102297009
CS5

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Ques 1) Let $(X_1, X_2, \ldots)$, be a random sample of size n taken from a Normal Population with parameters: mean $= \theta_1$ and variance $= \theta_2$. Find the Maximum Likelihood Estimates of these two parameters.

Ans 1) mean $= \theta_1$ variance $= \theta_2$

$$f(x_i; \theta_1, \theta_2) = \frac{1}{\sqrt{\theta_2}\sqrt{2\pi}} \exp\left[-\frac{(x_i - \theta_1)^2}{2\theta_2}\right]$$

$-\infty < \theta_1 < \infty$ and $0 < \theta_2 < \infty$

Likelihood function:

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2) = \theta_2^{-n/2} (2\pi)^{-n/2} \exp\left[-\frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2\right]$$

and therefore the log of the likelihood function:

$$\log L(\theta_1, \theta_2) = -\frac{n}{2} \log \theta_2 - \frac{n}{2} \log(2\pi) - \frac{\Sigma (x_i - \theta_1)^2}{2\theta_2}$$

Now, upon taking the ~~partial~~ derivative of log likelihood with respect to θ_1, and setting to 0, we see that a few things cancel each other out, leaving us with:

$$\frac{d \log L(\theta_1, \theta_2)}{d\theta_1} = \frac{-2 \Sigma (x_i - \theta_1)(-1)}{2\theta_2} \equiv 0$$

Multiplying through by θ_2, and distributing the summation, we get

$$\Sigma x_i - n\theta_1 = 0$$

Now, solving for $\hat{\theta}_1$, ~~and putting on its~~ MLE of θ_1 is:

$$\hat{\theta}_1 = \frac{\Sigma x_i}{n} = \bar{x}$$

Now, for θ_2 :-

$$\frac{d \log L(\theta_1, \theta_2)}{d\theta_2} = -\frac{n}{2\theta_2} + \frac{\Sigma (x_i - \theta_1)^2}{2\theta_2^2} \equiv 0$$

Multiplying by $2\theta_2^2$:

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$$= \left[\frac{-n}{2\theta_2} + \frac{\Sigma(x_i - \theta_1)^2}{2\theta_2^2}\right] \times 2\theta_2^2$$

we get: $-n\theta_2 + \Sigma(x_i - \theta_1)^2 = 0$

$$\hat{\theta}_2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

Ques 2) Let $X_1, X_2 \ldots X_n$ be a random sample from $B(m,\theta)$ distribution, where $\theta \in \Theta = (0,1)$ is unknown and 'm' is a known positive integer. Compare value of θ using the MLE.

Sol 2) pdf of Binomial $= {}^mC_x \, \theta^x (1-\theta)^{n-x}$

$$L(\theta | x_1, \ldots x_n) = \text{Joint pdf}(x_1, \ldots x_n | \theta)$$
$$= \prod_{i=1}^{n} {}^mC_{x_i} \, \theta^{x_i} (1-\theta)^{m-x_i}$$

Taking ln both sides:

$$= \sum_{i=1}^{n} \left[\ln({}^mC_{x_i}) + x_i \ln\theta + ((m-x_i)\ln(1-\theta))\right]$$

Differentiating with respect to θ:

$$= \frac{\Sigma x_i}{\theta} + \frac{nm - \Sigma x_i}{1-\theta} \cdot (-1)$$

$$\frac{\Sigma x_i}{nm} = \theta$$

$$\left[\theta = \frac{\bar{x}}{m}\right]$$

where $\bar{x}$ = mean of sample.