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On the hidden hazards of adaptive behavior

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A B S T R A C T

Adaptive behavior has been observed in almost all aspects of real-world. One of the main advantages of acting adaptively is its stabilizing effect on dynamic equilibrium, associated with which are three favorable features: (a) non-destabilizing characteristics, (b) low-speed effectiveness and (c) the convexity of the stabilization regime in terms of the adaptive parameter. It is shown either in theory or by counter-examples that these advantages may not be preserved if the adaptive mechanism is applied to multi-dimensional processes. The necessary and sufficient conditions for the relevant phenomena are provided for two-dimensional dynamic processes with application to duopolistic dynamics. Our findings not only help to clarify hidden misconceptions and prevent potential abuse of adaptive mechanisms, but also illustrate the possible pitfalls arising from generalizing well-known characteristics of low dimensional and/or homogeneous agent models to high-dimensional and heterogenous agent models.

1. Introduction

Adaptation is one of the most widely observed economic behaviors and business strategies in economic activities.

One of the favorable features of adopting adaptive actions relates to its stabilizing effect on a dynamical system (Lucas, 1986). As early as in Nerlove (1958), it was already illustrated that an originally unstable linear cobweb model can be stabilized if adaptive expectations are utilized. Since then, the exploration of the impact of adaptive expectations on dynamic stability has emerged in many fields of economic analysis such as oligopoly dynamics (Okuguchi, 1970; Szidarovszky and Li, 2000), cobweb dynamics (Chiarella, 1988), adaptive learning (Honkapohja, 1993), adaptive tactics in games (Conlisk, 1993; Rassenti et al., 2000) and other contexts (Naish, 1993; Garratt and Hall, 1997). Heiner (1989) showed with a general decision-making framework that the adaptive adjustment of decision variables can lead to their convergence to optimal targets. Huang (2000a, b, 2001a, b, 2002) formally utilized adaptive adjustment as a measure to stabilize an equilibrium or to control chaos in various discrete dynamic models.

A closely related and desirable aspect of adaptation is its low-speed effectiveness, by which we mean that if an originally unstable dynamic process can be stabilized when the adjustment speed is high, it should also be stabilized if the speed is sufficiently low. This special advantage assures that an economic agent is able to undergo a trial and error process safely.

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Unfortunately, we shall show both in theory and by examples that these nice properties may not always be preserved when the adaptive behavior is embedded in a multi-dimensional dynamic process. In other words, adaptation may bring about negative effects on dynamic stability: (i) adjusting part of the state variables adaptively in a multi-dimensional dynamic process may turn an originally stable economy into an unstable one and (ii) reducing the speed of adjustment to a certain extent may destroy an otherwise stable economic process.

To make matters worse, the stabilization regime in terms of each adaptive parameter may no longer be convex when the adaptation is implemented in a multi-dimensional dynamic process. If a dynamic process can be stabilized when a particular adjustment speed takes either one of two "stabilizing" values, then it may fail to do so if the adjustment speed takes an average of these two values instead. Geometrically, there may exist many "holes" in the stabilizing regime. This counter intuitive fact makes those studies in which adaptive parameters are set arbitrarily and dynamic stability is taken for granted invalid.

Fortunately, none of these undesirable traits of adaptive behavior appear in one-dimensional dynamic processes, nor do they appear in multi-dimensional dynamic processes with uniform adaptations. This may account for why they have never been addressed formally and systematically in the relevant literature. Our study thus provides further evidence of the potential pitfalls arising from generalizing well-known characteristics of low dimensional homogeneous agent models to multi-dimensional heterogeneous agent models.

The paper is organized as follows. The ensuing section briefly discusses the quantitative characterization of adaptive behavior. Section 3 is devoted to the investigation of two-dimensional dynamic processes in detail. While the convexity of adaptive regime is preserved in general, both non-destabilization characteristics and low-speed effectiveness are lost. The theoretical results derived are then applied to a duopoly in Section 4. It is shown that while all the nice characteristics of adaptive behavior are preserved if both firms adopt Cournot best-response strategy, they may not be retained if firms behave heterogeneously. The pitfall of non-convexity of stabilizing regimes is further illustrated in Section 5. Possible areas for further research and concluding remarks are addressed in Section 6.

2. Adaptive behavior

Among the reasons for acting adaptively or making adaptive economic decisions are cautious nature, resource limitations, external policy restrictions and stability consideration. According to Day (1975), adaptive behavior takes two distinct forms. The first is essentially a servomechanism, referred to by Day as "determined homeostasis", by which he meant that actions (in the broader sense) are adjusted on the basis of an observed discrepancy between a desired or targeted value of one or more critical variables and their experienced values. Adaptive expectations, adaptive adjustments and some adaptive learnings/selections may belong to this form. The second form of adaptive behavior involves bounded rationality and was referred to as "optimizing with feedback" since explicit, as opposed to implicit, optimizing is pursued in economic decisions. At a given point in real time, an economic agent perceives a set of feasible actions and selects the best member in this set based on an objective function or preference pre-ordering. The perceived feasible set, or the objective function, or both, are then adjusted in response to experience. Most types of adaptive learning, adaptive selection of strategies in game theory, adaptive programming (which includes recursive programming) and adaptive control belong to the latter category.

2.1. Quantitative characterization

Quantitatively, most of the first form and a part of the second form of adaptive behaviors can be abstractly characterized or simplified to the updating of certain economic variables based on their historical data.

Definition 1. *Adaptive adjustment*: By which we mean that a variable vector $\mathbf{x}_t = (x_{1,t}, x_{2,t}, \ldots, x_{n,t})$ is adaptively adjusted to its target value, say $\mathbf{x}^*_{t+1} = (x^*_{1,t+1}, x^*_{2,t+1}, \ldots, x^*_{n,t+1})$, according to a specific rule:

$$x_{i,t+1} - x_{i,t} = \gamma_i \cdot (x^*_{i,t+1} - x_{i,t}), \quad i = 1,2,\ldots,n, \tag{1}$$

where the γ_i's are parameters that take values between zero and unity and are conventionally referred to as the *speeds of adjustment*.[1] Eq. (1) suggests that the rates of adjustment to the target values are proportional to their differences to the current values.

Generally, the target values $x^*_{i,t}$ are formed in reaction to their own historical data, the current and/or historical data of other economic variables (either endogenous or exogenous), that is,

$$x^*_{i,t+1} = f_i(\{\mathbf{x}_{t+1-j}\}_{j=1}^{\infty}, \{\mathbf{y}_{t+1-j}\}_{j=0}^{\infty}), \quad i = 1,2,\ldots,n, \tag{2}$$

with $\mathbf{y} = \{y_1, y_2, \ldots, y_m\}$ being the vector of all exogenous economic variables available at the time t.

[1] The speeds of adjustment in this study is assumed to take value in the conventional range ($0 < \gamma_i \le 1$) for all i. It is shown in Huang (2000a, 2001a, 2001b) that this range is sufficient to stabilize any equilibrium at which the real parts of the original eigenvalues are all less than the unity. The possible generalization to the unconventional range ($\gamma > 1$) is explored in Huang (2001a).

Combining (1) and (2) leads to

$$x_{i,t+1} = (1-\gamma_i)\cdot x_{i,t} + \gamma_i \cdot f_i(\{\mathbf{x}_{t+1-j}\}_{j=1}^{\infty}, \{\mathbf{y}_{t+1-j}\}_{j=0}^{\infty}), \quad i = 1,2,\ldots,n, \tag{3}$$

which implies that *the current decision $x_{i,t+1}$ is a convex combination of the last decision $x_{i,t}$ and the current target value $x^*_{i,t+1}$*.

Since our focus will be the dynamic stability of an equilibrium, some simplifications are in order to make the subsequent presentation easier.

The first simplification is to drop the explicit inclusion of exogenous variables in (3) since asymptotic stability demands that the all exogenous economic variables must approach the respective constants.

Secondly, by suitably introducing new variables, any finite order discrete process can be alternatively expressed as a system of first-order discrete processes, so we shall assume that only one-lag historical data are included in the f_i's of (2).

With these simplifications and modifications, (3) is recast to

$$x_{i,t+1} = (1-\gamma_i)\cdot x_{i,t} + \gamma_i \cdot f_i(x_{1,t}, x_{2,t}, \ldots, x_{n,t}) \tag{4}$$

for $\gamma_i \in I, i = 1,2,\ldots,n$, where $I \triangleq (0,1]$.

