Exercise 4.6 *For a second-order discrete system given by*

$$x_t + a_1 x_{t-1} + a_2 x_{t-2} = 0,$$

show that the following inequalities constitute a set of necessary and sufficient conditions for the convergency of the system:

$$\left.\begin{array}{c} 1 + a_1 + a_2 > 0, \\ 1 - a_2 > 0, \\ 1 - a_1 + a_2 > 0. \end{array}\right\} \tag{4.51}$$

and illustrate the above inequalities in the a_1-a_2 plane.

$$x_{t+2} + a_1 x_{t+1} + a_2 x_t = 0.$$

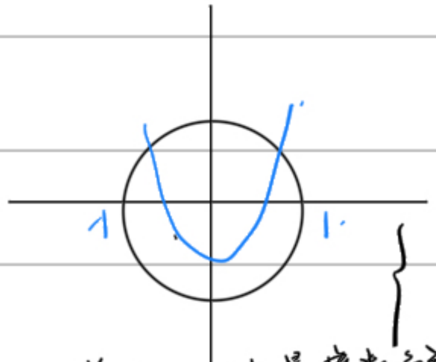
$$x_{t+1} = y_t. \qquad x_{t+2} = y_{t+1}.$$

$$\begin{cases} x_{t+1} = y_t \\ y_{t+1} = -a_2 x_t - a_1 y_t. \end{cases}$$

$$J = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \qquad |\lambda I - J| = \begin{vmatrix} \lambda & -1 \\ a_2 & \lambda + a_1 \end{vmatrix}.$$

$$= \lambda(\lambda + a_1) + a_2$$

characteristic equation. $\quad = \lambda^2 + a_1\lambda + a_2 = 0.$

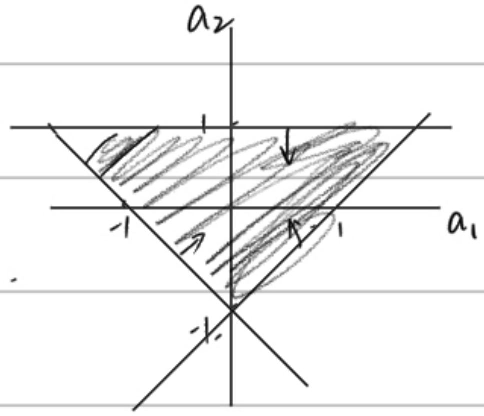


$$\begin{cases} \lambda = 1 & 1 + a_1 + a_2 > 0 \\ \lambda = -1 & 1 - a_1 + a_2 > 0. \end{cases}$$

当 $a_2>0$ 时是虚数解. $\lambda_{1,2} = \dfrac{-a_1 \pm i\sqrt{4a_2 - a_1^2}}{2}$

$$|\lambda|^2 = \frac{a_1^2}{4} + \frac{4a_2 - a_1^2}{4} = a_2. \quad \sqrt{a_2} < 1.$$

$$a_2 < 1.$$



Exercise 4.11 *In a dynamical duopoly game, two firms X and Y produce a homogeneous product with output x_t and y_t, respectively. The inverse market demand is $p_t = 1/(x_t + y_t)$, where p_t is the market price, and the cost function is identical for both firms: $C(q_t) = \frac{\sigma^2}{2} q_t^2$, $q = x, y$. X is a price-taker who determines its output by equating its marginal cost to the last period's price level. Y maximizes its profit with full-knowledge of X's current production.*

i) Verify that the output trajectory $\{x_t, y_t\}$ follows the following recursive relationship:

$$x_t = f(x_{t-1}, y_{t-1}) = \frac{1}{\sigma^2 (x_{t-1} + y_{t-1})}$$

$$y_t = g(x_t),$$

i). $MC_x = P_{t-1}$.

$MC_x = \frac{d\,C(x)}{dx} = \sigma^2 x = P_{t-1}$.

$\sigma^2 x_t = \frac{1}{x_{t-1} + y_{t-1}}$

$\therefore \quad x_t = f(x_{t-1}, y_{t-1}) = \frac{1}{\sigma^2 (x_{t-1} + y_{t-1})}$

y_t : max profit $\Pi_y = P_t y_t - C(y_t)$

$= \frac{y_t}{x_t + y_t} - \frac{\sigma^2}{2} y_t^2$

$\frac{d\Pi_y}{dy_t} = \frac{x_t + y_t - y_t}{(x_t + y_t)^2} - \sigma^2 y_t = \frac{x_t}{(x_t + y_t)^2} - \sigma^2 y_t = 0$.

