CG MODULE 2

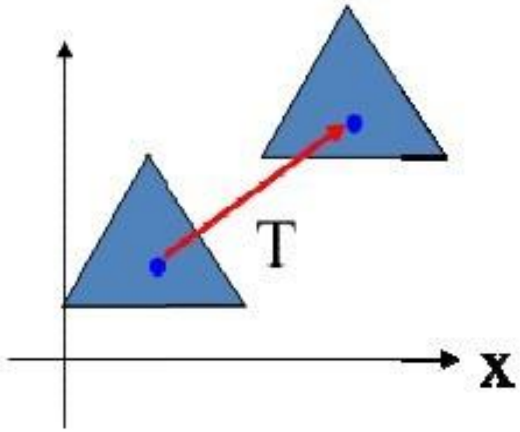
2D & 3D Transformations

Two dimensional transformations

- Transformation means changing graphics into something else by applying rules.
- Transformation plays an important role in computer graphics to reposition the graphics on the screen and change their size or orientation.
- Basic geometric transformations:
  - ✓ **Translation**
  - ✓ **Rotation**
  - ✓ **Scaling**

Translation

- A translation is applied to an object by repositioning it along a straight-line path from one coordinate location to another.
- We can translate a point in 2D by adding translation distances, (tx, ty) to the original coordinate (X, Y) to move the point to a new position (X’, Y’).



$$X' = X + tx$$

$$Y' = Y + ty$$

- The pair (tx, ty) is called the translation vector or shift vector.
- The above equations can also be represented using the column vectors.

$$P = \begin{bmatrix} X \\ Y \end{bmatrix} \qquad P = \begin{bmatrix} X' \\ Y' \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

- So, the two-dimensional translation equations in the matrix form:

$$P' = P + T$$

- Polygons are translated by adding the translation vector to the coordinate position of each vertex and regenerating the polygon using the new set of vertex coordinates and the current attribute settings.

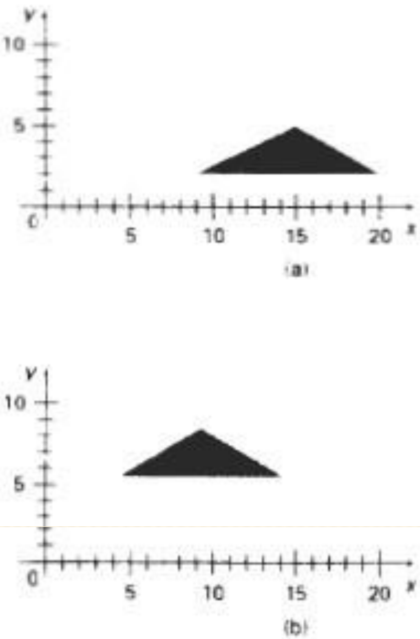
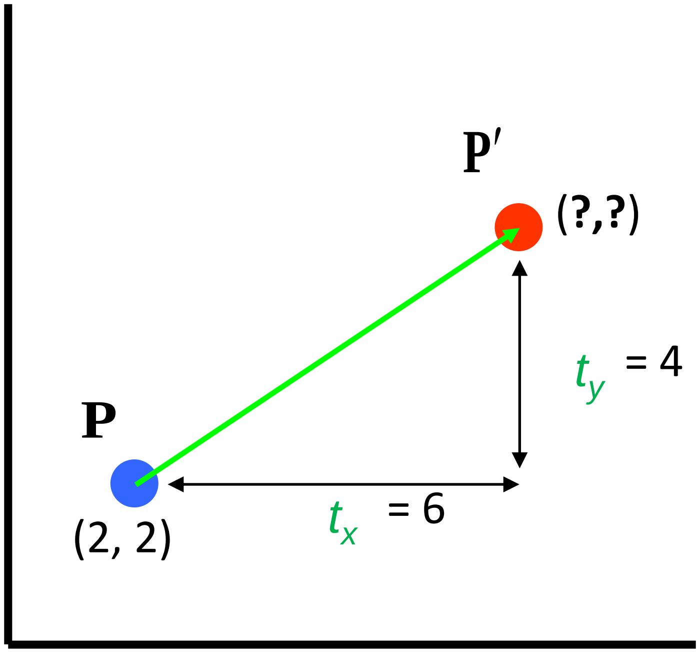


Figure 5-2
Moving a polygon from position (a) to position (b) with the translation vector (-5.50, 3.75).

- Similar methods are used to translate curved objects. To change the position of a circle or ellipse, we translate the center coordinates and redraw the figure in the new location.

