the same process.

As shown in Figure 1.3, when the background rate is smaller than 1, events are less likely to occur, but when they do, they tend to form avalanches of activity thanks to the self-excitation. On the other hand, when the background rate is greater than 1, events occur more frequently, forming avalanches of activity more frequently and longer, as shown in Figure 1.4. If we ignore the time of occurrence of the events and we focus only on the structure of λ and therefore of the events, we can see that the process with $\mu = 0.01$ has a bursty structure, while the process with $\mu = 10$ has a more regular structure. This phenomenon is exposed in Figures 1.5 and 1.6.

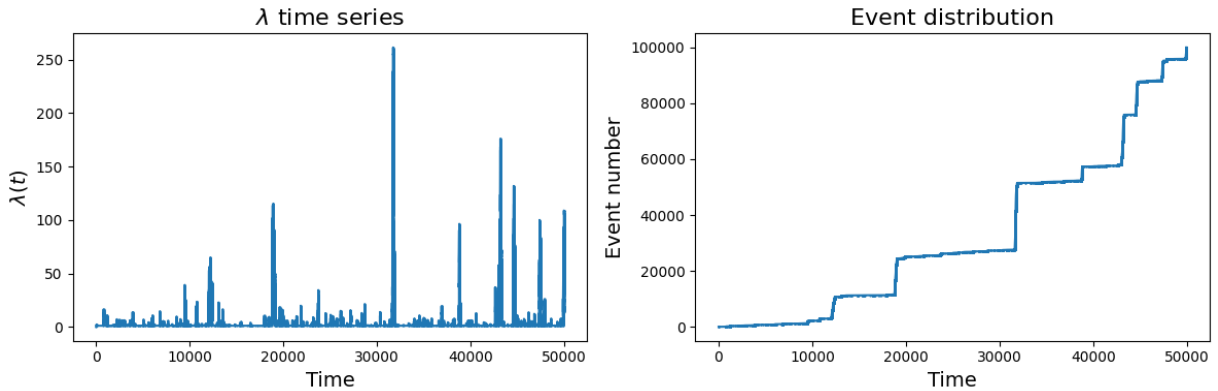


Figure 1.5. First, a temporal series of $K = 10^5$ events of a Hawkes process with $\mu = 0.01$, on the right, the event distribution.

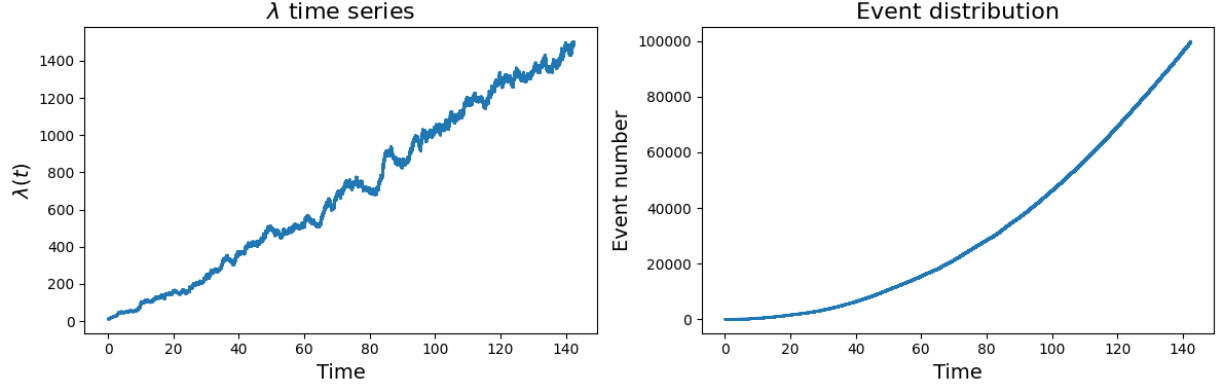


Figure 1.6. First, a temporal series of $K = 10^5$ events of a Hawkes process with $\mu = 10$, on the right, the event distribution.

In most cases, the motivation of study of point processes is counting the events, but in our case we also are interested in the time of occurrence of the events which will let us define bursts or avalanches of activity that we will use to describe the dynamics of the system. Additionally,

Unless otherwise stated, $n = 1$ HABLAR DE CRITICIDAD A PARTIR DE AQUÍ, EJEMPLOS DE CRITICAL BRANCHING PROCESSES, ETC.

Chapter 2

Objectives

Ordenar los objetivos una vez escrito el trabajo para que coincidan con como se presenta.

The main objectives of this Master's thesis are:

- To understand what Hawkes processes are, where we can find them, how to generate them computationally and relate them with neuroscience.
- To understand the importance of time binning and reproduce the results of the original paper [5] and compare them with the results obtained in this work.
- ¿Criticality?
- To study the behaviour of a self-exciting process with $n = 2$ and compare it with the case $n = 1$.
- To study the behaviour of an inhibitory and excitatory neuron coupled.

Chapter 3

Methodology

In the following sections, the methodology for data generation, management and analysis will be presented. To address these issues, we will use Python [9, 10] due to its versatility and the wide range of libraries available. The two used will be NumPy [11] and Matplotlib [12] for the visualization.

3.1 Time series factory

The first step is the generation of time series, there are two ways to do this: the slow one and the fast one. The first one is discretizing the time and calculating the rate at each time step according with Eq 1.2, then accept or reject the event if $p < \lambda \cdot dt$ for a random number $p \in \mathcal{U}[0, 1]$. This method works for small time series, but for large ones is not efficient because the summation of the kernel function has to be done at each time step. The pseudo-code for this method is presented in Algorithm 1.

Algorithm 1 Slow method to generate Hawkes processes.