For the convenience of reference, we shall refer the dynamic process in which all γ_i's equal to unity as the *original dynamic process*.

2.2. Adaptive adjustment for one-dimensional discrete system

In order to gain some intuition on the advantages of adaptive mechanisms, we examine first the one-dimensional dynamical process (n=1). Eq. (4) reduces to

$$x_{t+1} = F(x_t) = (1-\gamma)\cdot x_t + \gamma \cdot f(x_t).$$

For an equilibrium $\overline{x}$ determined from $f(\overline{x}) = \overline{x}$, its local stability is ensured when the multiplier defined by

$$\sigma(\gamma) \triangleq |F'(\overline{x})| = |1-\gamma+\gamma f'(\overline{x})|$$

is less than the unity. It is then straightforward to verify that, so long as $-\infty < f'(\overline{x}) < 1$, the inequality $\sigma(\gamma) < 1$ can be achieved by setting a suitable $\gamma \in I$. Moreover, for all $-\infty < f'(\overline{x}) < 1$, we have

$$\sigma(1) = |f'(\overline{x})| \quad \text{with } \left.\frac{\partial \sigma(\gamma)}{\partial \gamma}\right|_{\gamma=1} = \begin{cases} <0 & \text{if } 0 < f'(\overline{x}) < 1, \\ >0 & \text{if } f'(\overline{x}) < 0, \end{cases}$$

and $\sigma(0) = 1$ with $\partial\sigma(\gamma)/\partial\gamma|_{\gamma=0} < 0$.

These analytical properties lead to the following favorable characteristics:

(i) *Non-destabilizing characteristics*: If $-1 < f'(\overline{x}) < 1$, that is, the economic dynamic process is originally stable ($\gamma = 1$), we have $\sigma(\gamma) < 1$ for all $\gamma \in I$. In other words, an adjustment with any speed $\gamma \in I$ will never destroy an originally stable system.

(ii) *Low-speed effectiveness*: If $\sigma(\overline{\gamma}) < 1$ for any $\overline{\gamma} \in I$, then we have $\sigma(\gamma) < 1$ for all $\gamma \in (0,\overline{\gamma}]$. This is to say, if the local stability is achieved with a certain speed, then decreasing the speed will not destroy such stability. In particular, an adaptive dynamic process can always be stabilized when the speed is sufficiently low.

(iii) *Convexity of the stabilizing regime*: If $\sigma(\overline{\gamma}_i) < 1$ for $0 < \overline{\gamma}_i \le 1$, i=a, b, then for any $\gamma_\alpha = (1-\alpha)\overline{\gamma}_a + \alpha\overline{\gamma}_b$, where $0 \le \alpha \le 1$, we have $\sigma(\gamma_\alpha) < 1$.

Apparently, for a one-dimensional process, these three properties are related. In fact, property (ii) implies both property (i) and property (iii). However, the significance of these characteristics can never be undermined. The first characteristic implies that, if an equilibrium is stable when an economic variable is updated directly to its target value, then such stability will not be destroyed if the same variable is updated adaptively with an arbitrary speed. Such advantage guarantees the safety of adaptive adjustment with a relatively high speed. In contrast, the second characteristic allows an economic agent to start a trial and error procedure with a relatively low speed of adjustment whenever stability is the main concern. This speed can then be increased gradually to pursue faster convergence without sacrificing dynamic stability.[2] Such trials are particularly important in many economic applications like adaptive learning. Finally, if two speeds $\overline{\gamma}_a$ and $\overline{\gamma}_b$, with $\overline{\gamma}_a < \overline{\gamma}_b$, are proved to be effective either from historical experience or learning from others, a choice of a suitable γ in the range $[\overline{\gamma}_a, \overline{\gamma}_b]$ will be effective as well.

[2] Mathematically, it also implies that, to utilize adaptive adjustment to stabilize an economic system, the inequality $\sigma(0^+) < 1$ is a necessary and sufficient condition.

2.3. Adaptive adjustment for n-dimensional discrete systems

Unfortunately, the above mentioned nice characteristics may not be preserved when adaptive adjustments are implemented in an n-dimensional dynamic process (4) with heterogeneous speeds.[3] Let $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ be an equilibrium of the original dynamic process ($\gamma_i = 1$ for all i). Let $f'_{ij} \triangleq \partial f_i / \partial x_j|_{\overline{\mathbf{x}}}$. Then the Jacobian matrix of (4) at an equilibrium $\overline{\mathbf{x}}$ is given by

$$\mathcal{J} \triangleq \begin{pmatrix} (1-\gamma_1)+\gamma_1 f'_{11} & \gamma_1 f'_{12} & \cdots & \gamma_1 f'_{1n} \\ \gamma_2 f'_{21} & (1-\gamma_2)+\gamma_2 f'_{22} & \cdots & \gamma_2 f'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_n f'_{n1} & \gamma_n f'_{n2} & \cdots & (1-\gamma_n)+\gamma_n f'_{nn} \end{pmatrix}. \tag{5}$$

Given a $\mathbf{\Gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in I^n \triangleq I \times I \times \cdots \times I$, let $\rho(\mathcal{J})$ denote the spectral radius of the Jacobian $\mathcal{J}$, that is, the largest module of eigenvalues of $\mathcal{J}$. Then the equilibrium $\overline{\mathbf{x}}$ is local stable if and only if $\rho(\mathcal{J}) < 1$.

In order to get a picture of how adaptive behavior affects the local stability of an equilibrium, we shall isolate the impact of each individual adaptive action by assuming that the adaptive mechanism is implemented for one of the variables (or, the sequential implementation of adaptive behavior, should all variables be adjusted). Without loss of generality, we assume that only the last variable $x_{n,t}$ is adaptively adjusted and other variables are not, that is, $\gamma_n = \gamma < 1$ and $\gamma_i = 1$, i=1, 2,…,$n-1$. That is,

$$\begin{aligned} x_{1,t+1} &= f_1(x_{1,t}, x_{2,t}, \ldots, x_{n,t}), \\ x_{2,t+1} &= f_2(x_{1,t}, x_{2,t}, \ldots, x_{n,t}), \\ &\cdots \\ x_{n,t+1} &= (1-\gamma) \cdot x_{1,t} + \gamma \cdot f_n(x_{1,t}, x_{2,t}, \ldots, x_{n,t}). \end{aligned}$$

Such simplification enables us to treat the Jacobian $\mathcal{J}$ defined in (5) as a function of γ, $\gamma \in I$, and denote it as $\mathcal{J}(\gamma)$.

Definition 2. *Destabilizing effect*: If we have $\rho(\mathcal{J}(1)) < 1$ but $\rho(\mathcal{J}(\gamma)) \geq 1$ for at least one $\gamma \in I$, then adaptive adjustment to the nth variable is not free from the destabilizing effect.

Definition 3. *Low-speed failure*: If we have $\rho(\mathcal{J}(\gamma)) < 1$ for at least some $\gamma \in I$ but $\rho(\mathcal{J}(0)) > 1$, then setting the nth adjustment speed to zero fails to stabilize the process.

Definition 4. Any interval A in I=(0,1] will be referred to as the *effective regime* for γ if $\rho(\mathcal{J}(\gamma)) < 1$ for all $\gamma \in A$. The union of all possible effective regimes, denoted as Λ, is referred to as the *stabilizing regime* for nth variable. The *stabilizing regime* is said to exhibit convexity if Λ consists of one single interval only.

With these definitions, we are ready to move to the multi-dimensional dynamic processes.

3. Destabilizing effect and low-speed failure

To see the destabilizing effect and low-speed failure of adaptive behavior, it is sufficient to illustrate with a two-dimensional discrete process.