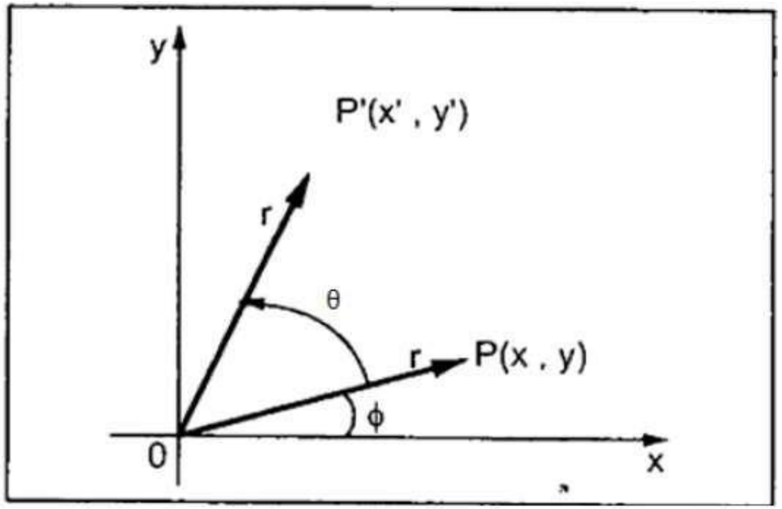
Example of Translation



(8,6)

Rotation

- In rotation, we rotate the object at particular angle θ (theta) from its origin.
- To generate a rotation, we specify a rotation angle θ and the position (x_r , y_r) of the rotation point (or pivot point) about which the object is to be rotated.



- Suppose we want to rotate point P at the angle θ. After rotating it to a new location, we will get a new point $P'(X', Y')$
- In this figure, r is the constant distance of the point from the origin, angle ϕ is the original angular position of the point from the horizontal, and θ is the rotation angle.
- Positive values for the rotation angle define counter-clockwise rotations about the pivot point and negative values rotate objects in the clockwise direction.

Using standard trigonometric the original coordinate of point P(X, Y) can be represented as –

$$X = r\cos\phi \ldots\ldots (1)$$

$$Y = r\sin\phi \ldots\ldots (2)$$

Same way we can represent the point P′ (X′, Y′) as –

$$x' = r\cos(\phi + \theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta \ldots\ldots\ldots (3)$$

$$y' = r\sin(\phi + \theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta \ldots\ldots\ldots (4)$$

Substituting equation (1) & (2) in (3) & (4) respectively, we will get

$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

- or in matrix form:

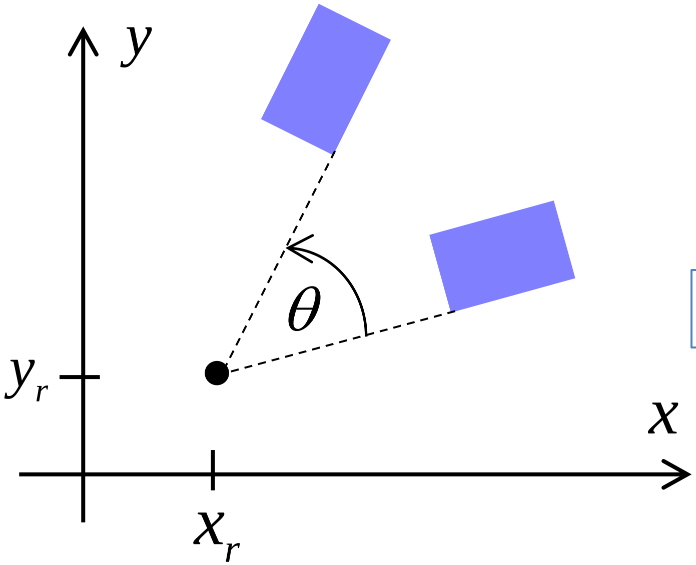
P' = R • P

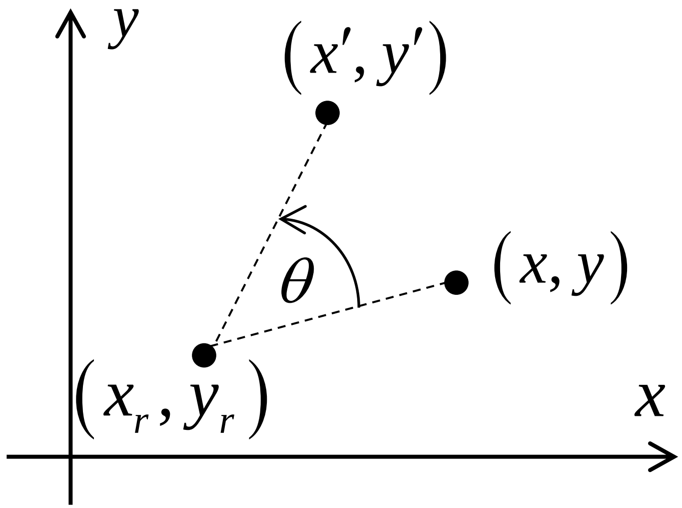
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Rotation matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation about a fixed reference point (x_r, y_r)





$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$

$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$$

OR

$$\mathbf{P}' = \mathbf{P}_r + \mathbf{R}\cdot(\mathbf{P} - \mathbf{P}_r)$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