Require: t_{max} , $n_{intervals}$, $\lambda(t_0) = \mu$, p

$$dt \leftarrow \frac{t_{max}}{n_{intervals}}$$

for $i = 0$ to $n_{intervals}$ **do**

$$\lambda(t_k) \leftarrow \mu + n \sum_{t_i < t_k} \phi(t_k - t_i)$$

$$\triangleright t_i = i \cdot dt$$

if $\lambda(t_k) \cdot dt > p$ **then**

$$t_{event} \leftarrow t_k$$

end if

end for

The fast method takes advantage of Monte Carlo methods [13] to generate the time series. The idea of this procedure consists in computing the inter-event time instead of the time of the event. To get to the algorithm, we start from the following expression:

$$PDF(\text{inter-event time} = \Delta t) = \lambda(t + \Delta t) e^{-\int_t^{t+\Delta t} \lambda(t') dt'} \quad (3.1)$$

To demonstrate this, we have to take a look at the Figure 3.1 and recall that λ is a probability per unit of time.

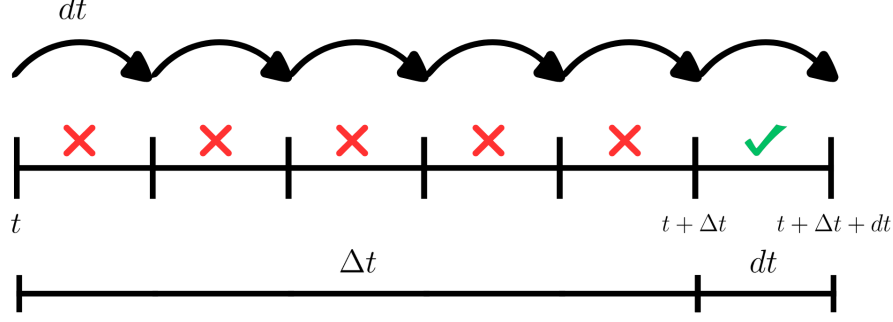


Figure 3.1. Diagram to calculate the cumulative probability of the inter-event time.

The probability per unit of time of having an event in the interval $[t + \Delta t, t + \Delta t + dt]$ is the probability of no events in the interval $[t, t + \Delta t]$ times the probability of happening in the interval $[t + \Delta t, t + \Delta t + dt]$. Putting words into mathematics, we have that the probability of not having an event in the interval $[t, t + \Delta t]$ is:

$$\begin{aligned}
 P(\text{event} \in [t + \Delta t, t + \Delta t + dt]) &= (1 - \lambda(0) \cdot dt) (1 - \lambda(dt) \cdot dt) (1 - \lambda(2dt) \cdot dt) \dots \\
 &= \prod_{k=0}^{\infty} \underbrace{(1 - \lambda(kdt) \cdot dt)}_{e^{\ln(1 - \lambda(kdt)dt)}} = e^{\sum_{k=0}^{\infty} \ln(1 - \lambda(kdt)dt)} = \dots \quad \text{Using } \ln(1 - \varepsilon) \approx -\varepsilon \\
 &= e^{-\sum_{k=0}^{\infty} \lambda(kdt)dt} \underset{dt \rightarrow 0}{=} e^{-\int_t^{t+\Delta t} \lambda(t')dt'}.
 \end{aligned} \tag{3.2}$$

Knowing that the probability of having an event in the interval $[t + \Delta t, t + \Delta t + dt]$ is $\lambda(t + \Delta t)dt$, we have:

$$P(\text{event} \in [t + \Delta t, t + \Delta t + dt])d\mathcal{t} = \lambda(t + \Delta t)dt \cdot e^{-\int_t^{t+\Delta t} \lambda(t')dt'} PDF(\text{inter-event time} = \Delta t)d\mathcal{t}. \tag{3.3}$$

Having that we can calculate the inter-event time following the next steps.

$$PDF(\text{inter-event time} = \Delta, t) = \lambda(t + \Delta t) \underbrace{e^{-\int_t^{t+\Delta t} \lambda(t')dt'}}_{\text{No events during } (t, t+\Delta t)}$$

In order to generate Δt , we will use the inverse transform method [14], therefore we have to calculate the cumulative probability of the inter-event time:

$$\begin{aligned}
 \text{accum}(\Delta t) &= \int_0^{\Delta t} PDF(\Delta t') d\Delta t' = u \in \mathcal{U}[0, 1] \\
 &= \int_0^{\Delta t} \underbrace{\lambda(t + \Delta t') e^{-\int_t^{t+\Delta t'} \lambda(t')dt'}}_{-\frac{d}{d\Delta t'} \left[e^{-\int_t^{t+\Delta t} \lambda(t')dt'} \right]} d\Delta t' = u \quad \text{Using Barrow rule} \\
 &= e^{-\int_t^{t+\Delta t} \lambda(t')dt'} \Big|_0^{\Delta t} = 1 - e^{-\int_t^{t+\Delta t} \lambda(t')dt'} = u \quad \text{Taking logarithms} \\
 &\int_t^{t+\Delta t} \lambda(t')dt' = -\ln(1 - u) = \ln(\bar{u})
 \end{aligned} \tag{3.4}$$

To compute the inter-event time, we have to generate $\bar{u} \sim$ and solve the equation. Having in mind this relation and using Eq 1.4 we have: EN EL LA ECUACIÓN REFERENCIADA Y LA SEGUNDA EXPONENCIAL SE PUEDE PONER $\lambda(t_k^-)$ en lugar de $\lambda(t_k)$? .

