Inverse z-transform:

The procedure for transforming from the z-domain to the time domain is called inverse z-transform. It's formula can be derived usin[g] cauchy integral theorem.

IF,

$$X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \quad \text{------} \quad (1)$$

Now ① Multiply both side by z^{n-1}

② Integrate both sides over a closed contour $[\oint_c]$ within the ROC of $X(z)$, which encloses the origin. Then equation ① becomes:

$$\oint_c X(z) z^{n-1} dz = \oint_c \sum_{k=-\infty}^{\infty} x(k) z^{+n-k-1} dz \quad \cdots ②$$

or, $$\oint_c X(z) z^{n-1} dz = \sum_{k=-\infty}^{\infty} x(k) \oint_c z^{n-k-1} dz \quad \cdots ③$$

Cauchy Integral theorem:

$$\frac{1}{2\pi j} \oint z^{n-1-k} dz = \begin{cases} 1, & k=n \\ 0, & k \neq n \end{cases} \quad \cdots ④$$

From ③ and ④

$$\oint_c X(z) z^{n-1} dz = 2\pi j x(n)$$

$$\boxed{\therefore x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz}$$

is the Inverse z-transform.

Property : 12. Multiplication of two sequences / Parseval's relation :

If,

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

Then,

$$x(n) = x_1(n) \cdot x_2(n) \xleftrightarrow{z} X(z) = \frac{1}{2\pi j} \oint_c X_1(v) X_2\left(\frac{z}{v}\right) v^{-1} dv$$

Proof:

We have,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{\left[\frac{1}{2\pi j} \oint_c X_1(v) v^{n-1} dv\right]}_{\text{IZT of } x_1(n)} x_2(n) z^{-n}$$

$$= \frac{1}{2\pi j} \oint_c X_1(v) \underbrace{\sum_{n=-\infty}^{\infty} x_2(n) \left(\frac{z}{v}\right)^{-n}}_{\text{z-transform of } X_2\left(\frac{z}{v}\right)} v^{-1} dv$$

or, $$\boxed{X(z) = \frac{1}{2\pi j} \oint X_1(v) X_2\left(\frac{z}{v}\right) v^{-1} dv}$$

Whis is the desired relation.

Note: For a complex-valued sequence $x_1(n)$ and $x_2(n)$. The product Sequence is,

$$x(n) = x_1(n) \cdot x_2^*(n). \text{ Then}$$

$$x(n) = x_1(n) \cdot x_2^*(n) \xleftrightarrow{z} X(z) = \frac{1}{2\pi j} \oint_c X_1(v) X_2^*\left(\frac{z^*}{v^*}\right) v^{-1} dv$$

at $z = 1$,

$$\boxed{X(z) = 1/2\pi j \oint_c X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv}$$

is called parseval's relation,

Transient and steady state Response:

Response of system with rational system function can be expressed as,

$H(z) = \frac{B(z)}{A(z)}$ where, $B(z)$ = numerator polynomial that contains zeroes.

$A(z)$ = denominator polynomial that contains poles.

If $x(n)$ is the input and its z-transform is expressed as:

$$X(z) = \frac{N(z)}{Q(z)}$$

If the system is initially relaxed i.e. the initial condition for the differential equation is zero. i.e. $y(-1) = y(-2) = y(-3) = \cdots = y(-N) = 0$.

Then, $X(z) = \frac{Y(z)}{H(z)}$

$$\Rightarrow Y(z) = H(z) \cdot X(z)$$

$$= \frac{B(z)}{A(z)} \times \frac{N(z)}{Q(z)}$$

We assume that, the zeros of the numerator polynomial $B(z)$ and $N(z)$ does not coinside with the poles of denominator polynominal $A(z)$ and $Q(z)$. [There is no pole-zero cancellation).

Then, a partial fraction expansion gives.