STEPS

1. Translate to Origin
2. Rotate
3. Translate to (x_r, y_r)

Example : 2D Rotation

Example - Find the transformed point, P', caused by rotating P= (5, 1) about the origin through an angle of 90°.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cdot\cos\theta - y\cdot\sin\theta \\ x\cdot\sin\theta + y\cdot\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot\cos 90 - 1\cdot\sin 90 \\ 5\cdot\sin 90 + 1\cdot\cos 90 \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot 0 - 1\cdot 1 \\ 5\cdot 1 + 1\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Scaling

- A scaling transformation alters the size of an object.
- This operation can be carried out for polygons by multiplying the coordinate values ***(x, y)*** of each vertex by scaling factors S_x and S_y to produce the transformed coordinates **(x', y').**

$$\mathbf{x' = x \cdot S_x}$$

$$\mathbf{y' = y \cdot S_y}$$

- Scaling factor S_x, scales objects in the x direction, while S_y scales in the y direction.

- Scaling equations:

$$x' = s_x \cdot x$$

$$y' = s_y \cdot y$$

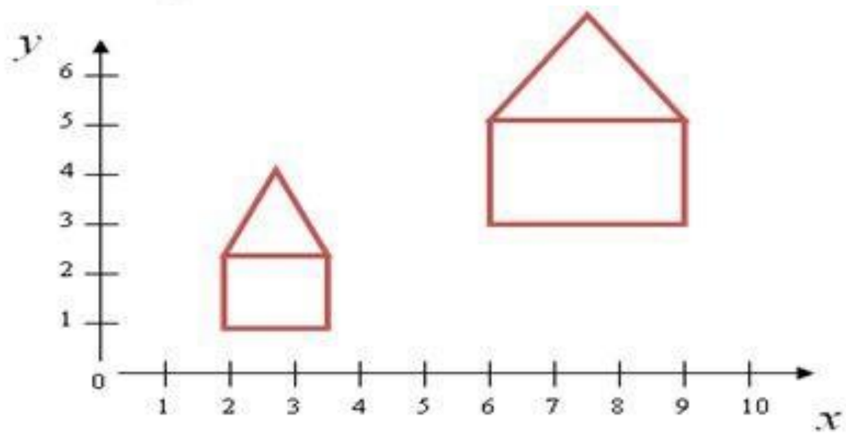
or in matrix form:

$$\mathbf{P'} = \mathbf{S} \cdot \mathbf{P}$$

Scale matrix as:

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

figure.

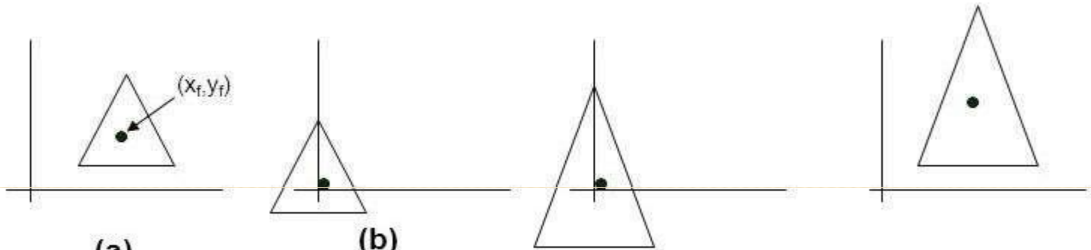


- If S_x & S_y >1, the size of the object will increase. Also objects move away from origin
- If S_x & S_y <1, the size of the object is reduced. Also objects move towards origin
- If S_x & S_y =1, the size of the object will remain unchanged. Also there will be no change in location.

- If S_x & S_y have same values, it is called **uniform scaling**.
- If S_x & S_y have different values, it is called **differential scaling**.

Scaling with respect to a fixed point (x_f,y_f)

- Translate object so that the fixed point coincides with the coordinate origin
- Scale the object with respect to the coordinate origin
- Use the inverse translation of step 1 to return the object to its original position



(a) Original Position of Object and Fixed point

(b) Translation of object so that fixed point (x_f,y_f) is at origin

(c) scaling was about origin

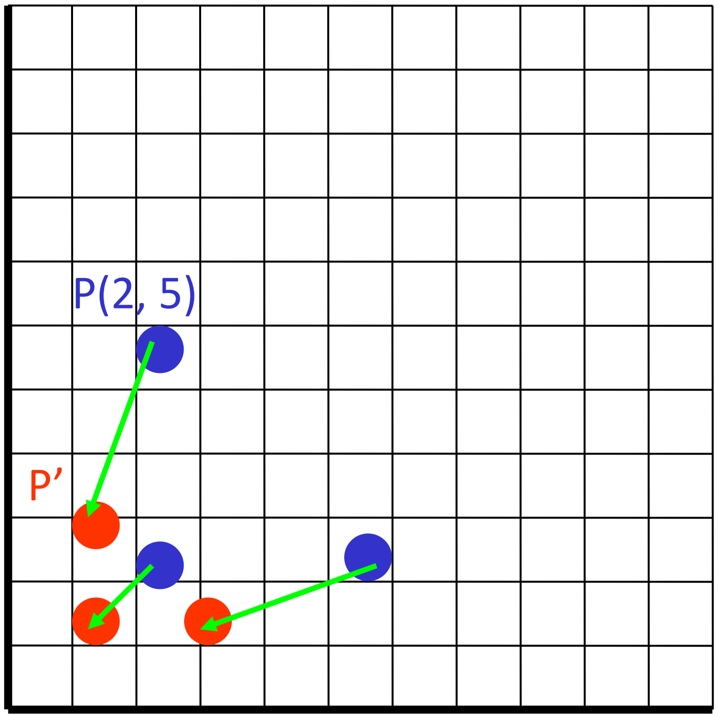
(d) Translation of the object so that the Fixed point is returned to position (x_f,y_f)