Let us consider an original dynamic process given by

$$\begin{cases} x_{t+1} = f(x_t, y_t), \\ y_{t+1} = g(x_t, y_t), \end{cases} \tag{6}$$

where f and g are the target values to which the two variables are to be updated. Then the local stability of an equilibrium of (6), denoted by $(\overline{x}, \overline{y})$, is determined by the eigenvalues of the Jacobian $\mathcal{J}$ evaluated at $\overline{\mathbf{x}}$:

$$\mathcal{J} = \begin{bmatrix} f'_x & f'_y \\ g'_x & g'_y \end{bmatrix}_{(\overline{x}, \overline{y})}. \tag{7}$$

Let $\mathcal{T} \triangleq f'_x + g'_y$ be the trace of $\mathcal{J}$ and $\mathcal{D} \triangleq f'_x g'_y - f'_y g'_x$ be the determinant of $\mathcal{J}$, respectively. To make the following discussion meaningful, we assume that the *linkage sensitivity* at the equilibrium $\delta \triangleq f'_y g'_x \neq 0$. The two eigenvalues related to $\mathcal{J}$ are

$$\lambda_{1,2} = \frac{1}{2}\mathcal{T} \pm \frac{1}{2}\sqrt{\mathcal{T}^2 - 4\mathcal{D}}. \tag{8}$$

[3] It is equivalent to require that the adjustments to the different variables should be neither synchronized nor coordinated. It is shown in Huang (2000a, 2001a) that all favorable characteristics of adaptive adjustments can still be preserved in n-dimensional dynamic processes if the adjustments are uniform.

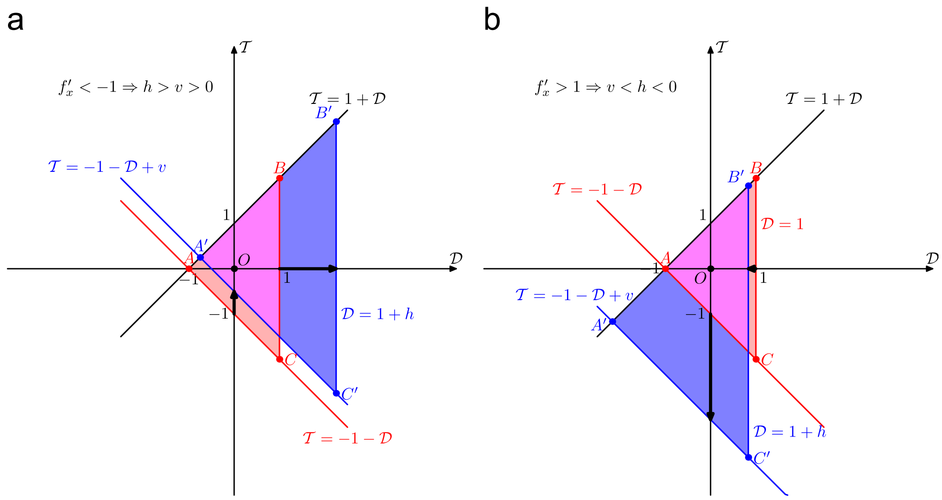

Fig. 1. Destabilizing effects of adaptive adjustment. (a) $f'_x < -1$ and (b) $f'_x > 1$.

The necessary and sufficient condition for the local stability of $(\overline{x},\overline{y})$ is that both eigenvalues given by (8) must lie in the unit circle in the complex plane,[4] which is guaranteed if the following inequalities are satisfied:

$$\mathbf{S}: \ 1 > \mathcal{D} > \max\{-\mathcal{T}-1, \mathcal{T}-1\}.$$

Therefore, as illustrated in Fig. 1, the stability region **S** is thus given by the triangle ABC in $(\mathcal{T},\mathcal{D})$ space formed by the line AB $(\mathcal{T}=\mathcal{D}+1)$, the line AC $(\mathcal{T}=-\mathcal{D}-1)$ and the line BC $(\mathcal{D}=1)$.

Now consider the case in which the variable y is adaptively adjusted with a speed $\gamma_y = \gamma \in I$ so that (6) is modified into

$$\begin{cases} x_{t+1} = f(x_t, y_t), \\ y_{t+1} = \tilde{g}(x_t, y_t) = (1-\gamma)\cdot y_t + \gamma \cdot g(x_t, y_t). \end{cases} \tag{9}$$

The trace and the determinant of the Jacobian matrix for (9) change to

$$\begin{cases} \tilde{\mathcal{T}}(\gamma) \triangleq \mathcal{T} + (1-\gamma)\cdot(1-g'_y), \\ \tilde{\mathcal{D}}(\gamma) \triangleq \gamma \cdot \mathcal{D} + (1-\gamma)\cdot f'_x. \end{cases} \tag{10}$$

For a given $\gamma \in I$, the stability region in terms of the original trace $\mathcal{T}$ and original determinant $\mathcal{D}$ can be recast as

$$\mathbf{S}(\gamma): \ 1 + h(\gamma) > \mathcal{D} > \max\{-\mathcal{T} - 1 + v(\gamma), \mathcal{T} - 1\}, \tag{11}$$

where $v(\gamma) \triangleq -2(1-\gamma)(1+f'_x)/\gamma$ and $h(\gamma) \triangleq (1-\gamma)(1-f'_x)/\gamma$ are values representing the vertical shift for the line AC (changes to the line $A'C'$) and the horizontal shift for the line BC (changes to the line $B'C'$), respectively, in Fig. 1.

Depending on the relative signs of v and h, which in turn depends on both γ and f'_x, the stability region $\mathbf{S}(\gamma)$ varies as the new triangle slides along the line AB: $\mathcal{T} = \mathcal{D} + 1$.

First, we consider the general case in which γ does not take an extremely small value, i.e., $\gamma \neq 0^+$.

- $f'_x < -1$: We have $h(\gamma) > 0$ and $v(\gamma) > 0$ for all $\gamma \in I$, so that the line AC moves upward and the line BC moves rightward. The overall effect of adaptive adjustment to y amounts to a combination of varying the size of the triangle and sliding upward along the line AB. As illustrated in Fig. 1(a), there exists a part of area of ΔABC that is not covered in $\Delta A'B'C'$. That is to say, there exist cases in which *an originally stable equilibrium becomes unstable when y is adaptively adjusted*, an indication of the destabilizing effect.

  Notice that, if $f'_x < -3$, we have $v(\gamma) > h(\gamma)$ for all $\gamma \in I$. That is, the upward shift of the line AC is more than the rightward shift of the line BC. Can $\Delta A'B'C'$ completely disappear? The answer is definitely " YES". Such a possibility occurs when $v(\gamma) > h(\gamma) + 4$, or, equivalently:

  $$\gamma \leq 1 - 4/(1 - f'_x) \quad \text{for } f'_x < -3.$$

  Consequently, when $f'_x < -3$, regardless of the values of other partial derivatives (i.e., f'_y, g'_x and g'_y) and regardless of whether the equilibrium is originally stable or not, setting the speed γ small enough always makes the equilibrium unstable, an indication of *low-speed failure.*

[4] Here we focus solely on the hyperbolic equilibrium ($|\lambda_{1,2}| \neq 1$) since for the non-hyperbolic equilibrium, the stability conditions are much more involved.