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1-q_k z^{-1}}$$

$$\Big\updownarrow \text{IZT}$$

$$\boxed{y(n) = \underbrace{A_k (p_k)^n u(n)}_{\text{natural response}} + \underbrace{Q_k (q_k)^n u(n)}_{\text{Forced response}}}$$

Causality and stability:

A causal linear time-invariant system is one whose unit sample response $h(n)$ satisfies the condition

$$h(n) = 0, \quad n < 0$$

We also know that the ROC of the z-transform of a causal sequence is the exterior of a circle. Consequently, a linear time-invariant system is causal if and only if the ROC of the system function is the exterior of a circle of radius $r < \infty$, including the point $z = \infty$.

The stability of a linear time-invariant system can also be expressed in terms of the characteristics of the system function. A necessary and sufficient condition for a linear time-invariant system to be BIBO stable is.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In turn, this condition implies that $H(z)$ must contain the unit circle within its ROC.

Indeed, since $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$

it follows that, $|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|$

When evaluated on the unit circle (i.e., $|z| < 1$),

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n)|$$

Hence, if the system is BIBO stable, the unit circle is contained in the ROC of $H(z)$. The converse is also true. Therefore a linear time-invariant system is BIBO stable if and only if the ROC of the system function includes the unit circle.

For a causal system, however, the condition on stability can be narrowed to some extent. Indeed, a causal system is characterized by a system function $H(z)$ having as a ROC the exterior of some circle of radius r. For a stable system, the ROC must include the unit circle. Consequently, a causal and stable system must have a system function that converges for $|z| > r < 1$. Since the ROC can not contain any poles of $H(z)$, it follows that a causal linear time-invariant system is BIBO stable if and only if all the poles of $H(z)$ are inside the unit circle.

Pole-zero cancellation:

When a z-transform has a pole that is at the same location as a zero, the pole is canceled by the zero and, consequently, the term containing that pole in the inverse z-transform vanishes. Such pole-zero cancellations are very important in the analysis of pole-zero systems.

Pole-zero cancellations can occur either in the system function itself or in the product of the system function with the z-transform of the input signal. In the first case we say that the order of the system is reduced by one. In the latter case we say that the pole of the system is suppressed by the zero in the input signal, or vice versa. Thus, by properly selecting the position of the zeros of the input signal, it is possible to suppress one or more system modes (pole factors) in the response of the system. Similarly, by proper selection of the zeros of the system function, it is possible to suppress one or more modes of the input signal from the response of the system.

When the zero is located very near the pole but not exactly at the same location, the term in the response has a very small amplitude.

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Properties of Z-Transform

① Linearity Property

If $x_1[n] \xleftrightarrow{Z.T} X_1(z)$ with ROC $= R_1$

$x_2[n] \xleftrightarrow{Z.T} X_2(z)$ " ROC $= R_2$

Then,

$$a x_1[n] + b x_2[n] \xleftrightarrow{Z.T} aX_1(z) + bX_2(z).$$

$\boxed{ROC = R_1 \cap R_2}$

Proof: We know that,

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Then,

$$Z\{a x_1[n] + b x_2[n]\} = \sum_{n=-\infty}^{\infty} \{a x_1[n] + b x_2[n]\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a x_1[n] z^{-n} + \sum_{n=-\infty}^{\infty} b x_2[n] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1[n] z^{-n} + b \sum_{n=-\infty}^{\infty} x_2[n] z^{-n}$$

$$= a X_1(z) + b X_2(z), \quad \boxed{ROC: R_1 \cap R_2}$$

Hence,

$$a x_1[n] + b x_2[n] \xleftrightarrow{Z.T} aX_1(z) + bX_2(z)$$

proved.

② Time Shifting Property

If, $x[n] \xleftrightarrow{Z.T} X(z)$

Then,

$$x[n-k] \xleftrightarrow{Z.T} z^{-k} X(z)$$

Proof:

