NOTES ON SIMULATION OF HAWKES PROCESSES:

Inhomogeneous Poisson process $\lambda \to \lambda(t)$:

$$\text{PDF}(\text{inter-event time } \Delta t, t) = \lambda(t+\Delta t)\, \underbrace{e^{-\int_t^{t+\Delta t} \lambda(t')dt'}}_{\text{no events during } (t,\, t+\Delta t)} .$$

(see Chapter 22 anonymous note).

↳ In order to generate Δt:

$$\text{accum}(\Delta t) = \int_0^{\Delta t} \text{PDF}(\Delta t')\, \Delta t' = u \in [0,1].$$

$$\int_0^{\Delta t} \underbrace{\lambda(t+\Delta t') \cdot e^{-\int_t^{t+\Delta t'} \lambda(t')dt'}}_{-\frac{d}{d\Delta t'}\left[e^{-\int_t^{t+\Delta t'} \lambda(t')dt'}\right]} d\Delta t' = u .$$

Barrow ↓

$$-e^{-\int_t^{t+\Delta t} \lambda(t')dt'} + \cancel{e^{-\int_t^{t} \lambda(t')dt'}}^{\,1} = u$$

log ↓

$$\int_t^{t+\Delta t} \lambda(t')dt' = -\log(1-u) = -\log(\bar{u}).$$

↳ Generate $\bar{u} \sim \mathcal{U}[0,1]$

↳ Solve last eq. for Δt.

Poisson Process as a Markov Process (exp. kernels)

$$\lambda(t) = \mu + \sum_{t_i < t} \alpha e^{-\beta(t-t_i)}$$

$$= \mu + \sum_{\substack{t_i \leq t_k \\ (t_k \text{ last event})}} \alpha e^{-\beta(t-t_k+t_k-t_i)} =$$

$$= \mu + e^{-\beta(t-t_k)} \underbrace{\sum_{t_i \leq t_k} \alpha e^{-\beta(t_k-t_i)}}$$

Σ goes for $t_i < t_k$, so it is $\lambda(t_k^-)$.

$$\lambda(t_k) = \mu + \sum_{t_i < t_k} \alpha e^{-\beta(t_k-t_i)}$$

$$\Downarrow$$

$$\sum_{t_i \leq t_k} \alpha e^{-\beta(t_k-t_i)} = \lambda(t_k) - \mu + \alpha. \quad (\lambda(t_k^-))$$

$$= \mu + e^{-\beta(t-t_k)} (\lambda(t_k) - \mu + \alpha).$$

→ Note that after t_k, we just need:

* $\lambda(t_k)$ $(= \lambda(t_k^-))$
* time elapsed $(t-t_k)$

} (it doesn't have memory after t_k).

Simulation method

Recall that in order to find next t after t_k:

$$-\log(1-u) = \int_{t_k}^{t} \lambda(t')\,dt'.$$

That combined with $\lambda(t) = \mu + e^{-\beta(t-t_k)}(\lambda(t_k^-)+\alpha-\mu)$:

$$u = 1 - e^{-\mu(t-t_k)}\, e^{-(\lambda(t_k^-)+\alpha-\mu)\cdot \underbrace{\int_{t_k}^{t} e^{-\beta(t'-t_k)}dt'}_{-\frac{1}{\beta}\left[e^{-\beta(t-t_k)}-1\right]}}$$

$$u = 1 - \underbrace{e^{-\mu(t-t_k)}}_{P_1(t_{k+1}^{(1)}>t)\,\cdot}\ \underbrace{e^{-\left[(\lambda(t_k^-)+\alpha-\mu)\beta^{-1}\left(1-e^{-\beta(t-t_k)}\right)\right]}}_{P(t_{k+1}^{(2)}>t)}$$

Then we apply the composition method.

If we take $t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)})$; then $t_{k+1} \sim P(t_{k+1}>t)$.

$$\text{Prob}(t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)}) \leq t) = 1 - \text{Prob}(\min(t_{k+1}^{(1)}, t_{k+1}^{(2)}) > t)$$

$$= 1 - \text{Prob}(t_{k+1}^{(1)}>t)\cdot \text{Prob}(t_{k+1}^{(2)}>t)$$

la prob. de que el más pequeño sea mayor que t es que cada uno por separado sea más grande que t. (porque ambos tienen que ser mayores).

which is the expresion above.

Then:

*) Generate $t_{k+1}^{(1)} \sim P(t_{k+1}^{(1)} > t) = e^{-\mu(t-t_k)}$.

$\{ P(t_{k+1}^{(1)} \leq t) = 1-\underbrace{e^{-\mu(t-t_k)}}_{\bar{u}_1 \in [0,1]} = u_1 \in [0,1]. = 1-\bar{u}_1$ } equivalent!