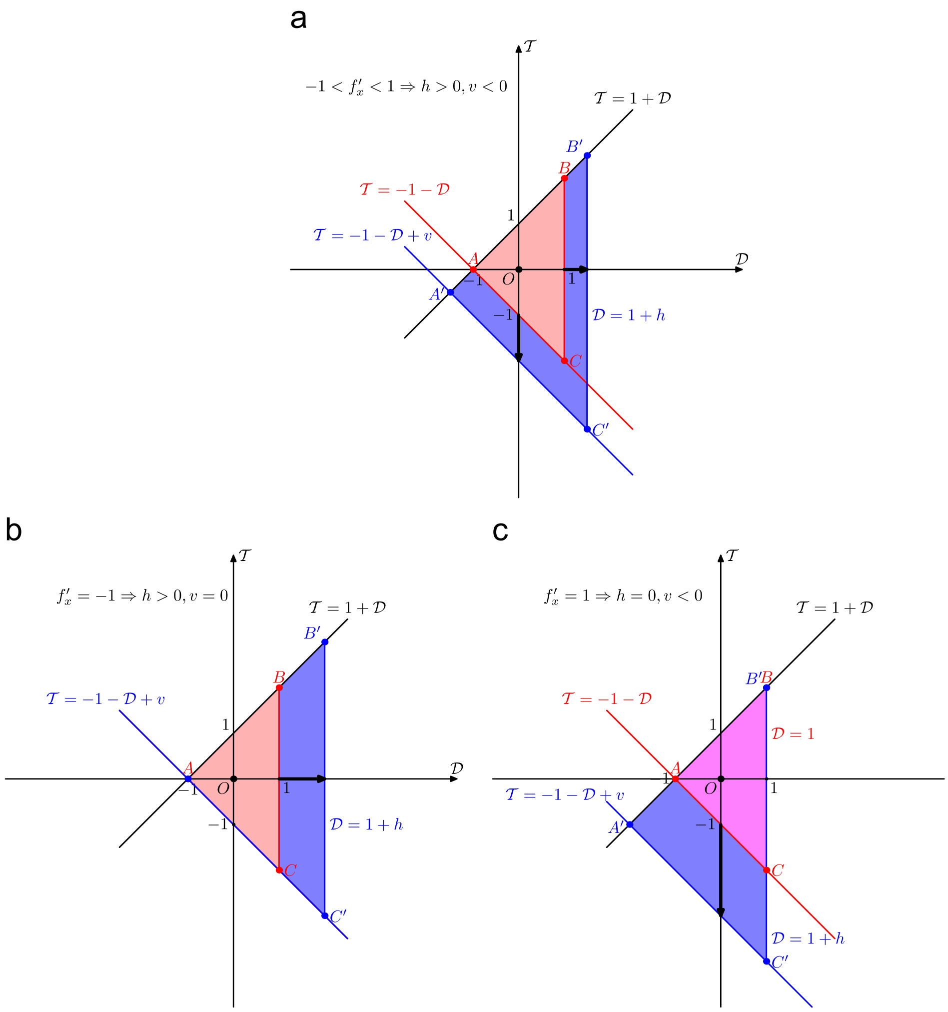

Fig. 2. Stabilizing effects of adaptive adjustment. (a) $-1 < f_x' < 1$, (b) $f_x' = -1$ and (c) $f_x' = 1$.

- $f_x' > 1$: We have $h(\gamma) < 0$ and $v(\gamma) < 0$ with $|v(\gamma)| > |h(\gamma)|$ for all $\gamma \in I$., so that the downward shift of the line *AC* is always greater than the leftward shift of the line *BC*. So $\Delta A'B'C'$ can never disappear and the area of $\Delta A'B'C'$ is always greater than the area of ΔABC. Needless to say, the destabilizing effect appears as well, as illustrated in Fig. 1(b).
- $-1 < f_x' < 1$: We have $h(\gamma) > 0$ and $v(\gamma) < 0$ for all $\gamma \in I$. This is a case in which the line AC moves downward and the line BC moves rightward. Now ΔABC is completely included in $\Delta A'B'C'$. Therefore, adaptive adjustment expands the stability region and there is no destabilizing effect. This case is depicted in Fig. 2(a).
- $f_x' = -1$: We have $v(\gamma) = 0$ and $h(\gamma) > 0$ for all $\gamma \in I$. The effect of adaptive adjustment becomes a simple enlargement of the original stability region so that the destabilizing effect does not appear, as shown in Fig. 2(b).
- $f_x' = 1$: We have $h(\gamma) = 0$ and $v(\gamma) < 0$ for all $\gamma \in I$. As illustrated in Fig. 2(c), only the line *AC* moves downward. Again ΔABC is again completely included in $\Delta A'B'C'$, another case in which the adaptive adjustment expands the stability region.

The above discussion leads to the following propositions.

Proposition 1. *For the dynamic process* (9) *with* $\delta \neq 0$ *and* $\gamma \neq 0^+$, *we have the following*:

(i) *when* $-1 \le f_x' \le 1$, *the stability region in the* $(\mathcal{T}, \mathcal{D})$ *space is expanded and hence the destabilizing effect can never appear. Moreover, the smaller the adjustment speed is, the more the stability region expands and*

(ii) *when* $f_x' < -1$ *or* $f_x' > 1$, *the stability region in the* $(\mathcal{T}, \mathcal{D})$ *space is shifted. While the stability region expands along one direction, it shrinks in the other direction. Moreover, decreasing the adjustment speed intensifies such displacement.*

In summary, for the case of $\gamma \neq 0^+$, adopting adaptive adjustment can either stabilize or destabilize a process. The overall effect relies critically on the speed γ. In particular, when γ approaches the zero, the effect is expected to be "maximized", which forces us to explore the extreme case of $\gamma \to 0^+$.

Proposition 2. *For the dynamic process* (9) *with* $\delta \neq 0$, *the necessary conditions that an equilibrium can be stabilized when* γ *approaches zero are* $-1 < f'_x < 1$ *and* $\mathcal{T} < \mathcal{D}+1$.

Proof. When $\gamma \to 0^+$, we have

$$\tilde{\mathcal{T}}(0^+) = 1+f'_x \quad \text{and} \quad \tilde{\mathcal{D}}(0^+) = f'_x,$$

so the stability region, denoted as $\boldsymbol{\Gamma}(0^+)$, is roughly given by $-1 < f'_x < 1$. In fact, for $f'_x \neq 1$, we have $\lambda_1(0^+) = f'_x$, $\lambda_2(0^+) = 1$, and the dynamic process is *conditionally stable* (or *not wholly-stable*, see Gandolfo, 1996) only if $-1 < f'_x < 1$. Notice that

$$\left.\frac{\partial \lambda_1(\gamma)}{\partial \gamma}\right|_{\gamma = 0^+} = -\frac{f'_y g'_x}{1-f'_x} = \frac{-\delta}{1-f'_x},$$

and

$$\left.\frac{\partial \lambda_2(\gamma)}{\partial \gamma}\right|_{\gamma = 0^+} = \frac{1}{1-f'_x}(\mathcal{T}-\mathcal{D}-1),$$

the local stability demands $\partial\lambda_2(\gamma)/\partial\gamma|_{\gamma=0^+} < 0$, or, equivalently, $\mathcal{T} < \mathcal{D}+1$. $\square$

Remark 1. The stability region for $\gamma \to 0^+$ can be alternatively expressed in $(\mathcal{T},\mathcal{D})$ space as

$$\mathbf{S}(0^+): \; -1+g'_y < \mathcal{T} < \min\{\mathcal{D}+1, 1+g'_y\},$$

which is depicted in Fig. 3.

Remark 2. It should be emphasized that, to characterize the stability region for the adjusted process in the original $(\mathcal{T},\mathcal{D})$ space with two displacement parameters h and v, the assumption of $\delta \neq 0$ is indispensable. Otherwise, f'_x itself as well as h and v themselves can be treated as a function of $\mathcal{T}$ and $\mathcal{D}$. In this case, the stability boundaries of the adapted process cannot be simply treated as parallel shifts of the stability boundaries for the original process in $(\mathcal{T},\mathcal{D})$ space. Instead, the stability analysis should be carried out in (f'_x, g'_y) space directly because the local stability of the two-dimensional process (9) is essentially characterized by the joint effect of the two independent one-dimensional processes.

Apparently, the conclusions in Proposition 1 apply to the case in which the variable x is adjusted with a speed $\gamma_x \in I$ as well since we need just replace f'_x with $\hat{f}'_x = (1-\gamma_x)+\gamma_x f'_x$ and f'_y with $\hat{f}'_y = \gamma_x f'_y$ and then substitute $\mathcal{T}$ and $\mathcal{D}$ with $\hat{\mathcal{T}} \triangleq T+(1-\gamma_x)(1-f'_x)$ and $\hat{\mathcal{D}} \triangleq \gamma_x \mathcal{D}+(1-\gamma_x)g'_y$, respectively. As a result, the destabilizing effect and low-speed failure can both occur for a two-dimensional dynamic process with either one variable or both variables being (independently) adjusted. Formally, we have

Theorem 1. *For a two-dimensional discrete process* (6) *with* $\delta \neq 0$, *when either one variable or both variables is* (*are independently*) *adjusted*,

(i) *destabilizing effects may occur*;
(ii) *low-speed failure may occur*; and
(iii) *the stabilizing regime in terms of each individual speed* γ_i, i=x, y, *is always convex.*