$$\begin{aligned}
u &= 1 - e^{-\mu(t-t_k)} e^{-\underbrace{(\lambda(t_k)+\alpha-\mu) \cdot \int_{t_k}^t e^{-\beta(t'-t_k)} dt'}_{-\frac{1}{\beta}[e^{-\beta(t-t_k)}-1]}} \\
u &= 1 - \underbrace{e^{-\mu(t-t_k)}}_{P(t_{k+1}^{(1)} > t)} e^{-\underbrace{[(\lambda(t_k)+\alpha-\mu)\beta^{-1}(1-e^{-\beta(t-t_k)})]}_{P(t_{k+1}^{(2)} > t)}}
\end{aligned} \tag{3.5}$$

Then we apply the composition method [7]. If we take $t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)})$; then $t_{k+1} \sim P(t_{k+1} > t)$, hence:

$$\begin{aligned}
\text{Prob}(t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)}) \leq t) &= 1 - \text{Prob}(\min(t_{k+1}^{(1)}, t_{k+1}^{(2)}) > t) \\
&= 1 - \text{Prob}(t_{k+1}^{(1)} > t) \cdot \text{Prob}(t_{k+1}^{(2)} > t)
\end{aligned} \tag{3.6}$$

where we have used that the probability that the smaller is greater than t is that each separately is greater than t because both have to be greater than t . As we can see the expressions in Eqs 3.5 and 3.6 are the same, so we can use the composition method to generate the inter-event time. Then, the algorithm to generate the inter-event time is:

1. Generate $t_{k+1}^{(1)} \sim P(t_{k+1}^{(1)} > t) = e^{-\mu(t-t_k)}$ using

$$P(t_{k+1}^{(1)} \leq t) = 1 - \underbrace{e^{-\mu(t-t_k)}}_{\bar{u}_1 \in \mathcal{U}[0,1] \Rightarrow u_1 - \bar{u}_1} = u_1 \in \mathcal{U}[0,1]$$

This is done by generating $u_1 \in \mathcal{U}[0,1]$ and solving the equation.

$$\begin{aligned}
u_1 &= 1 - e^{-\mu(t_{k+1}^{(1)} - t_k)} \\
\ln(u_1) &= -\mu(t_{k+1}^{(1)} - t_k) \Rightarrow t_{k+1}^{(1)} = t_k - \frac{\ln(u_1)}{\mu}
\end{aligned} \tag{3.7}$$

2. Generate $t_{k+1}^{(2)} \sim P(t_{k+1}^{(2)} > t) = e^{-\left((\lambda(t_k)+\alpha-\mu)\beta^{-1}\left(1-e^{-\beta(t_{k+1}^{(2)}-t_k)}\right)\right)}$ in a similar way as before:

$$\begin{aligned}
u_2 &= 1 - e^{-\left((\lambda(t_k)+\alpha-\mu)\beta^{-1}\left(1-e^{-\beta(t_{k+1}^{(2)}-t_k)}\right)\right)} \\
-\ln(u_2) &= \left((\lambda(t_k) + \alpha - \mu) \beta^{-1} \left(1 - e^{-\beta(t_{k+1}^{(2)} - t_k)}\right)\right) \\
1 + \frac{\beta \ln u_2}{\lambda(t_k) + \alpha - \mu} &= e^{-\beta(t_{k+1}^{(2)} - t_k)} \\
t_{k+1}^{(2)} &= t_k - \beta^{-1} \ln \underbrace{\left(1 + \frac{\beta \ln u_2}{\lambda(t_k) + \alpha - \mu}\right)}_{\text{This number must be positive}}
\end{aligned} \tag{3.8}$$

3. Choose $t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)})$

4. Calculate the rate at t_{k+1} using Eq 1.3 and go back to step 1.

With this method, we can generate time series efficiently. The pseudo-code for this method is presented in Algorithm 2.

Algorithm 2 Algorithm to generate K Hawkes events.

Require: $\alpha, \beta, \lambda(t_0) = \mu, K$
for $k = 0$ to K **do**
 $u_1, u_2 \leftarrow \mathcal{U}[0, 1]$
 $t_{k+1}^{(1)} \leftarrow \frac{\ln(u_1)}{\mu}$
 $t_{k+1}^{(2)} \leftarrow \beta^{-1} \ln \left(1 + \frac{\beta \ln u_2}{\lambda(t_k) + \alpha - \mu} \right)$
 $t_{k+1} \leftarrow \min(t_{k+1}^{(1)}, t_{k+1}^{(2)})$
 $\lambda(t_{k+1}) \leftarrow \mu + e^{-\beta(t_{k+1} - t_k)} (\lambda(t_k) - \mu + n)$
end for

3.2 When physics and cooking merge

Once we have a method to generate time series, we can start analyzing them. We should establish a control parameter to distinguish between different regimes of the process. In this case, we will define resolution parameter $\Delta > 0$ as a time interval which will help us to identify clusters of activities. Our time series will have K events that occur in times $\{t_1, \dots, t_K\}$. Considering this, a cluster of events starts at time t_i and ends at time t_j if $t_j - t_i \leq \Delta$. The number of events in the cluster (cluster size) is the number of events in the interval $[t_i, t_j]$ and its duration is $t_j - t_i$. The extreme cases are when Δ is smaller than the minimum inter-event time, in this case, each event is a cluster of size 1 and duration 0. On the other hand when Δ is greater than the largest inter-event time, all the events are in the same cluster of size K and duration $t_K - t_1$. Between these two extremes, we will have different regimes of the process. To identify these regimes, we need a phase diagram, in our case, it will be a percolation diagram where we will plot the percolation strength P_∞ as a function of the resolution parameter Δ . The percolation strength is defined as the fraction of events that are in the largest cluster over the total number of events. Three different set of parameters will be used to generate the time series in order to compare them. The parameters α and β will be fixed to 1 in all cases, the other parameters are shown in Table 3.1.