$$x' = x \cdot s_x + x_f(1 - s_x)$$

$$y' = y \cdot s_y + y_f(1 - s_y)$$

Scaling

- Example :
  - P(2, 5), Sx = 0.5, Sy = 0.5
  - Find P' ?



OTHER TRANSFORMATIONS

- **Reflection**
- **Shear**

REFLECTION

- A reflection is a transformation that produces a mirror image of an object.
- In other words, we can say that it is a rotation operation with 180° about the reflection axis.
- In reflection transformation, the size of the object does not change.

Reflection about the x axis (y = 0)

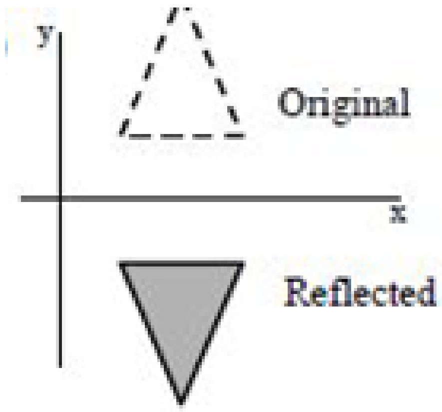
- Change the sign of y coordinate but x coordinate remains same.

$$x' = x$$

$$y' = -y$$

- Equivalent to 180° rotation about x axis.
- In matrix form

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



Reflection about the y axis (x = 0)

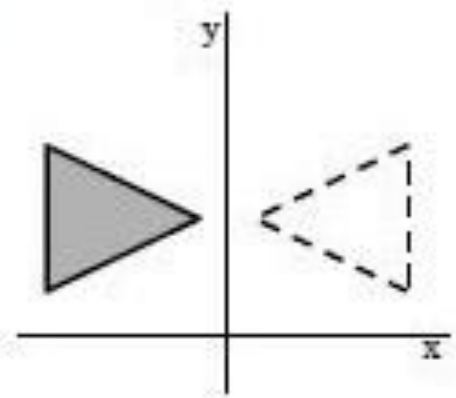
- Change the sign of x coordinate but y coordinate remains same.

$$x' = - x$$

$$y' = y$$

- Equivalent to 180° rotation about y axis.
- In matrix form

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



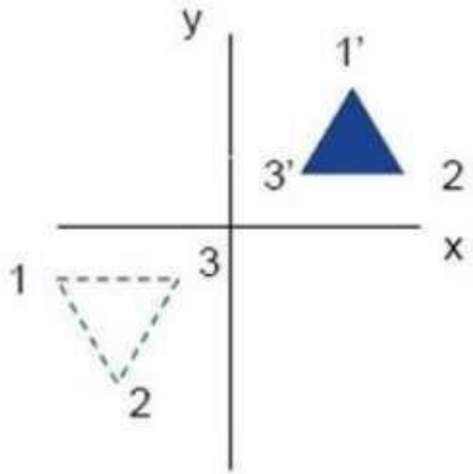
Reflection relative to the coordinate origin

- Reflection relative to an axis that is perpendicular to the xy plane and that passes through the coordinate origin.
- Change the sign of both x coordinate and y coordinate.

$$x' = -x$$
$$y' = -y$$

- Equivalent to 180° rotation about origin.
- In matrix form

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



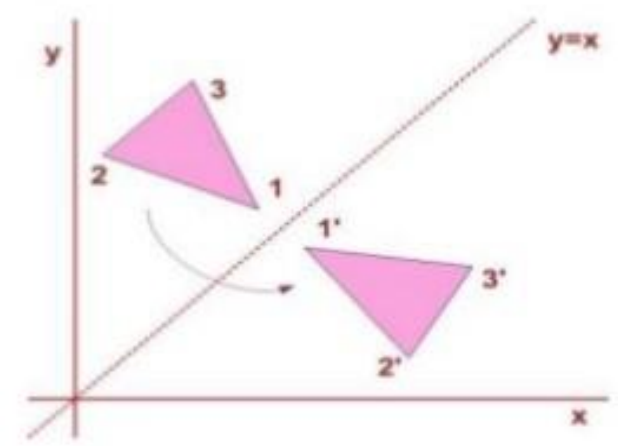
Reflection with respect to diagonal line , line y = x