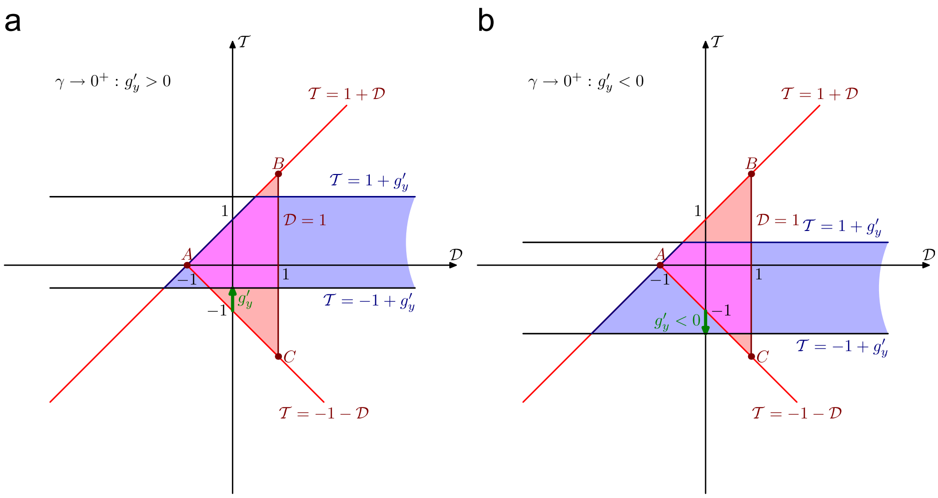

Fig. 3. Change of stability region when γ is extremely small. (a) $g'_y > 0$ and (b) $g'_y < 0$.

Proof. What remains to be proved is (iii). Without loss of generality, consider the case in which $\gamma_x = 1$ and $\gamma_y = \gamma > 0$. Then what we need to prove is that, if both $\overline{\gamma}_1$ and $\overline{\gamma}_2 \in \Lambda^{(y)}$, where $\Lambda^{(y)}$ denotes the stabilizing regime for variable y, then we always have $\gamma_\alpha \in \Lambda^{(y)}$ for all

$$\gamma_\alpha = (1-\alpha)\overline{\gamma}_1 + \alpha\overline{\gamma}_2 \quad \text{where } 0 \le \alpha \le 1. \tag{12}$$

This is straightforward since $\tilde{\mathcal{T}}(\gamma)$ and $\tilde{\mathcal{D}}(\gamma)$ given in (10) are both linear functions of γ. If $\overline{\gamma}_i$, i=1,2, satisfy the inequalities:

$$1 > \tilde{\mathcal{D}}(\overline{\gamma}_i) > \max\{-\tilde{\mathcal{T}}(\overline{\gamma}_i)-1, \tilde{\mathcal{T}}(\overline{\gamma}_i)-1\},$$

so does γ_α given by (12). □

Now we are ready to examine a typical application in duopolistic dynamics.

4. An economic application

Since adaptive behaviors are widely observed in economic modeling, the applications of our findings should be quite broad. We shall just provide one of the most popular applications, from which we can see further how adaptive behavior fails to preserve the nice characteristics in heterogeneous agent models.

We start with the traditional Cournot model,[5] in which two firms X and Y, with cost function C_x and C_y, produce an identical product with quantity x_t and y_t at period t, respectively. The inverse market demand for the product is given by p_t=$D(q_t^d)$, where $D'(\cdot) \le 0$, with equality holding only at a finite number of points. The conventional assumption that q_t^d=x_t+y_t, i.e., *the actual market price adjusts to the demand so as to clear the market at every period* applies. It is assumed that each firm knows the market demand and behaves as a Cournot optimizer in the sense that it maximizes its expected profit based on its rival's expect output. Let $\hat{\pi}_t^x$ and $\hat{\pi}_t^y$ be the expected profits for Firms X and Y, respectively, that is,

$$\hat{\pi}_t^x = D(x_t + \hat{y}_t)x_t - C_x(x_t) \quad \text{and} \quad \hat{\pi}_t^y = D(\hat{x}_t + y_t)y_t - C_y(y_t).$$

Based on the first-order derivative conditions

$$D(x_t + \hat{y}_t) + D'(x_t + \hat{y}_t)x_t = C_x'(x_t),$$

and

$$D(\hat{x}_t + y_t) + D'(\hat{x}_t + y_t)y_t = C_y'(y_t),$$

the well-known Cournot reaction functions can be, respectively, derived:

$$\begin{cases} x_t = R_x(\hat{y}_t), \\ y_t = R_y(\hat{x}_t). \end{cases} \tag{13}$$

In the traditional Cournot model, both firms are assumed to take naive expectations of the next period's output, that is, $\hat{x}_t = x_{t-1}$ and $\hat{y}_t = y_{t-1}$, which when substituted into (13) yields a two-dimensional discrete dynamic process

$$\begin{cases} x_t = f(y_{t-1}) = R_x(y_{t-1}), \\ y_t = g(x_{t-1}) = R_y(x_{t-1}). \end{cases} \tag{14}$$

At the Cournot equilibrium ($\overline{x}$,$\overline{y}$), the Jacobian matrix takes a simple form due to f_x'=g_y'=0.

When two firms have different costs, that is, $C_x \ne C_y$, f_y' and g_x' may have opposite signs because

$$f_y' = \frac{D'(\overline{x}+\overline{y}) + \overline{x}D''(\overline{x}+\overline{y})}{C_x''(\overline{x}) - (2D'(\overline{x}+\overline{y}) + \overline{x}D''(\overline{x}+\overline{y}))},$$

$$g_x' = \frac{D'(\overline{x}+\overline{y}) + \overline{y}D''(\overline{x}+\overline{y})}{C_y''(\overline{y}) - (2D'(\overline{x}+\overline{y}) + \overline{y}D''(\overline{x}+\overline{y}))}.$$

It turns out that $\delta = -f_y'g_x' = -\mathcal{D}$ and $\mathcal{T} = 0$, so that the intertemporal equilibrium is stable if $|\delta| < 1$.

For the unstable equilibrium, consider the case in which $\delta < -1$, adopting adaptive adjustment by either or both firm(s) (with appropriate adjustment speeds γ_x and γ_y, respectively) may stabilize the equilibrium.[6] However, due to the fact that f_x'=g_y'=0 (corresponding to (7)), regardless of the values of f_y' and g_x', Propositions 1 and 2 together imply

[5] Since the model is discussed purely for the purpose of illustration, we neglect the second-order conditions, which may be beneficial to economic interpretations of the relevant conclusions.

[6] For the other possibility of unstable equilibrium ($\delta > 1$), it can be verified straightforwardly that no combinations of γ_x and γ_y are able to stabilize the equilibrium.

Proposition 3. *For the traditional duopolistic dynamics* (14), *at any intertemporal equilibrium* $(\overline{x},\overline{y})$, *the adaptive adjustment always improves the local stability of the equilibrium. Moreover, all favorable characteristics of the adaptive adjustment are well preserved. That is*,

(i) *destabilizing effects can never appear*;
(ii) *adaptive adjustment exhibits low-speed effectiveness*; *and*
(iii) *for each individual adjustment speed*, *the stabilizing regime is convex.*

Now consider a heterogenous duopoly model where two firms adopt different production strategies.

At first, we assume that, either due to deficiency in market information or being less strategic in business competition, Firm X behaves as a price-taker in the sense that its target output x_t is determined through equating the marginal cost $C_x'(x_t)$ to the naive price expectation $\hat{p}_t^x = p_{t-1} = D(x_{t-1}+y_{t-1})$ so that its output is determined from

$$x_t = f(x_{t-1},y_{t-1}) = MC_x^{-1}(D(x_{t-1}+y_{t-1})), \tag{15}$$

where MC_x^{-1} indicates the inverse function of the marginal cost $C_x'(x_t)$.

Secondly, Firm Y has a general reaction function *depending on the last periods outputs of both firms*, that is,

$$y_t = g(x_{t-1},y_{t-1}). \tag{16}$$

If Firm Y intends to implement an adaptive adjustment with a speed γ to stabilize an equilibrium of (14), we can have the following conclusions.