Table 3.1. Configuration of the parameters for the simulations

Configuration	μ	n
First	1	0
Second	10^{-4}	1
Third	10^2	1

The percolation diagram will be generated by generating 1000 time series for each configuration and calculating the percolation strength for each one because in general they are not stationary processes as we can observe in Figure 3.2.

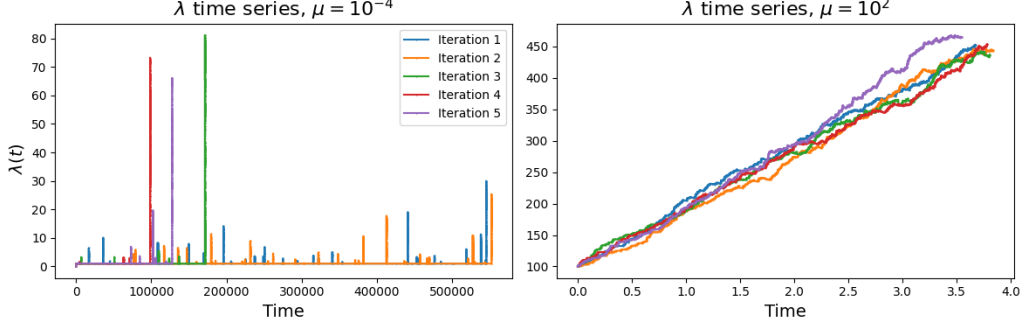


Figure 3.2. Five a temporal series of $K = 10^5$ events of Hawkes processes with $\mu = 10^{-4}$ on the left side and $\mu = 10^2$ on the right one.

Beginning with the first configuration, we have an homogeneous Poisson process which we know that its interevent time will be distributed randomly with a probability of having an inter-event time x_i given by $P(x_i) = \mu e^{-\mu x_i}$. Consequently, two consecutive events will be a part of a cluster fixing the resolution parameter to Δ with a probability of

$$P(x_i \leq \Delta) = 1 - e^{-\mu\Delta} \quad \forall i. \quad (3.9)$$

This represents the probability in a homogeneous 1D percolation model [15], where we can identify a non percolant phase and a percolant phase separated by the critical point Δ^* . We can calculate this parameter if we know the maximum inter-event time of the time series. Let us assume that our time series has K events, therefore, it will percolate if the condition we have just established is satisfied. We can calculate this threshold as the average of the maximum inter-event time in K samples from the inter-event time distribution solving the following equation:

$$\begin{aligned} K \int_{\Delta^*}^{\infty} P(x) dx &= 1 \\ K \int_{\Delta^*}^{\infty} \mu e^{-\mu x} dx &= 1 \\ -K [e^{-\mu x}]_{\Delta^*}^{\infty} &= K \left[e^{-\mu\Delta^*} - e^{-\mu\infty} \right] = 1 \\ K e^{-\mu\Delta^*} &= 1 \\ \Delta^*(K) &= -\frac{\ln(K)}{\mu} \end{aligned} \quad (3.10)$$

Chapter 4

Results

This section provides the main results of the investigation. First, the results reproduced from the original paper [5] are presented. Then, the results of the analysis with $n = 2$ are shown. Finally, we have studied the behaviour of an inhibitory and excitatory neuron coupled.

4.1 Results from the original paper

The first result is the percolation phase diagram is shown in Figure 4.1. It displays the percolation strength P_∞ versus the resolution parameter Δ .

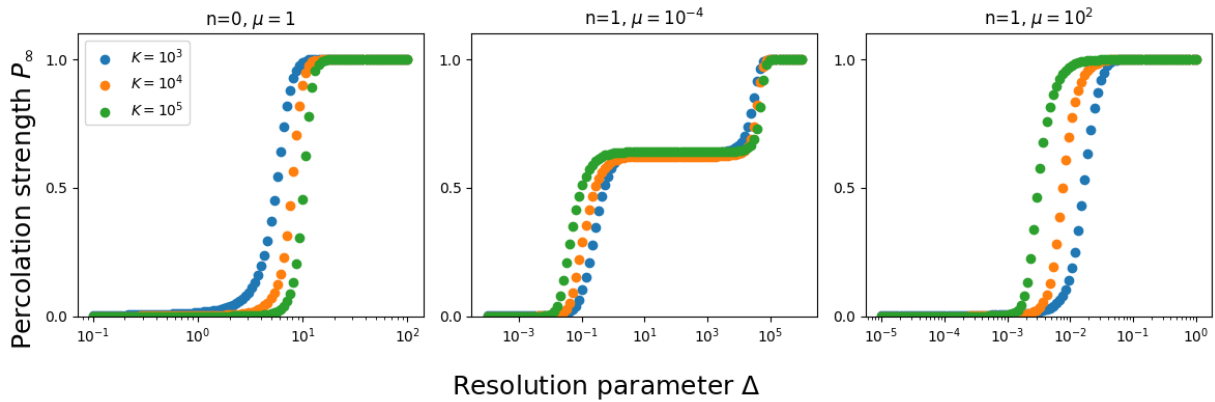


Figure 4.1. Percolation phase diagrams for different event number K taking average values of $R = 1000$ realizations.

The first plot configuration is a Markovian ($n = 0$) Poisson process with rate μ . This is the simplest case, where the inter-event time $x = t_i - t_{i-1}$ follows an exponential distribution $P(x_i) = \mu e^{-\mu x_i}$. The other two plots are Hawkes processes for $\mu \ll 1$ and $\mu \gg 1$ that are also Markovian as we have chosen an exponential kernel (REFERENCIAR AQUÍ A LA PARTE EN LA QUE SE EXPLICA EN METODOLOGÍA.) . In one hand, a double transition is observed when $\mu = 10^{-4}$, in the other hand, a single transition occurs when $\mu = 10^2$.