We know that,

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

So,

$$Z\{x[n-k]\} = \sum_{n=-\infty}^{\infty} x[n-k] Z^{-n}$$

Let, $n-k = p \Rightarrow n = p+k$

So,

$$Z\{x[n-k]\} = \sum_{p=-\infty}^{\infty} x[p] Z^{-(p+k)}$$

$$= \sum_{p=-\infty}^{\infty} x[p] Z^{-p} Z^{-k}$$

$$= Z^{-k} \sum_{p=-\infty}^{\infty} x[p] Z^{-p}$$

$$= Z^{-k} X(z)$$

So,

$$x[n-k] \xleftrightarrow{Z.T} Z^{-k} X(z)$$

Similarly,

$$x[n+k] \xleftrightarrow{Z.T} Z^{k} X(z)$$

proved

③ Scaling Property

If $x[n] \xleftrightarrow{Z.T} X(z)$

Then,

$$a^n x[n] \xleftrightarrow{Z.T} X(Z/a)$$

Proof: We know that,

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n] Z^{-n}$$

So,

$$Z\{a^n x[n]\} = \sum_{n=-\infty}^{\infty} \{a^n x[n]\} Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \times \frac{1}{a^{-n}} \times Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \times \left(\frac{Z}{a}\right)^{-n}$$

$$= X(Z/a)$$

$$\therefore \quad a^n x[n] \xleftrightarrow{Z} X(Z/a)$$

proved

④ Time Reversal Property

→

If $x[n] \xleftrightarrow{Z.T} X(z)$

Then,

$$x[-n] \xleftrightarrow{Z.T} X(Z^{-1}) = X(1/Z)$$

Proof:

We know that,

$$Z\{x[n]\} = X(Z) = \sum_{n=-\infty}^{\infty} x[n] Z^{-n}$$

Now,

$$Z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] Z^{-n}$$

put $-n = p$

Then,

$$Z\{x[-n]\} = \sum_{p=-\infty}^{\infty} x[p] Z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x[p] \, (z^{-1})^{-p}$$

$$= X(z^{-1})$$

$$\therefore \quad x[-n] \xleftrightarrow{Z.T} X(z^{-1})$$

proved

⑤ <u>Differentiation property</u>

→

If $x[n] \xleftrightarrow{Z.T} X(z)$

Then,

$$n\,x[n] \xleftrightarrow{Z.T} -z \frac{d}{dz} X(z)$$

Proof:

We know that,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \, z^{-n}$$

Differentiating above eqⁿ w.r.t. z, we get,

$$\frac{d}{dz} X(z) = \frac{d}{dz} \left\{ \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right\}$$

$$= \left\{ \sum_{n=-\infty}^{\infty} x[n] \cdot \frac{d}{dz} z^{-n} \right\}$$

$$= \left\{ \sum_{n=-\infty}^{\infty} x[n] \cdot (-n) z^{-n-1} \right\}$$

$$= -n \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} \cdot z^{-1}$$

or, $\frac{d}{dz}X(z) = -nz^{-1}\sum_{n=-\infty}^{\infty} x[n]z^{-n}$

or, $-z\frac{d}{dz}X(z) = n\sum_{n=-\infty}^{\infty} x[n]z^{-n}$

or $n\sum_{n=-\infty}^{\infty} x[n]z^{-n} = -z\frac{d}{dz}X(z)$

$\therefore\ n\,x[n] \xleftrightarrow{Z.T} -z\frac{d}{dz}X(z)$

proved

⑥ Convolution Property

→ It states that,

$$y[n] = x[n] * h[n] \xleftrightarrow{Z.T} Y(z) = X(z)\cdot H(z)$$

Where,

$$y[n] = x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Proof:

We know that,

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$$

or, $Y(z) = \sum_{n=-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right\}z^{-n}$

Changing the order of summation we get

$$Y(z) = \sum_{k=-\infty}^{\infty} x[k]\sum_{n=-\infty}^{\infty} h[n-k]z^{-n}$$

put, $n-k=p \Rightarrow n=p+k$

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Then,

$$Y(z) = \sum_{k=-\infty}^{\infty} x[k] \sum_{p=-\infty}^{\infty} h[p] Z^{-(p+k)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] Z^{-k} \cdot \sum_{p=-\infty}^{\infty} h[p] Z^{-p}$$

$$= X(z) \cdot H(z)$$

Hence,

$$y[n] = x[n] * h[n] \xleftrightarrow{Z.T} X(z) \cdot H(z) = Y(z)$$

proved