$\Rightarrow y_t = g(x_t)$

where g is an implicit function solved from the identity

$$\frac{x_t}{(x_t+y_t)^2} = \sigma^2 y_t.$$

ii) Show that the system has an unique equilibrium $(\bar{x}, \bar{y})$ such that $0 < \bar{y} < \bar{x}$.
iii) Show that the equilibrium $(\bar{x}, \bar{y})$ is always stable regardless of the value σ;

(ii). $x_t = \frac{1}{\sigma^2(x_{t-1}+y_{t-1})}$ $\Rightarrow$ $\bar{x}, \bar{y}$. $\bar{x} = \frac{1}{\sigma^2(\bar{x}+\bar{y})}$ $\bar{x} = \frac{\bar{x}+\bar{y}}{\sigma^2(\bar{x}+\bar{y})^2}$.

$\sigma^2 y_t = \frac{x_t}{(x_t+y_t)^2}$. $\sigma^2\bar{y} = \frac{\bar{x}}{(\bar{x}+\bar{y})^2}$. $\bar{y} = \frac{\bar{x}}{\sigma^2(\bar{x}+\bar{y})^2}$

$\frac{\bar{x}}{\bar{y}} = \frac{\bar{x}+\bar{y}}{\bar{x}} > 1$ $\sigma^2\bar{y} = \frac{\bar{x}}{\left(\frac{1}{\sigma^2\bar{x}}\right)^2}$.

$\bar{x} + \bar{x}^3\sigma^2 = \frac{1}{\sigma^2\bar{x}}$ $\sigma^2\bar{y} = \bar{x}^3\sigma^4$.

$\sigma^2\bar{x}^2 + \sigma^4\bar{x}^4 = 1$. $\bar{y} = \bar{x}^3\sigma^2$.

$\bar{x}^2 = m$ $\sigma^4 m^2 + \sigma^2 m - 1 = 0$.

$$m = \frac{-\sigma^2 \pm\sqrt{\sigma^4+4\sigma^4}}{2\sigma^4} = \frac{\sigma^2\pm\sqrt{5}\sigma^2}{2\sigma^4}$$

$\because m = \bar{x}^2 > 0$ $\therefore m = \bar{x}^2 = \frac{(\sqrt{5}-1)}{2\sigma^2}$. $\bar{x} = \sqrt{\frac{\sqrt{5}-1}{2\sigma^2}}$ $\bar{y} = \bar{x}\cdot\sigma^2\bar{x}^2 = \frac{\sqrt{5}-1}{2}\bar{x}$.

$\frac{\sqrt{5}-1}{2} < 1$ $\therefore \bar{y} < \bar{x}$.

(iii). $\frac{\partial x_t}{\partial x_{t-1}} = \frac{\partial x_t}{\partial y_{t-1}} = \frac{-1}{\sigma^2(x_{t-1}+y_{t-1})^2}$ at $(\bar{x},\bar{y})$ $= \frac{-1}{\sigma^2\left(\frac{1}{\sigma^2\bar{x}}\right)^2} = -\sigma^2\bar{x}^2$

$= \frac{\sqrt{5}-1}{2}$

$= -0.618$

$\sigma^2 y_t = \frac{x_t}{(x_t+y_t)^2}$ 两边对 x_t 求导.

$$\sigma^2\frac{\partial y_t}{\partial x_t} = \frac{(x_t+y_t)^2 - x_t 2(x_t+y_t)\left(1+\frac{\partial y_t}{\partial x_t}\right)}{(x_t+y_t)^4}$$

$$\sigma^2\frac{\partial y_t}{\partial x_t} = \frac{(x_t+y_t) - 2x_t\left(1+\frac{\partial y_t}{\partial x_t}\right)}{(x_t+y_t)^3}$$

$$\sigma^2(x_t+y_t)^3\frac{\partial y_t}{\partial x_t} = x_t+y_t - 2x_t\left(1+\frac{\partial y_t}{\partial x_t}\right)$$

$$\frac{\partial y_t}{\partial x_t} = \frac{y_t - x_t}{\sigma^2(x_t+y_t)^3+2x_t}$$ $\sigma^2\bar{x}^2 = 0.618$