Once we have the phase diagram, we can study avalanche statistics. Given a resolution parameter Δ , we can spot clusters or avalanches of activity. A cluster starts when a neuron fires and ends if

the neuron does not fire for a time greater than Δ . We define the size of a cluster as the number of spikes it contains and the duration as the time between the first and last spike. We have studied the avalanches for $K = 10^5$ events and $R = 1000$ realizations to obtain the average values since the process is highly not stationary. We will study the size and duration of the avalanches for the three different regions of the phase diagram for $\mu = 10^{-4}$ and the two regions of the phase diagram for $\mu = 10^2$. These regions are separated by two thresholds, a pseudocritical threshold Δ_1^* and the threshold of the second transition at Δ_2^* . We can compute these with the following formulas [5]:

$$\Delta_1^* \simeq \frac{\log(K)}{\langle \lambda \rangle} = \frac{\log(K)}{\mu + \sqrt{2\mu K}} \quad (4.1)$$

$$\Delta_2^* = \frac{\log(K)}{\mu} \quad (4.2)$$

Once we have the thresholds, we can study the avalanches for the different regions of the diagram. The results are shown in Figure 4.2.

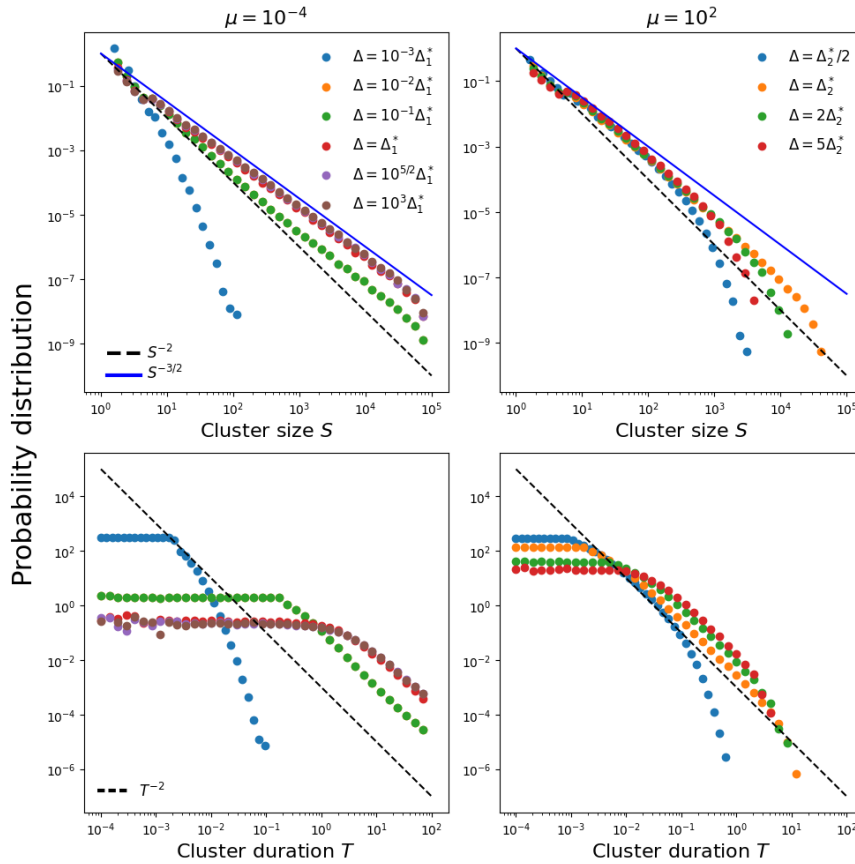


Figure 4.2. Avalanche statistics for a self-exciting Hawkes process with $n = 1$ for $K = 10^5$ events averaged over $R = 1000$ realizations.

For $\mu = 10^{-4}$, the results show a power-law distribution for the size and duration of the avalanches. In the case of duration, the exponent is $\tau = 2$ and for the size, we can notice a transition of the exponent from $\alpha = 2$ to $\alpha = 3/2$ as we increase the resolution parameter Δ . The first exponent corresponds to the universality class of 1D percolation, whereas the second is compatible with the universality class of mean-field branching process. However, if $\Delta \ll \Delta_1^*$, the behaviour is subcritical for the size and duration of the avalanches.

For $\mu = 10^2$, the result shows another power-law distribution for both size and duration of the avalanches unless $\Delta \ll \Delta_2^*$, where the behaviour is subcritical. In this case, the exponents are $\alpha = \tau = 2$ corresponding to the universality class of 1D percolation.

HABLAR AQUÍ DE LA INFLUENCIA DEL MENOR NÚMERO DE EVENTOS, SE SIGUE PRODUCIENDO LA TRANSICIÓN, PERO PARA OTROS VALORES DE DELTA

4.2 Results for $n=2$

In the article, the authors have studied a process which is critical itself because the parameter n is fixed to $n = 1$. We have studied the case $n = 2$ to see if the process is still critical. In the Figure 4.3 two time series for $n = 1$ and $n = 2$ are shown.

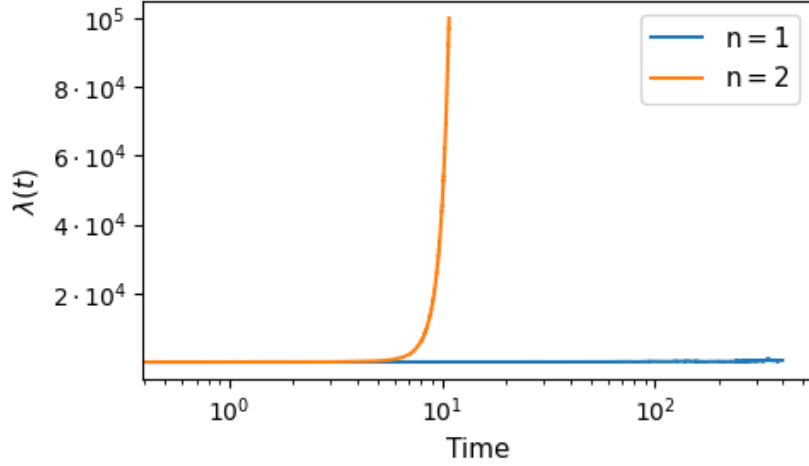


Figure 4.3. Time series for $n = 1$ and $n = 2$.