- can be obtained by performing following sequence of transformations:

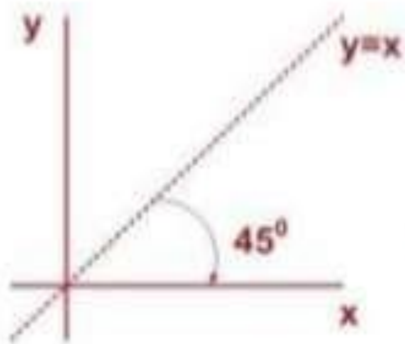
1. Clock wise rotation with 45°
2. Reflection with respect to x axis
3. Inverse rotation with an angle 45°

- In matrix form

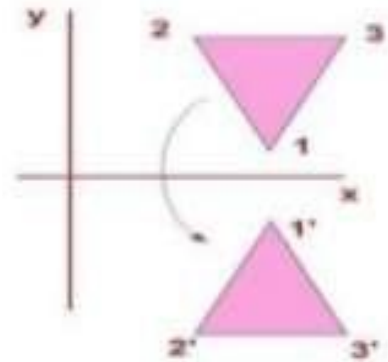
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

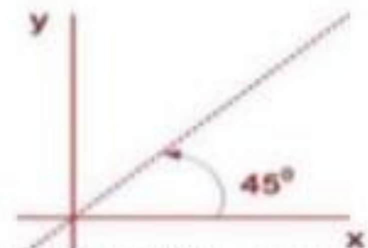


1. first perform a clockwise rotation through a 45" angle

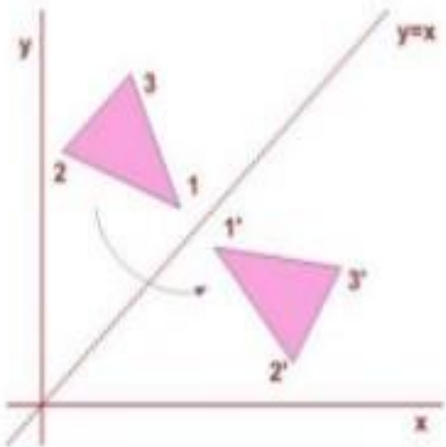


2.Next, we perform a reflection with respect to the x axis.





3. back to its original position with a counterclockwise rotation through 45".



Reflection about the diagonal y = -x

- can be obtained by performing following sequence of transformations:
  1. Clockwise rotation by 45°
  2. Reflection about the y axis
  3. Counterclockwise rotation by 45°
- In matrix form

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

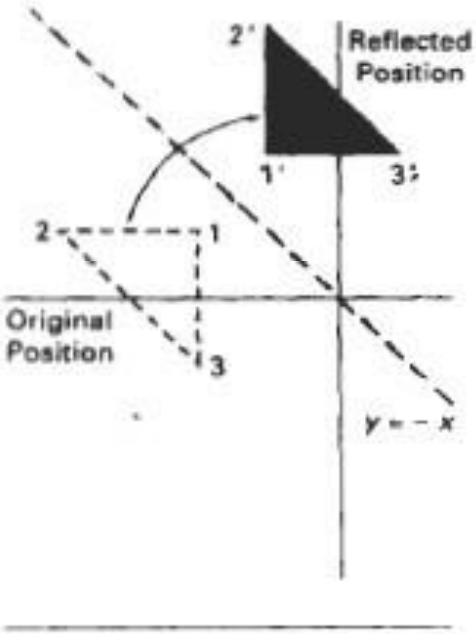


Figure 5-22
Reflection with respect to the line $y = -x$.

SHEAR

- A transformation that distorts the **shape** of an object
- There are two shear transformations **X-Shear** and **Y-Shear**.
- One shifts X coordinates values and other shifts Y coordinate values.
- However, in both the cases only one coordinate changes its coordinates and other preserves its values.
- Shearing is also termed as **Skewing**.

X direction shear:

- An x-direction shear relative to the x axis is produced with the transformation matrix

$$\begin{pmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

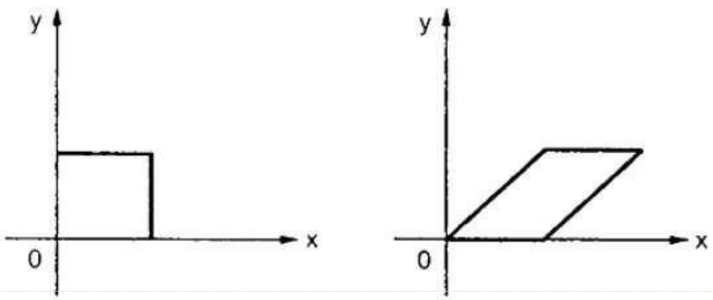
with coordinate positions transformed as