at $(\bar{x},\bar{y})$ $= \frac{\sigma^2\bar{x}^3-\bar{x}}{2\bar{x}+\sigma^2\left(\frac{1}{\sigma^2\bar{x}}\right)^3} = \frac{\sigma^2\bar{x}^3-\bar{x}}{2\bar{x}+\frac{1}{\sigma^4\bar{x}^3}} = \frac{\sigma^2\bar{x}^2-1}{2+\frac{1}{(\sigma^2\bar{x}^2)^2}} \approx -0.0828$

$$\left.\frac{\partial y_t}{\partial x_{t-1}}\right|_{(\bar{x},\bar{y})} = \left.\frac{\partial y_t}{\partial x_t}\cdot\frac{\partial x_t}{\partial x_{t-1}}\right|_{(\bar{x},\bar{y})} = -0.618\times(-0.0828) = 0.0512$$

$$|J| = \begin{vmatrix} -0.618 & -0.618 \\ 0.0512 & 0.0512 \end{vmatrix} \qquad |\lambda I - J| = \lambda^2 + 0.5668\lambda = 0.$$

$$\lambda_1 = 0 \quad |\lambda_2 = -0.5668| < 1.$$

iv) Verify that at the equilibrium $(\bar{x}, \bar{y})$, X makes higher profit than Y.

v) How will the above conclusions be changed if Y does not know the output x_t and makes its production decision basing on the assumption of $x_t^e = x_{t-1}$?

iv): at $(\bar{x}, \bar{y})$,

$$\Pi_x = p\bar{x} - \frac{\sigma^2}{2}\bar{x}^2 = \frac{\bar{x}}{\bar{x}+\bar{y}} - \frac{\sigma^2}{2}\bar{x}^2$$

$$\Pi_y = p\cdot\bar{y} - \frac{\sigma^2}{2}\bar{y}^2 = \frac{\bar{y}}{\bar{x}+\bar{y}} - \frac{\sigma^2}{2}\bar{y}^2.$$

$$\bar{y} = \frac{\sqrt{5}-1}{2}\bar{x}.$$

$$\rightarrow \Pi_x - \Pi_y = \frac{\bar{x}-\bar{y}}{\bar{x}+\bar{y}} - \frac{\sigma^2}{2}(\bar{x}^2 - \bar{y}^2) \qquad \bar{x}+\bar{y} = \frac{1}{\sigma^2 \bar{x}}$$

$$= (\bar{x}-\bar{y})\left(\sigma^2\bar{x} - \frac{\sigma^2}{2}(\bar{x}+\bar{y})\right).$$

$$= \frac{1}{2}(\bar{x}-\bar{y})(2\sigma^2\bar{x} - \sigma^2(\bar{x}+\bar{y}))$$

$$= \frac{1}{2}(\bar{x}-\bar{y})\sigma^2(\bar{x}-\bar{y}) = \frac{\sigma^2}{2}(\bar{x}-\bar{y})^2$$

$$> 0.$$

v).

Exercise 4.12 *In Puu (1991), a duopoly model is established*

$$x_t = f(y_{t-1}) = \sqrt{y_{t-1}/a} - y_{t-1},$$
$$y_t = g(x_{t-1}) = \sqrt{x_{t-1}/b} - x_{t-1},$$

where $a \neq b$.

i) Please identify the stability regime in terms of parameters.

ii) Show that, when the system is unstable, it can always be stabilized through adaptive adjustment:

$$x_t = \alpha x_{t-1} + (1-\alpha) f(y_{t-1})$$
$$y_t = \beta y_{t-1} + (1-\beta) g(x_{t-1})$$

with suitable choice of α and β, where $0 \le \alpha, \beta \le 1$.

i).

$$x_t = f(y_{t-1}) = \sqrt{\frac{y_{t-1}}{a}} - y_{t-1}$$

$$y_t = \sqrt{\frac{x_{t-1}}{b}} - x_{t-1}.$$

$$J = \begin{pmatrix} 0 & \frac{1}{2a}\left(\frac{y_{t-1}}{a}\right)^{-\frac{1}{2}} - 1 \\ \frac{1}{2b}\left(\frac{x_{t-1}}{b}\right)^{-\frac{1}{2}} - 1 & 0 \end{pmatrix}.$$

$$\lambda^2 - \left(\frac{1}{a}\left(\frac{y_{t-1}}{a}\right)^{-\frac{1}{2}} - 1\right)\left(\frac{1}{b}\left(\frac{x_{t-1}}{b}\right)^{-\frac{1}{2}} - 1\right) = 0.$$