Similarly to the previous section, first we obtain the phase diagram in order to observe the transitions. In this case, Eqs 4.1-4.2 are not valid. Therefore, we will obtain this parameter graphically from the phase diagrams shown in Figure 4.4.

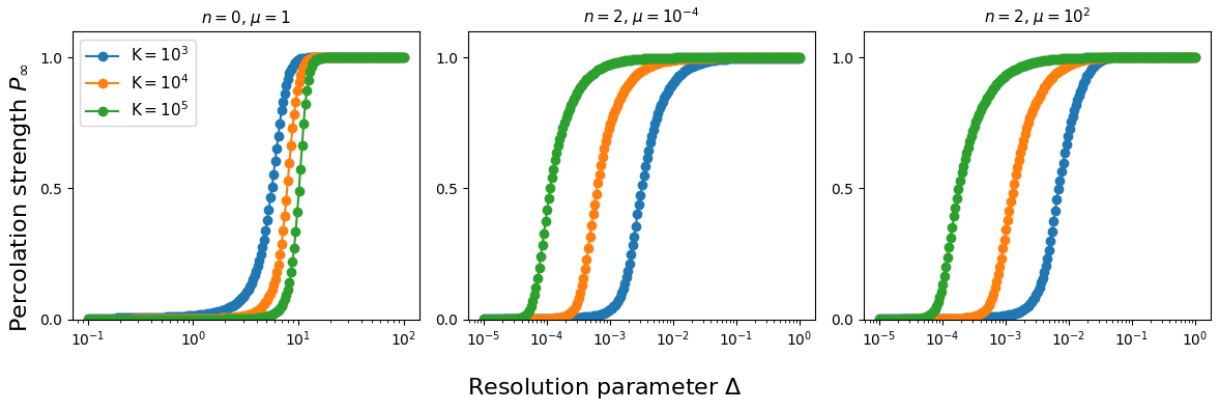


Figure 4.4. Percolation phase diagrams for a Hawkes process with $n = 2$.

As we can see, now we have a single transition for $\mu = 10^{-4}$ and $\mu = 10^2$ corresponding to 1D

percolation, consequently, the exponents for the size and duration should be $\alpha = \tau = 2$. In a similar way, we have studied the avalanches for $K = 10^5$ events and $R = 1000$ realizations to obtain the average values. The statistics of the avalanches are shown in Figure 4.5.

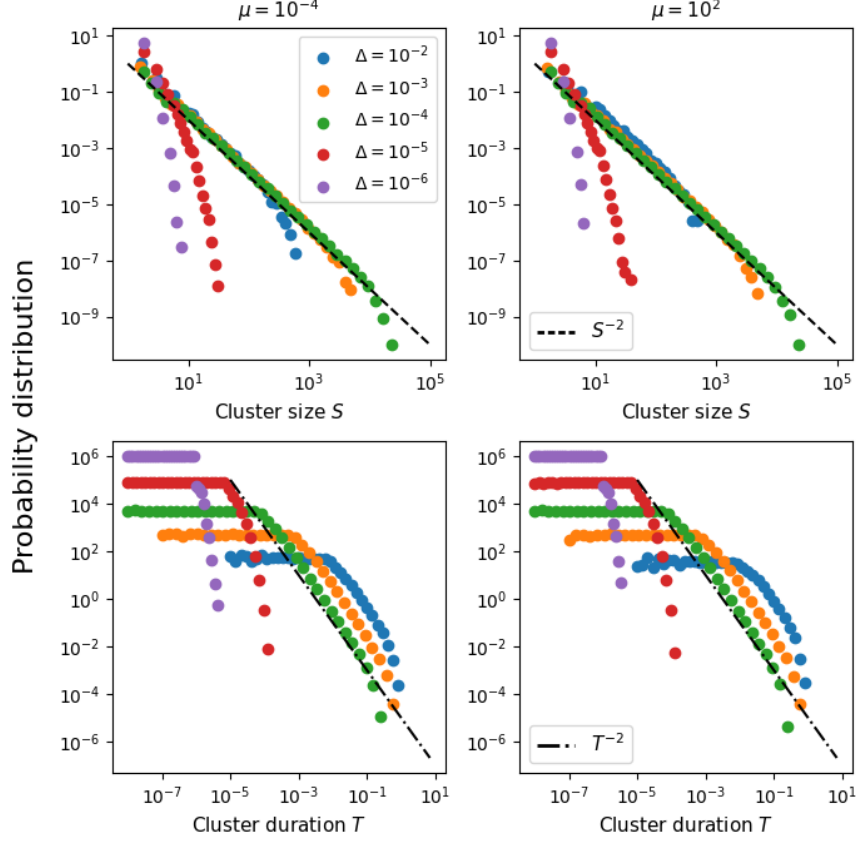


Figure 4.5. Avalanche statistics for a self-exciting Hawkes process with $n = 2$ for $K = 10^5$ events averaged over $R = 1000$ realizations.

4.3 Inhibitory and excitatory neurons coupled

Chapter 5

Conclusions

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Anexo

REVISAR LOS CÓDIGOS PARA QUE ESTÉN ACTUALIZADOS

Script 5.1. Script with the main functions.