This is done with:

$$u_1 = e^{-\mu(t_{k+1}^{(1)}-t_k)}$$

$$\log u_1 = -\mu(t_{k+1}^{(1)}-t_k) \Rightarrow \boxed{t_{k+1}^{(1)} = t_k - \frac{1}{\mu}\log u_1.}$$

*) Generate $t_{k+1}^{(2)} \sim P(t_{k+1}^{(2)} > t) = e^{-(\lambda(t_k^-)+\alpha-\mu)\beta^{-1}(1-e^{-\beta(t_{k+1}^{(2)}-t_k)})}$

$$-\log u_2 = [\lambda(t_k^-)+\alpha-\mu]\beta^{-1}(1-e^{-\beta(t_{k+1}^{(2)}-t_k)})$$

$$1+\frac{\beta\cdot\log u_2}{\lambda(t_k^-)+\alpha-\mu} = e^{-\beta(t_{k+1}^{(2)}-t_k)}$$

$$\boxed{t_{k+1}^{(2)} = t_k - \beta^{-1}\log\Bigg(1+\overbrace{\frac{\beta\log u_2}{\underbrace{\lambda(t_k^-)+\alpha}_{\lambda(t_k^+)}-\mu}}^{\text{this should be a positive number!}}\Bigg)}$$

*) Take $t_{k+1} = \min(t_{k+1}^{(1)}, t_{k+1}^{(2)})$.

*) Calculate $\lambda(t_{k+1}^-) = \mu + e^{-\beta(t_{k+1}-t_k)}(\lambda(t_k^-)-\mu+\alpha)$

$$= \mu + e^{-\beta(t_{k+1}-t_k)}(\lambda(t_k^+)-\mu)$$

$$\boxed{\lambda(t_{k+1}^+) = \mu+\alpha+e^{-\beta(t_{k+1}-t_k)}(\lambda(t_k^+)-\mu)}$$

This is in complete agreement with what is done in the Notermans paper.

Extension to a BIVARIATE process:

we would use $\neq$ β's.

*) Markovian?

$$\lambda_1(t) = \mu_1 + \sum_{t_i^{(1)}<t} \alpha_{1\to 1}\, e^{-\beta(t-t_i^{(1)})} + \sum_{t_i^{(2)}<t} \alpha_{2\to 1}\, e^{-\beta(t-t_i^{(2)})}$$

$$\lambda_2(t) = \mu_2 + \sum_{t_i^{(1)}<t} \alpha_{1\to 2}\, e^{-\beta(t-t_i^{(2)})} + \sum_{t_i^{(2)}<t} \alpha_{2\to 2}\, e^{-\beta(t-t_i^{(2)})}$$

Last event was $t_k^{(\ell)}$

$$\lambda_1(t) = \mu_1 + \sum_{t_i^{(1)}<t} \alpha_{1\to 1}\, e^{-\beta(t-t_k^{(\ell)}+t_k^{(\ell)}-t_i^{(1)})} + \sum_{t_i^{(2)}<t} \alpha_{2\to 1}\, e^{-\beta(t-t_k^{(\ell)}+t_k^{(\ell)}-t_i^{(2)})}$$

$$= \mu_1 + e^{-\beta(t-t_k^{(\ell)})}\left\{\sum_{t_i^{(1)}<t_k^{(\ell)}} \alpha_{1\to 1}\, e^{-\beta(t_k^{(\ell)}-t_i^{(1)})} + \sum_{t_i^{(2)}<t_k^{(\ell)}} \alpha_{2\to 1}\, e^{-\beta(t_k^{(\ell)}-t_i^{(2)})}\right\}$$

$$= \mu_1 + e^{-\beta(t-t_k^{(\ell)})}\left\{\lambda_1(t_k^{(\ell)-}) - \mu_1 + \alpha_{\ell\to 1}\right\}.$$

$$= \mu_1 + e^{-\beta(t-t_k^{(\ell)})}\left\{\lambda_1(t_k^{(\ell)+}) - \mu_1\right\}.$$

... to be finished.

Extension to a BIVARIATE process:

*) According to Dassios & Zhao, $\Delta_{k+1} = t_{k+1} - t_k$ is taken as:

$$\Delta_{k+1} = \min \{ \Delta_{k+1}^{(1)}, \Delta_{k+1}^{(2)} \} \quad , \text{ with } \Delta_{k+1}^{(j)} \text{ generated as usual:}$$

$$\Delta_{k+1}^{(j)} = \min \left(-\frac{1}{\mu_j} \log u_1^{(j)} \, , \, -\beta \log \left(1 + \underbrace{\frac{\beta \log u_2^{(j)}}{\lambda_j(t_k^+) - \mu_j}}_{g_j} \right) \right)$$

[unless $g_j \leq 0$ (then take first argument).]

And check what process (ℓ) corresponds to Δ_{k+1} $(= \Delta_{k+1}^{(\ell)})$.

*) Update time for next event: $t_k \to t_{k+1} = t_k + \Delta_{k+1}^{(\ell)}$.

*) Update $\lambda_j(t_{k+1}^+) = (\lambda_j(t_k^+) - \mu_j)\, e^{-\beta(t_{k+1} - t_k)} + \mu_j + \alpha_{\ell \to j}$.

with $j = 1, 2$.