Proposition 4. *For the heterogenous duopoly model given by* (15) *and* (16), *at any intertemporal equilibrium* $(\overline{x},\overline{y})$ *with* $|D'(\overline{x}+\overline{y})| \neq 0$,

(i) *if* $|D'(\overline{x}+\overline{y})| < C_x''(\overline{x})$, *the adaptive adjustment for any speed* γ *always improves the local stability of the equilibrium and*
(ii) *if* $|D'(\overline{x}+\overline{y})| \geq C_x''(\overline{x})$, *an originally stable equilibrium may be destroyed if the adjustment speed* γ *is too slow.*

Proof. From

$$C_x'(f(x_{t-1},y_{t-1})) = D(x_{t-1}+y_{t-1})$$

we have

$$f_x'(x_{t-1},y_{t-1}) = f_y'(x_{t-1},y_{t-1}) = D'(x_{t-1}+y_{t-1})/C_x''(f(x_{t-1},y_{t-1})).$$

At the equilibrium $(\overline{x},\overline{y})$, It turns out

$$f_x'(\overline{x},\overline{y}) = f_y'(\overline{x},\overline{y}) = D'(\overline{x}+\overline{y})/C_x''(\overline{x}) < 0. \tag{17}$$

According to Propositions 1 and 2, so long as $f_x'(\overline{x},\overline{y}) > -1$, adaptive adjustment always expands the stabilizing regime. Otherwise, if $f_x'(\overline{x},\overline{y}) \leq -1$, both the destabilizing effect and low-speed failure can occur. □

The implications of Proposition 4 can be further illustrated with the following widely applied symmetric *linear model*, that is, linear market demand ($D(Q) = 1-Q$) and identically linear marginal cost for both firms ($C_x(q) = C_y(q) = C(q) = cq^2/2, c>0$). Since $|D'| \equiv 1$, Proposition 4 implies that the possibility of low-speed failure occurs only when $c<1$. We now consider two typical behaviors of Firm Y.

Case I (*Cournot best-response* (*with accurate information*)): Firm Y commands complete information about Firm X's output x_t and behaves as a traditional Cournot maximizer. In other words, its output is determined by

$$y_t = g(x_{t-1},y_{t-1}) = R_y(x_t) = R_y(f(x_{t-1},y_{t-1})), \tag{18}$$

where R_y is the Cournot best-response function in (13) and f is defined in (15).

For the linear model, Eqs. (15) and (18) yield

$$\left.\begin{aligned} x_t &= f(x_{t-1},y_{t-1}) = (1-x_{t-1}-y_{t-1})/c \\ y_t &= g(x_{t-1},y_{t-1}) = \frac{c-1+x_{t-1}+y_{t-1}}{c(c+2)} \end{aligned}\right\}, \tag{19}$$

with $c>0$ so that the Jacobian associated becomes

$$\mathcal{J} = \begin{bmatrix} f_x' & f_y' \\ g_x' & g_y' \end{bmatrix} = \begin{bmatrix} -1/c & -1/c \\ \dfrac{1}{c(c+2)} & \dfrac{1}{c(c+2)} \end{bmatrix},$$

which results in $\mathcal{T} = f_x' + g_y' = -(c+1)/(c(c+2))$ and $\mathcal{D} = f_x'g_y' - f_y'g_x' = 0$.

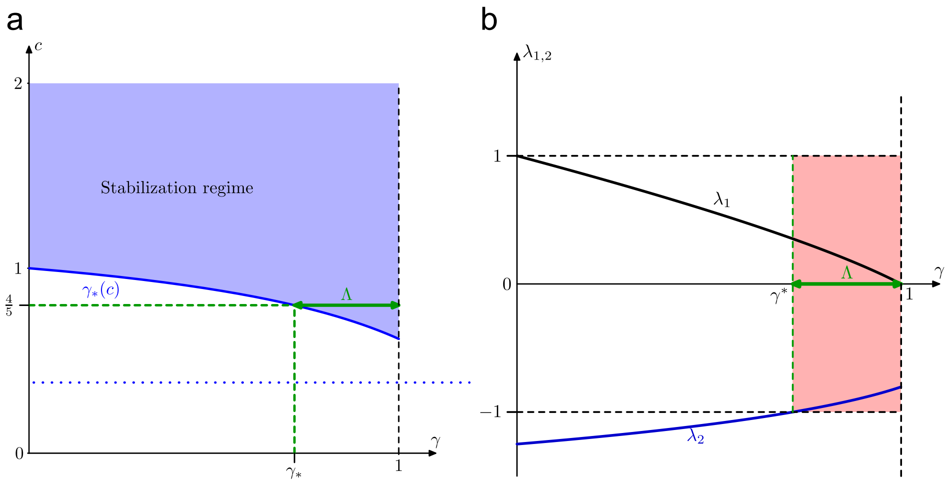

Fig. 4. Low-speed failure. (a) stabilizing regime and (b) eigenvalues for $c=4/5$.

If adaptive adjustment is implemented to y alone with a speed γ, then it follows from (11) that the stabilizing regime, denoted by $\mathbf{\Lambda}_1(\gamma)$, is given by

$$\mathbf{\Lambda}_1: \begin{cases} \gamma_*(c) \leq \gamma < 1 & \text{for } 1 > c > c_1, \\ 0 < \gamma < 1 & \text{for } c \geq 1, \end{cases}$$

where $\gamma_*(c) = 2(c+2)(1-c)/(3-c-c^2)$ and $c_1 = (\sqrt{5}-1)/2$.

Fig. 4(a) depicts the stabilizing regime Λ_1 in the γ-c space, which confirms that the destabilizing effect occurs with $c \leq 1$. To get a visual impression of such destabilizing effect, the evolution of the two eigenvalues against γ is plotted in Fig. 4(b) for $c=\frac{4}{5}$.

Case II (*Optimal mark-up*): Firm Y produces proportionally to the price-taker's output at last period, that is,

$$y_t = (1+\alpha)x_{t-1}, \tag{20}$$

where the ratio α is a positive constant that is set to maximize the sales revenue in equilibrium.[7] Let $(\overline{x}(\alpha), \overline{y}(\alpha))$ denote the equilibrium outputs. Then it can be verified that, for the linear model, we have $(\overline{x}(\alpha),\overline{y}(\alpha)) = (1/(c+\alpha+2),(1+\alpha)/(c+\alpha+2))$ so that the equilibrium sale's revenue of Firm Y is

$$S^y(\alpha) = \overline{y}D(\overline{x}+\overline{y}) = c(1+\alpha)/(c+\alpha+2)^2.$$

Maximizing $S^y(\alpha)$ demands $\alpha = c$. Substituting $\alpha = c$ into the original dynamics formulated with Eqs. (15) and (20) leads to

$$\left.\begin{aligned} x_t &= f(x_{t-1},y_{t-1}) = (1-x_{t-1}-y_{t-1})/c \\ y_t &= g(x_{t-1},y_{t-1}) = (1+c)x_{t-1} \end{aligned}\right\},$$

so that the Jacobian changes to

$$\mathcal{J} = \begin{bmatrix} -1/c & -1/c \\ 1+c & 0 \end{bmatrix}.$$

The stabilizing regime turns out to be

$$\mathbf{\Lambda_2}: \ \max\{0,\gamma_*(c)\} < \gamma < \min\{1,\gamma^*(c)\},$$

where $\gamma_*(c) = 1-c$ and $\gamma^*(c) = (c+1)/(c+2)$. In particular, $\gamma_*(c)$ and $\gamma^*(c)$ intersect at $c = c^* \triangleq \sqrt{2}-1$.

This case is interesting because it can be seen from Fig. 5(a) that the equilibrium is unstable either for low-speed or full speed. Without adaptive adjustment ($\gamma=1$), the equilibrium is unstable for all c. A simple examination of the eigenvalues help to provide a first-hand impression about what happens. In fact, for arbitrary c and γ, the two eigenvalues are given by

$$\lambda_{1,2}(\gamma) = \frac{1}{2c}\left(c(1-\gamma)-1 \pm \sqrt{c(1-\gamma)(4c+c(1-\gamma)+6)-4c^2-4c+1}\right),$$

and $\lambda_2(\gamma) = -1$ at $\gamma_*(c)$, which is only possible for $c<1$.

[7] Although this type of response is non-standard in the analysis of duopolistic competition, it is discussed here purely for its simplicity in terms of model structure.