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 def algorithm(rate, mu, n):
5     """
6     Algorithm that computes interevent times and Hawkes intensity for a self-exciting
7     process
8
9     #Output: rate x_k, x_k
10    """
11    # 1st step
12    u1 = np.random.uniform()
13    if mu == 0:
14        F1 = np.inf
15    else:
16        F1 = -np.log(u1) / mu
17
18    # 2nd step
19    u2 = np.random.uniform()
20    if (rate - mu) == 0:
21        G2 = 0
22    else:
23        G2 = 1 + np.log(u2) / (rate - mu)
24
25    # 3rd step
26    if G2 <= 0:
27        F2 = np.inf
28    else:
29        F2 = -np.log(G2)
30
31    # 4th step
32    xk = min(F1, F2)
33
34    # 5th step
35    rateTk = (rate - mu) * np.exp(-xk) + n + mu
36    return rateTk, xk
37
38 def generate_series(K, n, mu):
39     """
40     Generates temporal series for K Hawkes processes
41
```

```

42  ##Inputs:
43  K: Number of events
44  n: Strength of the Hawkes process
45  mu: Background intensity
46
47  ##Output:
48  times: time series the events
49  rate: time series for the intensity
50  """
51  times_between_events = [0]
52  rate = [mu]
53  for _ in range(K):
54      rateTk, xk = algorithm(rate[-1], mu, n)
55      rate.append(rateTk)
56      times_between_events.append(xk)
57  times = np.cumsum(times_between_events)
58  return times, rate
59
60 def identify_clusters(times, delta):
61     """
62     Identifies clusters in a temporal series given a resolution parameter delta
63
64     ## Inputs:
65     times: temporal series
66     delta: resolution parameter
67
68     ## Output:
69     clusters: list of clusters
70     """
71     clusters = []
72     current_cluster = []
73     for i in range(len(times) - 1):
74         if times[i + 1] - times[i] <= delta:
75             if not current_cluster:
76                 current_cluster.append(times[i])
77                 current_cluster.append(times[i + 1])
78             else:
79                 if current_cluster:
80                     clusters.append(current_cluster)
81                     current_cluster = []
82     return clusters
83
84 def generate_series_perc(K, n, mu):
85     """
86     Generates temporal series for K Hawkes processes
87
88     ##Inputs:
89     K: Number of events
90     n: Strength of the Hawkes process
91     mu: Background intensity
92
93     ##Output:
94     times_between_events: time series the interevent times
95     times: time series the events
96     rate: time series for the intensity
97     """
98     times_between_events = [0]
99     rate = [mu]
100    for _ in range(K):
101        rateTk, xk = algorithm(rate[-1], mu, n)
102        rate.append(rateTk)
103        times_between_events.append(xk)

```

```

104     times = np.cumsum(times_between_events)
105     return times_between_events, times, rate
106
107 def calculate_percolation_strength(times_between_events, deltas):
108     """
109     Calculate the percolation strength for a given set of deltas (resolution parameters)
110
111     ## Inputs:
112     times_between_events: time series of interevent times
113     deltas: list of resolution parameters
114
115     ## Output:
116     percolation_strengths: list of percolation strengths
117     """
118
119     percolation_strengths = []
120
121     for delta in deltas:
122         cluster_sizes = []
123         current_cluster_size = 1 # The first event is always a cluster
124
125         for time in times_between_events:
126             if time < delta:
127                 current_cluster_size += 1
128             else:
129                 if current_cluster_size > 1: # Only consider clusters with more than one
event
130                     cluster_sizes.append(current_cluster_size)
131                     current_cluster_size = 1 # The next event is always a cluster
132
133                 if current_cluster_size > 1: # Consider the last cluster if it ends at the last
event
134                     cluster_sizes.append(current_cluster_size)
135
136                 if len(cluster_sizes) != 0: # Check if cluster_sizes is not empty to avoid
errors
137                     max_cluster_size = max(cluster_sizes)
138                 else:
139                     max_cluster_size = 0
140
141                 percolation_strengths.append(max_cluster_size / len(times_between_events))
142
143     return percolation_strengths
144
145 """def calculate_percolation_strength(times_between_events, deltas):
146     percolation_strengths = []
147
148     for delta in deltas:
149         cluster_sizes = []
150         # Initialize the size of the current cluster
151         current_cluster_size = 1 # The first event is always a cluster
152
153         for i in range(len(times_between_events)):
154             if times_between_events[i] <= delta:
155                 current_cluster_size += 1
156             else:
157                 if current_cluster_size > 1: # Only consider clusters with more than one
event
158                     cluster_sizes.append(current_cluster_size)
159                     # Reset the size of the current cluster
160                     current_cluster_size = 1 # The next event is always a cluster

```

