Understanding the Markowitz Portfolio Optimization Problem and Its Modern Solution Approaches

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1 Introduction

The problem of constructing an optimal investment portfolio lies at the heart of financial decision-making. Investors seek to maximize returns while controlling the level of risk they are exposed to. Harry Markowitz's seminal work on portfolio selection provided the first rigorous mathematical framework for addressing this trade-off. By modeling portfolio construction as an optimization problem involving expected return and risk, Markowitz laid the foundation of modern portfolio theory.

This report presents a conceptual understanding and personal interpretation of the Markowitz portfolio optimization problem. It focuses on the two equivalent mathematical formulations of the model and discusses why, despite its elegance, the classical framework often requires extensions and alternative solution methods in modern financial practice.

2 Core Idea of Markowitz Portfolio Theory

The central insight of Markowitz portfolio theory is that portfolio risk depends not only on the riskiness of individual assets, but also on how asset returns move relative to one another. This idea formalizes the concept of diversification.

Let $\mu \in \mathbb{R}^n$ denote the vector of expected asset returns and $\Sigma \in \mathbb{R}^{n \times n}$ the covariance matrix of asset returns. For a portfolio with weight vector $w \in \mathbb{R}^n$, the expected portfolio return and

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variance are given by

$$\mathbb{E}[R] = \mu^\top w, \tag{1}$$

$$\mathrm{Var}(R) = w^\top \Sigma w. \tag{2}$$

The investor's objective is therefore not to select assets independently, but to choose portfolio weights that appropriately balance expected return against risk. By explicitly accounting for co-variances, the model shows that combining risky assets can reduce overall portfolio risk when asset returns are imperfectly correlated.

3 Two Equivalent Formulations of the Markowitz Problem

The Markowitz portfolio optimization problem is commonly expressed using two closely related formulations. While these formulations differ in interpretation, they are mathematically equivalent and generate the same set of efficient portfolios.

3.1 Maximizing Return for a Target Risk Level

The first formulation maximizes expected portfolio return subject to a constraint on portfolio risk:

$$\max_{w} \quad \mu^\top w \tag{3}$$

$$\text{subject to} \quad w^\top \Sigma w \leq \sigma_{\text{target}}^2, \tag{4}$$

$$\mathbf{1}^\top w = 1, \tag{5}$$

where σ_{target} is a target level of portfolio volatility.

This formulation emphasizes explicit risk control. The investor specifies a maximum acceptable level of risk and then selects the portfolio that delivers the highest expected return within that risk budget. Varying σ_{target} traces out the efficient frontier.

3.2 Maximizing Risk-Adjusted Return

The second formulation combines risk and return directly into the objective function:

$$\max_{w} \quad \mu^\top w - \gamma w^\top \Sigma w \tag{6}$$

$$\text{subject to} \quad \mathbf{1}^\top w = 1, \tag{7}$$

where $\gamma > 0$ is a risk aversion parameter.

In this formulation, higher values of γ place greater emphasis on risk reduction, leading to lower-risk portfolios, while smaller values allow for higher risk in pursuit of higher expected returns.

3.3 Equivalence of the Two Formulations

Although these formulations appear different, they are mathematically equivalent. The equivalence can be seen by forming the Lagrangian of the risk-constrained problem, where the Lagrange multiplier associated with the variance constraint plays the role of the risk aversion parameter γ in the risk-adjusted formulation.

For every feasible target volatility σ_{target}, there exists a corresponding value of γ that yields the same optimal portfolio, and vice versa. Varying either parameter traces out the same efficient frontier. The distinction between the two formulations is therefore interpretational rather than fundamental.

4 Personal Understanding of the Markowitz Framework

From my perspective, the Markowitz portfolio optimization problem is best understood as a general framework for reasoning about the trade-off between risk and return, rather than as a single rigid model. The two canonical formulations represent the same underlying optimization problem expressed in different mathematical forms.

The relationship between the target volatility σ_{target} and the risk aversion parameter γ highlights this equivalence: specifying one implicitly determines the other through the optimality conditions of the problem. Both approaches capture the same economic intuition regarding investor preferences.

I view the classical Markowitz model as a starting point for portfolio construction rather than a complete solution. Real-world investment problems often involve additional constraints, such as asset selection restrictions or structural requirements, which can make the optimization problem more complex. In such cases, alternative solution approaches are best understood as extensions of the original framework that preserve its core insight.

5 Solving the Markowitz Problem and Practical Extensions

Under its original assumptions, the Markowitz portfolio optimization problem has a clean and well-structured mathematical form, which allows it to be solved efficiently using standard optimization techniques. These classical methods form the foundation of many traditional portfolio construction approaches.

Monte Carlo simulation is often used as a simple and intuitive way to explore the problem. By randomly generating a large number of portfolios and evaluating their risk and return, this approach provides a visual approximation of the efficient frontier and helps build intuition. However, it does not alter the underlying optimization problem itself.

In contemporary applications, many portfolio optimization problems that follow the Markowitz framework are formulated as convex optimization problems and solved using quadratic programming techniques. High-level optimization libraries, such as CVXPY, allow these problems to be expressed in a transparent and flexible way without requiring detailed attention to numerical solver

implementation. This makes it easier to incorporate practical constraints, such as long-only portfolios or full investment requirements, while preserving the structure of the original model. At the same time, concerns about estimation error have motivated the use of more stable covariance estimates, such as shrinkage-based approaches, which aim to improve the robustness of optimized portfolios without changing the underlying risk–return trade-off.

Machine learning and neural network–based approaches have also been explored in this context, primarily to address practical challenges related to noisy data and changing market conditions. Rather than replacing the Markowitz framework, these methods are often used to improve the estimation of model inputs or to enhance robustness in high-dimensional settings.

More recently, portfolio optimization has been studied using quantum-inspired approaches. In this setting, the risk–return trade-off is rewritten in a form that allows quantum algorithms, such as the Quantum Approximate Optimization Algorithm (QAOA), to search for portfolio solutions under complex or discrete constraints. While these methods remain at an early stage, they provide an additional perspective on extending the Markowitz framework.

6   Limitations and Modern Relevance

Despite its theoretical importance, the Markowitz model has several well-known limitations. It is highly sensitive to estimation error in expected returns and covariances, which can lead to unstable portfolio weights. Additionally, variance treats upside and downside deviations symmetrically, even though investors typically care more about downside risk.

The model is also single-period and relies only on the first two moments of return distributions. Nevertheless, its central contribution formalizing the relationship between risk, return, and diversification remains highly relevant in modern portfolio management.

7   Conclusion

The Markowitz portfolio optimization problem represents a cornerstone of modern finance. Although its original assumptions limit its direct applicability, the framework provides a clear and rigorous way to think about the trade-off between risk and return.

The two canonical formulations of the problem are mathematically equivalent and offer complementary perspectives on portfolio choice. When understood as a foundational framework and extended to accommodate realistic constraints, Markowitz portfolio theory continues to play a central role in contemporary investment decision-making.

References

[1] S. Boyd, K. Johansson, R. Kahn, P. Schiele, and T. Schmelzer. *Markowitz Portfolio Construction at Seventy.* 2024.