Communication-Efficient Distributed Second-Order Optimization Methods for Generalized Convex Problems

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Every Minute There Are:

~500,000 Tweets Sent

~50,000 Photos Posted

~100,000 Hours Streamed

~4,000,000 Searches

~750,000 Songs Streamed

~2,000,000 Snaps

The Problem

Centralized Computing Environment

Worker node i with $f_i : \mathbb{R}^d \to \mathbb{R}$

Driver node

Goal

Over m workers, solve:

$$\min_{w \in \mathbb{R}^d} \left\{ f(w) \stackrel{\text{def}}{=} \frac{1}{m} \sum_{i=1}^{m} f_i(w) \right\}.$$

Use Case: Big Data Regimes

Distributively Working With a Very Large Dataset $\{x_i\}_{i=1}^n$

Worker i stores data indexed by set $S_i \subseteq \{1, \ldots, n\}$, i.e., stores $\{x_j\}_{j \in S_i}$

Why use Second-Order Methods?

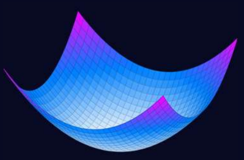
Second-order methods employ curvature (Hessian matrix) information to transform the gradient so that it is a more suitable direction to follow.

Benefits

- Perform more computations per iteration
- May take full advantage of available distributed computational resources
- May require significantly less communication costs
- Often require far fewer iterations to achieve similar results

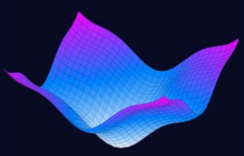
Related Work

Method	Applicable to Non-Convex Functions	Arbitrary Data Distribution	Arbitrary Form of f_i	Simple Sub-Problems	Not Sensitive to Hyper-Parameters
GIANT	✗	✗	✗	✓	✓
DiSCO	✗	✓	✓	✓	✓
DANE	✓	✓	✓	✗	✗
InexactDANE	✓	✓	✓	✗	✗
AIDE	✓	✓	✓	✗	✗
DINGO	✓	✓	✓	✓	✓



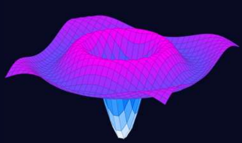
Convex

- Hessian is positive semidefinite.
- Local minima are global minima.



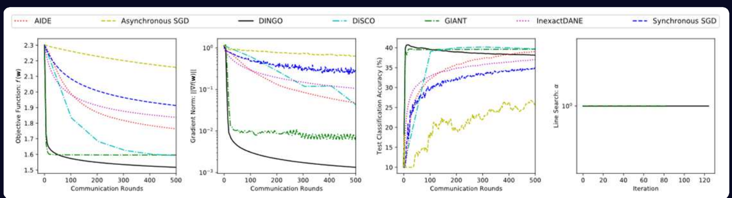
Invex

- Hessian can be indefinite and singular.
- Local minima are global minima.



Non-Convex

- Hessian can be indefinite and singular.
- Not all local minima are global minima.



Softmax regression, with regularization, problem on the CIFAR10 dataset.

Our Method: DINGO

Derived by optimization of the gradient's norm as a surrogate function, i.e.,

$$\min_{w \in \mathbb{R}^d} \left\{ \frac{1}{2} \|\nabla f(w)\|^2 = \frac{1}{2m^2} \left\| \sum_{i=1}^{m} \nabla f_i(w) \right\|^2 \right\}.$$

DINGO is for "**DI**stributed **N**ewton-type method for **G**radient-norm **O**ptimization". DINGO is particularly suitable for invex objectives. A strict linear-rate reduction in the gradient norm is always guaranteed.