```

162     # Add the size of the last cluster
163     if current_cluster_size > 1: # Only consider clusters with more than one event
164         cluster_sizes.append(current_cluster_size)
165
166     max_cluster_size = max(cluster_sizes)
167
168     percolation_strengths.append(max_cluster_size / len(times_between_events))
169 return percolation_strengths"""
170
171 def model(n_max, mu_E, mu_I, tau, n_EE, n_IE, n_EI, n_II, dt):
172     """
173     Solve the equations of the mena field model for a given number of iterations n_max
174
175     Inputs:
176     n_max: number of iterations
177     mu_E: Poisson rate of excitatory neurons
178     mu_I: Poisson rate of inhibitory neurons
179     tau: characteristic time of the system
180     n_EE: influence of excitatory neurons on excitatory neurons
181     n_IE: influence of excitatory neurons on inhibitory neurons
182     n_EI: influence of inhibitory neurons on excitatory neurons
183     n_II: influence of inhibitory neurons on inhibitory neurons
184     dt: time step
185
186     Outputs:
187     time: time series
188     t_events_E: times of events of excitatory neurons
189     t_events_I: times of events of inhibitory neurons
190     rates_E: rates of excitatory neurons
191     rates_I: rates of inhibitory neurons
192     """
193     n_E = n_I = n = 0
194     t_events_E = [0]
195     t_events_I = [0]
196     rates_E = [mu_E]
197     rates_I = [mu_I]
198     time = [0]
199     while n <= n_max:
200         # Excitation neurons
201         l_Enew = rates_E[-1] + dt * (mu_E - rates_E[-1])/tau
202         if np.random.uniform() < rates_E[-1]*dt:
203             l_Enew += n_EE
204             t_events_E.append(time[-1]+dt*np.random.uniform())
205             n_E += 1
206         if np.random.uniform() < rates_I[-1]*dt:
207             l_Enew -= n_IE
208             t_events_E.append(time[-1]+dt*np.random.uniform())
209             n_E += 1
210
211         # Inhibition neurons
212         l_Inew = rates_I[-1] + dt * (mu_I - rates_I[-1])/tau
213         if np.random.uniform() < rates_E[-1]*dt:
214             l_Inew += n_EI
215             t_events_I.append(time[-1]+dt*np.random.uniform())
216             n_I += 1
217         if np.random.uniform() < rates_I[-1]*dt:
218             l_Inew -= n_II
219             t_events_I.append(time[-1]+dt*np.random.uniform())
220             n_I += 1
221         rates_E.append(l_Enew)
222         rates_I.append(l_Inew)
223

```

```

224         time.append(time[-1]+dt)
225
226         n = n_E + n_I
227     return time, t_events_E, t_events_I, rates_E, rates_I
228
229 def identify_clusters_model(times, delta):
230     """
231     Identifies clusters in a temporal series given a resolution parameter delta
232     Computes the size and duration of clusters
233
234     ## Inputs:
235     times: temporal series
236     delta: resolution parameter
237
238     ## Output:
239     clusters: list of clusters
240     clusters_sizes: list of sizes of clusters
241     clusters_times: list of durations of clusters
242     """
243     clusters = []
244     current_cluster = []
245     for i in range(len(times) - 1):
246         if times[i + 1] - times[i] <= delta:
247             if not current_cluster:
248                 current_cluster.append(times[i])
249                 current_cluster.append(times[i + 1])
250             else:
251                 if current_cluster:
252                     clusters.append(current_cluster)
253                     current_cluster = []
254
255     clusters_sizes = [len(cluster) for cluster in clusters]
256     clusters_times = [cluster[-1] - cluster[0] for cluster in clusters]
257     return clusters, clusters_sizes, clusters_times
258
259 def bivariate_algorithm(rate1, rate2, muE, muI, nEE, nII, nEI, nIE):
260     """
261     Algorithm that computes interevent times and Hawkes intensity for a bivariate Hawkes
262     process
263
264     #Inputs:
265     rate1: Previous excitation rate
266     rate2: Previous inhibition rate
267     nEE: "Strength" of the autoexcitation process
268     nII: "Strength" of the autoinhibition process
269     nEI: "Strength" of the excitation to the inhibition
270     nIE: "Strength" of the inhibition to the excitation
271     muE: Background intensity of the excitation
272     muI: Background intensity of the inhibition
273
274     #Output: ratex_k, x_k, reaction (0 for excitatory events and 1 for inhibitory events)
275     """
276     _, xk1 = algorithm(rate1, muE, nEE)
277     _, xk2 = algorithm(rate2, muI, nII)
278
279     xks = [xk1, xk2]
280
281     if xk1 < 0:
282         print(xk1)
283         xk1 = 0
284     if xk2 < 0:

```

```

285     print(xk2)
286     xk2 = 0
287
288     reaction = np.argmin(xks)
289
290     if reaction == 0:
291         rate1_tk = (rate1 - muE) * np.exp(-xk1) + nEE + muE
292         rate2_tk = (rate2 - muI) * np.exp(-xk1) + nEI + muI
293     else:
294         rate1_tk = (rate1 - muE) * np.exp(-xk2) + nIE + muE
295         rate2_tk = (rate2 - muI) * np.exp(-xk2) + nII + muI
296
297     if rate1_tk <= muE:
298         rate1_tk = muE
299     if rate2_tk <= muI:
300         rate2_tk = muI
301
302     xk = xks[reaction]
303
304     return rate1_tk, rate2_tk, xk, reaction
305
306 def generate_series_bivariate(K, nEE, nII, nEI, nIE, muE, muI):
307     """
308     Generates temporal series for K bivariate Hawkes processes
309
310     ##Inputs:
311     K: Number of events
312     nEE: "Strength" of the autoexcitation process
313     nII: "Strength" of the autoinhibition process
314     nEI: "Strength" of the excitation to the inhibition
315     nIE: "Strength" of the inhibition to the excitation
316     muE: Background intensity of the excitation
317     muI: Background intensity of the inhibition
318
319     ##Output:
320     times_between_events: time series the interevent times
321     times: time series the events
322     rate1: time series for the intensity of process 1 (Excitation)
323     rate2: time series for the intensity of process 2 (Inhibition)
324     reactions: list the event type (0 for excitation. 1 for inhibition)
325     """
326     times_between_events = [0]
327     rate1 = [muE]
328     rate2 = [muI]
329     reactions = []
330     for _ in range(K):
331         rate1_tk, rate2_tk, xk, reaction = bivariate_algorithm(rate1[-1], rate2[-1], muE,
332         muI, nEE, nII, nEI, nIE)
333         rate1.append(rate1_tk)
334         rate2.append(rate2_tk)
335         reactions.append(reaction)
336         times_between_events.append(xk)
337     times = np.cumsum(times_between_events)
338     return times_between_events, times, rate1, rate2, reactions

```