$$x' = x + sh_x . y$$

$$y' = y$$

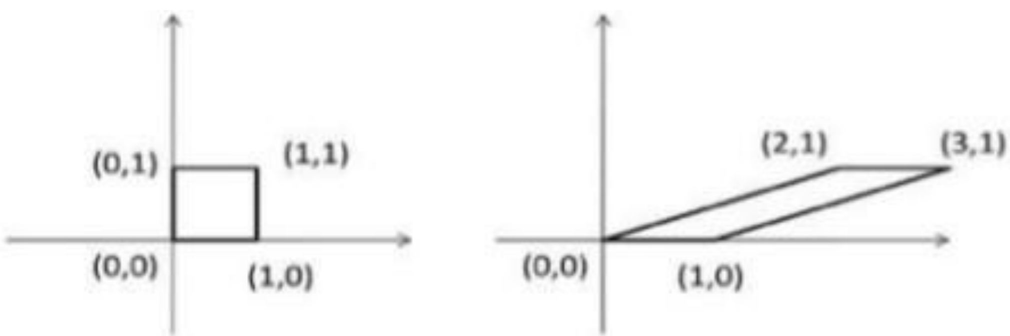
- Negative values for sh_x shift coordinate positions to the left.

- The X-Shear preserves the Y coordinate and changes are made to X coordinates, which causes the vertical lines to tilt right or left as shown in below figure.



(a) Original object (b) Object after x shear

- A unit square (a) is converted to a parallelogram (b) using the *x direction* shear matrix with $sh_x = 2$.



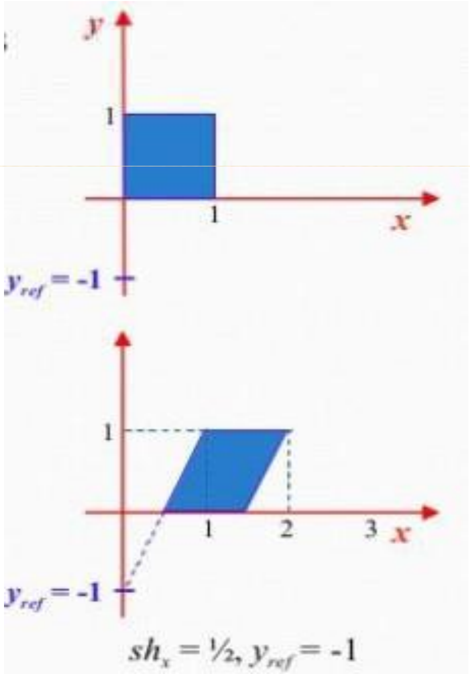
x-direction shears relative to other reference lines

$$\begin{pmatrix} 1 & sh_x & -sh_x.y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with coordinate positions transformed as

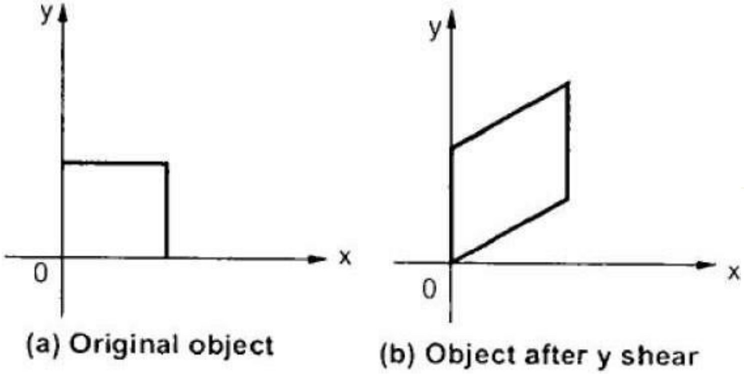
$$x' = x + sh_x(y - y_{ref})$$
$$y' = y$$

- An example of this shearing transformation for a shear parameter value of 1/2 relative to the line $y_{ref} = -1$.



Y-Shear

- The Y-Shear preserves the X coordinates and changes the Y coordinates which causes the horizontal lines to transform into lines which slopes up or down.



- The Y-Shear can be represented in matrix form as:

$$\mathbf{y' = y + sh_y \,.\, x}$$
$$\mathbf{x' = x}$$

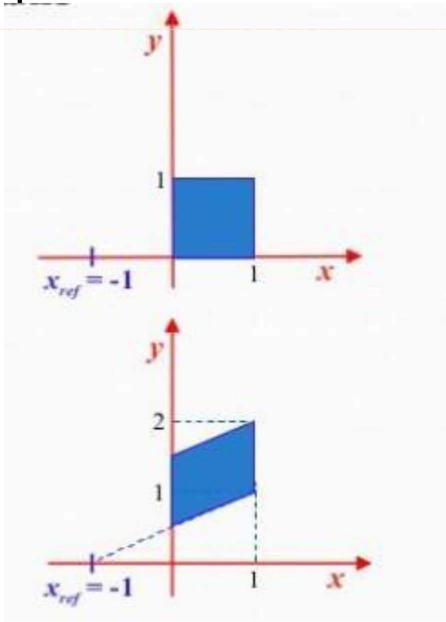
$$\begin{pmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **A y-direction shear relative to the line $x = x_{ref}$** is generated with the transformation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y . x_{ref} \\ 0 & 0 & 1 \end{pmatrix}$$

which generates transformed coordinate positions

$$x' = x$$
$$y' = y + sh_y (x - x_{ref})$$



Example1: Translate the given point (2,5) by translation vector (3,3)