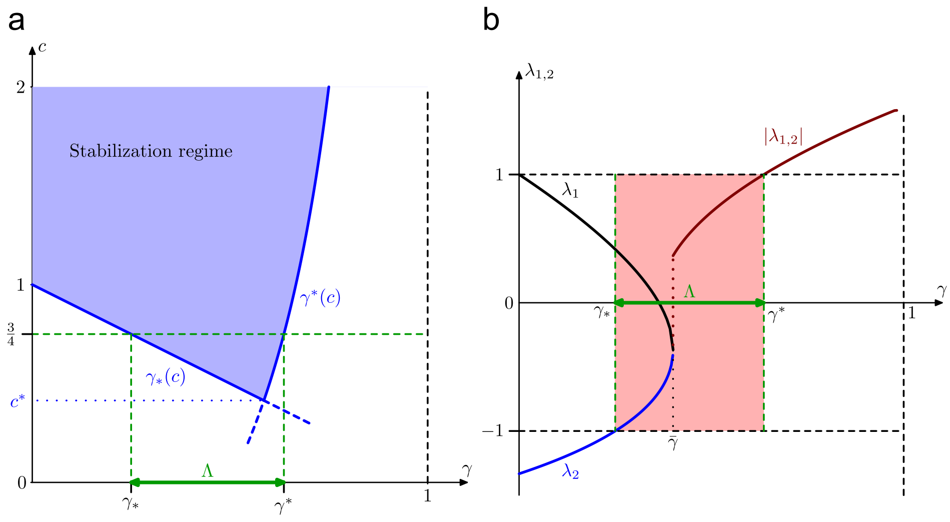

Fig. 5. Disjoint destabilization regimes. (a) stabilizing regime and (b) eigenvalues for $c = 3/4$.

When $\gamma > \overline{\gamma} = (3-2\sqrt{2})(c+1)/c$, which is only possible for $c > \overline{c} = (\sqrt{2}-1)/2$, these two characteristic roots become complex with a module given by $|\lambda_{1,2}(\gamma)| = \sqrt{c(c+1-\gamma(c+2))}/c$. Therefore, $|\lambda_{1,2}(\gamma)| \geq 1$ if $\gamma \geq \gamma^*(c)$. In particular, we have $\lambda_1(0) = 1$ and $\lambda_2(0) = -1/c < -1$ when $c < 1$. On the other hand, $\lambda_{1,2}(1) = (-1 \pm \sqrt{1-4c^2-4c})/(2c)$, which become complex when $c > \overline{c}$ with $|\lambda_{1,2}(1)| = \sqrt{c(c+1)}/c > 1$. Fig. 5(b) provides a plot of $\lambda_{1,2}$ against γ for $c = 3/4$.

5. Non-convexity of the stabilizing regime

Although the convexity of the stabilizing regime can still be preserved for $n=2$, it might not hold in general for $n \geq 3$. To show this, it is sufficient to construct a counter-example.

For a three-dimensional discrete process with a vector of adjustment speeds $\Gamma = (\gamma_x, \gamma_y, \gamma_z)$:

$$\left.\begin{aligned} x_{t+1} &= (1-\gamma_x)x_t + \gamma_x f(x_t, y_t, z_t) \\ y_{t+1} &= (1-\gamma_y)y_t + \gamma_y g(x_t, y_t, z_t) \\ z_{t+1} &= (1-\gamma_z)z_t + \gamma_z h(x_t, y_t, z_t) \end{aligned}\right\},$$

the Jacobian matrix at a given equilibrium $(\overline{x}, \overline{y}, \overline{z})$ takes the following form:

$$\mathcal{J}(\Gamma) = \begin{bmatrix} (1-\gamma_x)+\gamma_x f'_x & \gamma_x f'_y & \gamma_x f'_z \\ \gamma_y g'_x & (1-\gamma_y)+\gamma_y g'_y & \gamma_y g'_z \\ \gamma_z h'_x & \gamma_z h'_y & (1-\gamma_z)+\gamma_z h'_z \end{bmatrix}_{(\overline{x},\overline{y},\overline{z})}. \tag{21}$$

Let

$$P(\Gamma) = \lambda^3 + c_1(\Gamma)\lambda^2 + c_2(\Gamma)\lambda + c_3(\Gamma) \tag{22}$$

be the characteristic polynomial corresponding to (21). It is straightforward to verify that each coefficient c_i, $i=1, 2, 3$, can be expressed as a linear function of either element of Γ. In other words, $c_i(\Gamma)$, $i=1, 2, 3$, can be alternatively expressed as

$$c_i(\Gamma) = a_i^{(1)}(\gamma_y, \gamma_z) + b_i^{(1)}(\gamma_y, \gamma_z)\gamma_x = a_i^{(2)}(\gamma_x, \gamma_z) + b_i^{(2)}(\gamma_x, \gamma_z)\gamma_y = a_i^{(3)}(\gamma_x, \gamma_y) + b_i^{(3)}(\gamma_x, \gamma_y)\gamma_z.$$

The famous Samuelson necessary and sufficient conditions for all roots of $P(\Gamma) = 0$ to lie inside the unity circle of the complex space can be simplified to the following three inequalities (Wen et al., 2002):

$$\begin{cases} H_1(\Gamma) \triangleq 1 - |c_3(\Gamma)| > 0, \\ H_2(\Gamma) \triangleq 1 + c_2(\Gamma) - |c_1(\Gamma) + c_3(\Gamma)| > 0, \\ H_3(\Gamma) \triangleq 1 - c_2(\Gamma) + c_3(\Gamma)(c_1(\Gamma) - c_3(\Gamma)) > 0. \end{cases} \tag{23}$$

Fixing any two speeds of Γ, say $\gamma_x = \overline{\gamma}_x$ and $\gamma_y = \overline{\gamma}_y$, while $H_1(\Gamma)$ and $H_2(\Gamma)$ in (23) remain as linear functions of the remaining element γ_z, $H_3(\gamma)$ is generally a quadratic function of γ_z. Therefore, so long as $H_1(\Gamma) > 0$ and $H_2(\Gamma) > 0$ can be ensured for most value of γ_z and $H_3(\Gamma)$ becomes a convex function of γ_z that satisfies the following

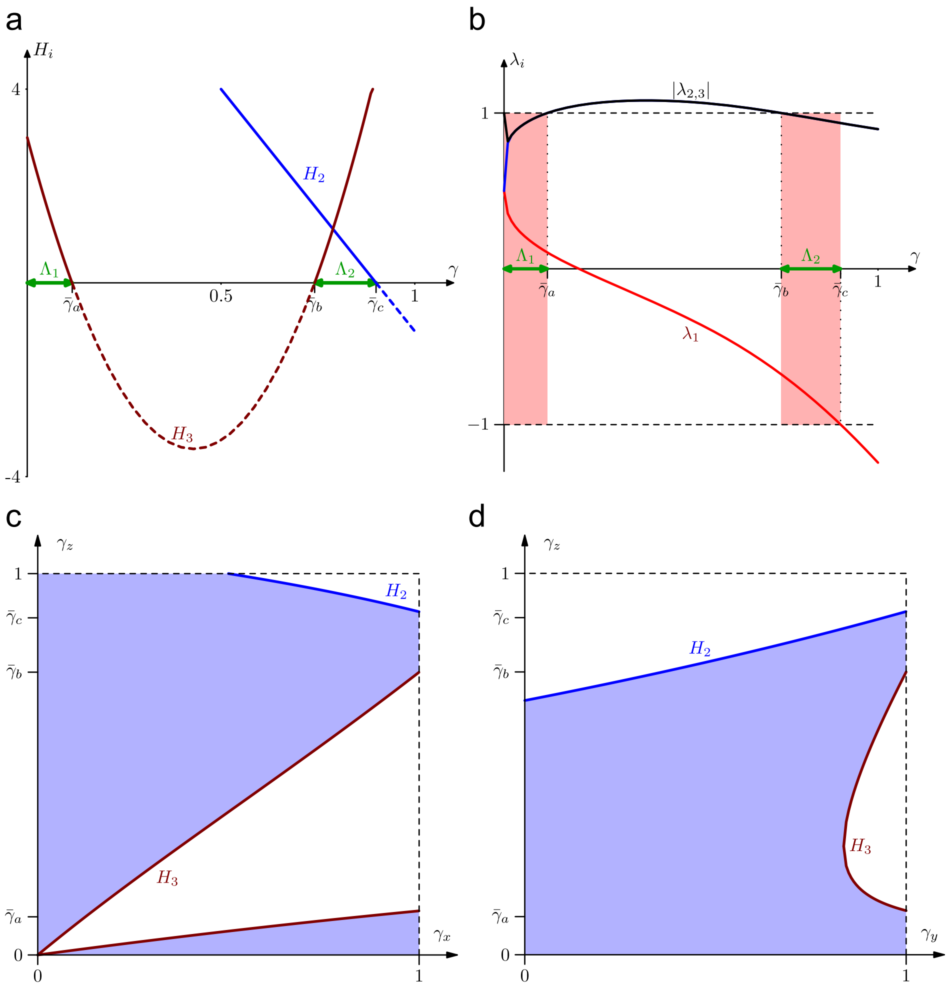

Fig. 6. Non-convexity of stabilizing regime. (a) stabilizing regime for (a) $\Gamma_1=(1,1,\gamma_z)$, (b) eigenvalues for $\Gamma_1=(1,1,\gamma_z)$, (c) stabilizing regime for $\Gamma_2=(\gamma_x,1,\gamma_z)$ and (d) stabilizing regime for $\Gamma_3=(1,\gamma_y,\gamma_z)$.