$\frac{1}{2b} a+b - 1.$

$$\bar{x} = \sqrt{\frac{\bar{y}}{a}} - \bar{y}$$
$$\bar{y} = \sqrt{\frac{\bar{x}}{b}} - \bar{x}.$$

$$\rightarrow \sqrt{\frac{\bar{y}}{a}} = \sqrt{\frac{\bar{x}}{b}}.$$

$$\frac{\bar{y}}{a} = \frac{\bar{x}}{b}.$$
$$\bar{y} = \frac{a\bar{x}}{b}.$$

$$\bar{x} = \sqrt{\frac{\bar{x}}{b}} - \frac{a\bar{x}}{b}.$$

$$\left(1 + \frac{a}{b}\right)^2 \bar{x}^2 = \frac{\bar{x}}{b}.$$

$$\bar{x} = \frac{b^2}{b(a+b)^2} = \frac{b}{(a+b)^2} \qquad \bar{y} = \frac{a}{(a+b)^2}.$$

$$J = \begin{pmatrix} 0 & \frac{b-a}{2a} \\ \frac{a-b}{2b} & 0 \end{pmatrix} \qquad |\lambda I - J| = \begin{vmatrix} \lambda & -\frac{b-a}{2a} \\ -\frac{a-b}{2b} & \lambda \end{vmatrix} = 0.$$

$$\lambda^2 - \frac{(a-b)(b-a)}{4ab} = 0.$$

$$\lambda = -\frac{(a-b)^2}{4ab}.$$

$$|\lambda| = \frac{|a-b|}{2\sqrt{ab}} < 1.$$

ii) Show that, when the system is unstable, it can always be stabilized through adaptive adjustment:

$$x_t = \alpha x_{t-1} + (1-\alpha) f(y_{t-1})$$
$$y_t = \beta y_{t-1} + (1-\beta) g(x_{t-1})$$

with suitable choice of α and β, where $0 \le \alpha, \beta \le 1$.

$$J = \begin{pmatrix} \alpha & \frac{1-\alpha}{2a}\left(\frac{y_{t-1}}{a}\right)^{-\frac{1}{2}} - (1-\alpha) \\ \frac{1-\beta}{2b}\left(\frac{x_{t-1}}{b}\right)^{-\frac{1}{2}} - (1-\beta) & \beta \end{pmatrix}$$

$$\frac{(a+b)(1-\alpha)}{2a} - (1-\alpha).$$

$$= \frac{(a+b)(1-\alpha) - (1-\alpha)2a}{2a}$$

$$\bar{x} = \alpha \bar{x} + (1-\alpha)\left(\sqrt{\frac{\bar{y}}{a}} - \bar{y}\right) \qquad \bar{x} = \sqrt{\frac{\bar{y}}{a}} - \bar{y}$$

$$\bar{y} = \beta \bar{y} + (1-\beta)\left(\sqrt{\frac{\bar{x}}{b}} - \bar{x}\right) \qquad \bar{y} = \sqrt{\frac{\bar{x}}{b}} - \bar{x}.$$

$$\bar{x} = \frac{b}{(a+b)^2} \quad \bar{y} = \frac{a}{(a+b)^2}$$

$$J = \begin{pmatrix} \alpha & \frac{(b-a)(1-\alpha)}{2a}. \\ \frac{(a-b)(1-\beta)}{2b} & \beta \end{pmatrix}$$

$$\lambda^2 - (\alpha+\beta)\lambda - \frac{(a-b)^2(1-\alpha)(1-\beta)}{4ab} = 0.$$

when original unstable

$$\frac{|a-b|}{2\sqrt{ab}} \geq 1.$$

$$\lambda_1 + \lambda_2 = \alpha + \beta < 2.$$

$$\lambda_1 \lambda_2 = -\frac{(a-b)^2(1-\alpha)(1-\beta)}{4ab}$$

$|\lambda_1 \lambda_2| < 1$ when $(a-b)^2 \geq 4ab$.

need $(1-\alpha)(1-\beta)$ to be small. $\alpha \to 1$ $\beta \to 1$.

by selecting α β close to 1, can adjust for stability.