Solution:

(x,y) = (2,5)

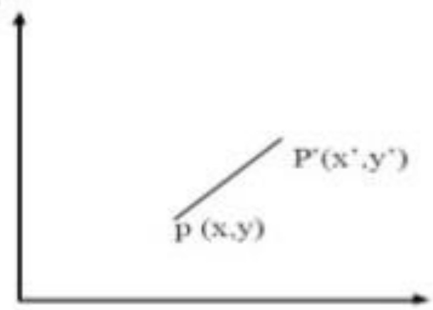
$T_x = 3$

$T_y = 3$

$X' = x + t_x$

$Y' = y + t_y$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$



- Example2: Translate a polygon with coordinates A(2,5), B(7,10) and C(10,2) by 3 units in x direction and 4 unit in y direction.
- Solution:

$$A' = A + T$$
$$= \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$
$$B' = B + T$$
$$= \begin{bmatrix} 7 \\ 10 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$
$$C' = C + T$$
$$= \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 13 \\ 6 \end{bmatrix}$$

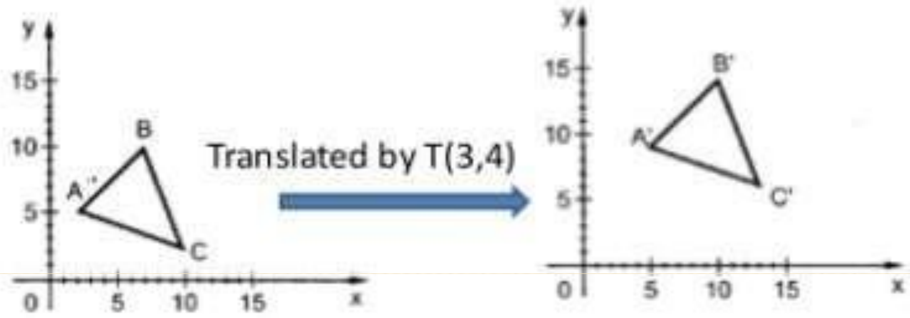


Fig: Before translation

Fig: After translation

- **Example:** A point (4,3) is rotated counterclockwise by an angle 45 degree. Find the rotation matrix and the resultant point.
- **Solution:**

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\therefore \quad P' = [4 \;\; 3] \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4/\sqrt{2} - 3/\sqrt{2} & 4/\sqrt{2} + 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 7/\sqrt{2} \end{bmatrix}$$

- **Example 1:** scale the polygon with coordinates A(2,5), B(7,10) and C(10,2) by two units in x-direction and two units in y-direction.

Sol. : Here $S_x = 2$ and $S_y = 2$. Therefore, transformation matrix is given as

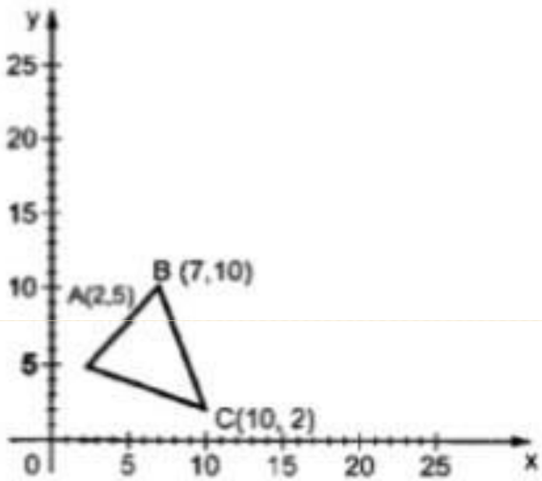
$$S = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The object matrix is :

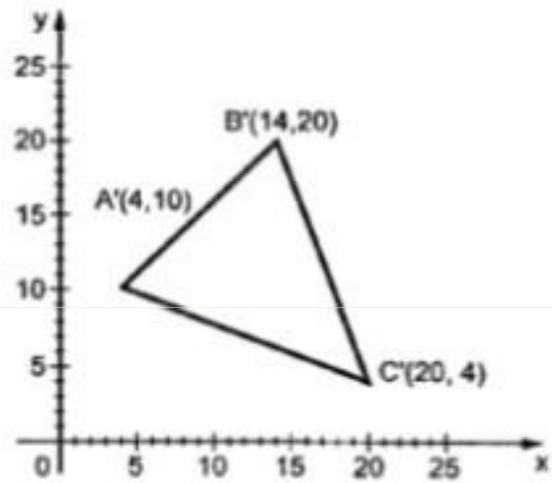
$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 10 & 2 \end{bmatrix} \end{array}$$

$$\therefore \quad \begin{array}{c} A' \\ B' \\ C' \end{array} \begin{bmatrix} x_1' & y_1' \\ x_2' & y_2' \\ x_3' & y_3' \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 \\ 14 & 20 \\ 20 & 4 \end{bmatrix}$$