additional inequalities:

$$H_3(\overline{\gamma}_x,\overline{\gamma}_y,0)>0,$$

$$H_3(\overline{\gamma}_x,\overline{\gamma}_y,1)>0,$$

$$\min_{\gamma_z\in(0,1)} H_3(\overline{\gamma}_x,\overline{\gamma}_y,\gamma_z)<0,$$

then the stabilizing regime in terms of γ_z consists of two disjoint convex subsets in I.

For instance, consider an original Jacobian matrix given by

$$\mathcal{J}=\begin{bmatrix} 0 & -\alpha/4 & -\alpha/3 \\ 1/\alpha & 1 & -3/2 \\ 0 & 3/2 & -2 \end{bmatrix}.$$

It can be verified that for arbitrary $\alpha\neq 0$, $\mathcal{J}$ produces an identical characteristic polynomial that is independent of α: $p(\lambda)=\lambda^3+\lambda^2+\lambda/2+1=0$, which has three distinct eigenvalues: $\lambda_1=-1.2442$ and $\lambda_{2,3}=0.1221\pm 0.8882\mathbf{i}$.

When all three variables are adjusted independently, the Jacobian changes to

$$J(\Gamma)=\begin{bmatrix} 1-\gamma_x & -\gamma_x\alpha/4 & -\gamma_x\alpha/3 \\ \gamma_y/\alpha & 1 & -3\gamma_y/2 \\ 0 & 3\gamma_z/2 & 1-3\gamma_z \end{bmatrix},$$

so that the characteristic polynomial (22) has the following coefficients:

$$c_1(\Gamma)=3\gamma_z+\gamma_x-3,$$

$$c_2(\Gamma)=9\gamma_y\gamma_z/4+\gamma_x\gamma_y/4-2\gamma_x+3\gamma_x\gamma_z-6\gamma_z+3,$$

$$c_3(\Gamma)=\gamma_x+3\gamma_z-3\gamma_x\gamma_z-\gamma_x\gamma_y/4-9\gamma_y\gamma_z/4+7\gamma_x\gamma_y\gamma_z/2-1.$$

Case (i): $\Gamma=\Gamma_1=(1,1,\gamma_z)$ (only z_t is adjusted).

When only z_t is adjusted, polynomial (22) simplifies to

$$P(\Gamma_1)=\lambda^3+(3\gamma_z-2)\cdot\lambda^2+(5-3\gamma_z)/4\cdot\lambda+(5\gamma_z-1)/4.$$

Since $|c_3(\Gamma_1)|=|(5\gamma_z-1)/4|<1$ for all $\gamma_z\in I$, the conditions given in (23) then simplify to

$$H_2(\Gamma_1)=9-10\gamma_z>0 \quad\text{and}\quad H_3(\Gamma_1)=3-30\gamma_z+35\gamma_z^2>0.$$

As illustrated in Fig. 6(a), the stabilizing regime in terms of γ_z is a union of two disjointed intervals, i.e.,

$$\mathbf{\Lambda}(\Gamma_1)=\mathbf{\Lambda}_1\cup\mathbf{\Lambda}_2=(0,\gamma_a)\cup(\gamma_b,\gamma_c).$$

The effects of γ_z on the eigenvalues λ_i, i=1,2,3, solved from $P(\Gamma_1)=0$ are illustrated in Fig. 6(b).

We can similarly examine the joint effects when two variables are *adjusted independently*. The subtle interplay between γ_z and other two adjustment speeds can be seen clearly from the next two cases.

Case (ii): $\Gamma=\Gamma_2=(\gamma_x,1,\gamma_z)$ (i.e., y_t is not adjusted).

The stability conditions (23) simplify to

$$H_2(\Gamma_2)>0\Longrightarrow\gamma_z<\frac{16-7\gamma_x}{5(3-\gamma_x)},$$

$$H_3(\Gamma_2)>0\Longrightarrow\gamma_z<\frac{\gamma_x(2\gamma_x+13-u(\gamma_x))}{(2\gamma_x+3)(9-2\gamma_x)}\ \text{or}\ \gamma_z>\frac{\gamma_x(2\gamma_x+13+u(\gamma_x))}{(2\gamma_x+3)(9-2\gamma_x)},$$

where $u(\gamma_x)=2\sqrt{4\gamma_x^2+4\gamma_x+22}$.

Case (iii): $\Gamma=\Gamma_3=(1,\gamma_y,\gamma_z)$ (i.e., x_t is not adjusted).

The stability conditions (23) simplify to

$$H_2(\Gamma_3)>0\Longrightarrow\gamma_z<\frac{8+\gamma_y}{2(6-\gamma_y)},$$

$$H_3(\Gamma_3)>0\Longrightarrow\begin{array}{l}\text{either } \gamma_z<\dfrac{v(\gamma_y)-w(\gamma_y)}{5\gamma_y(12-5\gamma_y)}\ \text{or}\ \gamma_z>\dfrac{v(\gamma_y)+w(\gamma_y)}{5\gamma_y(12-5\gamma_y)},\\ (\text{for } 0.83631\le\gamma_y<1),\end{array}$$

where $v(\gamma_y)=44\gamma_y-24-5\gamma_y^2$ and $w(\gamma_y)=2\sqrt{484\gamma_y^2-70\gamma_y^3-528\gamma_y+144}$.

The non-convexity of stabilizing regimes for Cases (ii) and (iii) are illustrated in Fig. 6(c) and (d), respectively.

6. Concluding remarks

We have shown either in theory or by example the possibility of the absence of the three favorable features of adaptive mechanisms in multi-dimensional dynamic processes: non-destabilizing effect, low-speed effectiveness and convexity of the stabilizing regime.

Instead of the endogenous variable themselves, adaptations are sometimes taken with some intermediate approximations of them (such as expectations, forecasting, and/or estimations). These intermediate variables in turn affect the endogenous variables. Mathematically, the so-called internally adaptive adjustment is characterized as a combination of two related recursive procedures:

$$x_{i,t+1}=g_i(x_{i,t+1}^e,\{\mathbf{x}_{t+1-j}\}_{j=1}^{\infty},\{\mathbf{y}_{t+1-j}\}_{j=0}^{\infty}),$$

$$x_{i,t+1}^e=x_{i,t}+\gamma_i(x_{i,t+1}^*-x_{i,t}),\quad i=1,2,\ldots,n,$$

for i=1, 2,…,n, where γ_i are adjustment speeds. The adaptive adjustment studied in the present paper is essentially of external characteristic and can be regarded a special case of the more general internally adaptive adjustment. In terms of complexity, the former is much simpler than the latter. Therefore, any salient feature manifested in the former is likely to be repeated in the latter. So all conclusions drawn in this study apply to internally adaptive adjustment mechanisms as well.

Further research may be directed to the exploration of the general necessary or sufficient conditions for observing and not observing the favorable characteristics in multi-dimensional dynamics. For instance, it is conjectured that $\mathrm{Re}(\lambda_j)<1$ for all $j=1, 2,\ldots,n$ are the necessary conditions for low-speed effectiveness. On the other hand, in many economic applications, Jacobian matrices often appear with special structures. Typical instances include non-negative matrix (or Metzlerian matrix), non-positive matrix, symmetric matrix and recursive matrix. Necessary and/or sufficient conditions can be identified for the preservation of the three favorable characteristics of adaptive mechanism.

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