Each Iteration of DINGO

w_t

$g_{t,i}$

$g_t = \frac{1}{m}\sum_{i=1}^m g_{t,i}$

$H_{t,i} g_t, H_{t,i}^\dagger g_t, \tilde{H}_{t,i}^\dagger \tilde{g}_t$

$\frac{1}{m}\sum_{i=1}^m H_{t,i}^\dagger g_t, \quad \frac{1}{m}\sum_{i=1}^m \tilde{H}_{t,i}^\dagger \tilde{g}_t, \quad H_t g_t = \frac{1}{m}\sum_{i=1}^m H_{t,i} g_t$

Update Direction: p_t

Case 1

If: $\langle \frac{1}{m}\sum_{i=1}^m H_{t,i}^\dagger g_t, H_t g_t \rangle \geq \theta \|g_t\|^2$ then $p_t = \frac{1}{m}\sum_{i=1}^m p_{t,i}$ with $p_{t,i} = -H_{t,i}^\dagger g_t$

Case 2

Else If: $\langle \frac{1}{m}\sum_{i=1}^m \tilde{H}_{t,i}^\dagger \tilde{g}_t, H_t g_t \rangle \geq \theta \|g_t\|^2$ then $p_t = \frac{1}{m}\sum_{i=1}^m p_{t,i}$ with $p_{t,i} = -\tilde{H}_{t,i}^\dagger \tilde{g}_t$

Case 3

Else: $p_{t,i} = -\tilde{H}_{t,i}^\dagger \tilde{g}_t$ for all $i \notin I_t \stackrel{\text{def}}{=} \{ i = 1, \ldots, m \mid \langle \tilde{H}_{t,i}^\dagger \tilde{g}_t, H_t g_t \rangle < \theta \|g_t\|^2 \}$

Send $H_t g_t$ to all workers $i \in I_t$

$p_{t,i} = -\tilde{H}_{t,i}^\dagger \tilde{g}_t - \lambda_{t,i} \left(\tilde{H}_{t,i}^T \tilde{H}_{t,i}\right)^{-1} H_t g_t, \qquad \lambda_{t,i} = \frac{-g_t^T H_t \tilde{H}_{t,i}^\dagger \tilde{g}_t + \theta \|g_t\|^2}{g_t^T H_t \left(\tilde{H}_{t,i}^T \tilde{H}_{t,i}\right)^{-1} H_t g_t}$

$p_t = \frac{1}{m}\sum_{i=1}^m p_{t,i}$

Line Search: α_t

Distributively choose largest $\alpha_t \in (0,1]$ such that $\|\nabla f(w_t + \alpha_t p_t)\|^2 \leq \|g_t\|^2 + 2\alpha_t \rho \langle p_t, H_t g_t \rangle$.

Example: Backtracking Line Search

p_t

Compute $\nabla f_i(w_t + \alpha p_t)$ $\forall \alpha \in \{1, \omega, \ldots, \omega^k\}$, where $\omega \in (0,1), k \in \mathbb{N}$

Let $\alpha_t = \max\{1, \ldots, \omega^k\}$ such that $\|\nabla f(w_t + \alpha_t p_t)\|^2 \leq \|g_t\|^2 + 2\alpha_t \rho \langle p_t, H_t g_t \rangle$

Update

$w_{t+1} = w_t + \alpha_t p_t$

The constants $\theta, \phi > 0$ and $\rho \in (0,1)$ are hyper-parameters. The vector $w_t \in \mathbb{R}^d$ denotes the point at iteration t. For notational convenience, we denote $g_{t,i} \stackrel{\text{def}}{=} \nabla f_i(w_t)$, $H_{t,i} \stackrel{\text{def}}{=} \nabla^2 f_i(w_t)$, $g_t \stackrel{\text{def}}{=} \nabla f(w_t)$, $H_t \stackrel{\text{def}}{=} \nabla^2 f(w_t)$. We also let

$$\tilde{H}_{t,i} \stackrel{\text{def}}{=} \begin{bmatrix} H_{t,i} \\ \phi I \end{bmatrix} \in \mathbb{R}^{2d \times d}, \qquad \tilde{g}_t \stackrel{\text{def}}{=} \begin{bmatrix} g_t \\ 0 \end{bmatrix} \in \mathbb{R}^{2d},$$

where I is the identity matrix and 0 is the zero vector. Green and purple rectangles represent the driver node and worker nodes, respectively.

References

1. Crane, R., & Roosta, F. (2019). DINGO: Distributed Newton-Type Method for Gradient-Norm Optimization. *arXiv preprint arXiv:1901.05134*.
2. https://www.domo.com/blog/data-never-sleeps-6