(a) Original object



(b) Scaled object

- A triangle ABC with coordinates A(0,0), B(6,5), C(6,0) is scaled with scaling factors Sx=2 and Sy=3 about the vertex C(6,0). Find the transformed coordinate points.

$$P' = \begin{bmatrix} S_x & 0 & x_f(1-s_x) \\ 0 & S_Y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix} P$$

$$P' = \begin{bmatrix} 2 & 0 & 6(1-2) \\ 0 & 3 & 0(1-3) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 & 6 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 12 & 6 \\ 0 & 15 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The new coordinates after scaling in A(0,0), B (12, 15), C (6, 0)

Homogeneous co-ordinates

- We can represent the point by 3 numbers instead of 2 numbers, which is called Homogenous Coordinate system.
- To express any two-dimensional transformation as a matrix multiplication, we represent each Cartesian coordinate position (x, y) with the homogeneous coordinate triple (x_h, y_h, h) where

$$x = \frac{x_h}{h} \qquad y = \frac{y_h}{h}$$

- For two-dimensional geometric transformations, we can choose the homogenous parameter h to be any nonzero value.
- A convenient choice is simply to set h = 1.
- Each two dimensional position is then represented with homogeneous coordinates *(x, y, 1)*.

- Expressing positions in homogeneous coordinates allows us to represent all geometric transformation equations as matrix multiplications.
- **Advantages of using homogenous coordinate:** We can perform all transformations using matrix/vector multiplications. This allows us to pre-multiply all the matrices together.

Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

- About the coordinate origin:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scaling

- Scaling transformation relative to the coordinate origin:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

COMPOSITE TRANSFORMATIONS

- We can set up a matrix for any sequence of transformations as a **composite transformation matrix** by calculating the matrix product of the individual transformations.
- Forming products of transformation matrices is often referred to as a **concatenation,** or **composition,** of matrices.
- For column-matrix representation of coordinate positions, we form composite transformations by multiplying matrices in order from right to left.
- That is, each successive transformation matrix premultiplies the product of the preceding transformation matrices.

Translations

- When two successive translations are applied to a point with translation vectors (tx1,ty1) and (tx2,ty2), then

$$\begin{bmatrix} 1 & 0 & t_{x_2} \\ 0 & 1 & t_{y_2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{x_1} \\ 0 & 1 & t_{y_1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x_1} + t_{x_2} \\ 0 & 1 & t_{y_1} + t_{y_2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{T}(t_{x_2}, t_{y_2}) \cdot \mathbf{T}(t_{x_1}, t_{y_1}) \cdot \mathbf{P} = \mathbf{T}(t_{x_1} + t_{x_2}, t_{y_1} + t_{y_2}) \cdot \mathbf{P}$$

- This demonstrates that successive translations are additive in nature.

Rotations

Q) Show that the composition of two successive rotations are additive?

- When two successive rotations are applied to a point ,

we can write rotation matrix $R(\theta_1)$ as

$$R(\theta_1) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \text{ and } R(\theta_2) = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$R(\theta_1) \cdot R(\theta_2) = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_1 \cdot \cos\theta_2 + \sin\theta_1 \cdot (-\sin\theta_2) & \cos\theta_1 \cdot \sin\theta_2 + \sin\theta_1 \cdot \cos\theta_2 \\ -\sin\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \cdot (-\sin\theta_2) & -\sin\theta_1 \cdot \sin\theta_2 + \cos\theta_1 \cdot \cos\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}$$

since,

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$\sin(\theta_1 + \theta_2) = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2$$

- This demonstrates that successive rotations are additive in nature.

Scaling

- When two successive scaling are applied to a point, with vectors (S_{x1}, S_{y1}) and (S_{x2}, S_{y2})

$$\begin{vmatrix} S_{x2} & 0 & 0 \\ 0 & S_{y2} & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} S_{x1} & 0 & 0 \\ 0 & S_{y1} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} S_{x1} \cdot S_{x2} & 0 & 0 \\ 0 & S_{y1} \cdot S_{y2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$S(S_{x2}, S_{y2}) \cdot S(S_{x1}, S_{y1}) = S(S_{x1} \cdot S_{x2}, S_{y1} \cdot S_{y2})$$

- This demonstrates that **two successive scalings are Multiplicative** in nature.

Rotation with respect to a fixed reference point

- We can generate rotations about any selected pivot point (xr, yr) by performing the following sequence of translate-rotate-translate operations:

  1. Translate the object so that the pivot-point position is moved to the coordinate origin.
  2. Rotate the object about the coordinate origin.
  3. Translate the object so that the pivot point is returned to its original position.

(a) Original Position of Object and Pivot Point

(b) Translation of Object so that Pivot Point (x_r, y_r) is at Origin

(c) Rotation about Origin

(d) Translation of Object so that the Pivot Point is Returned to Position (x_r, y_r)

Note the order of operations:

$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

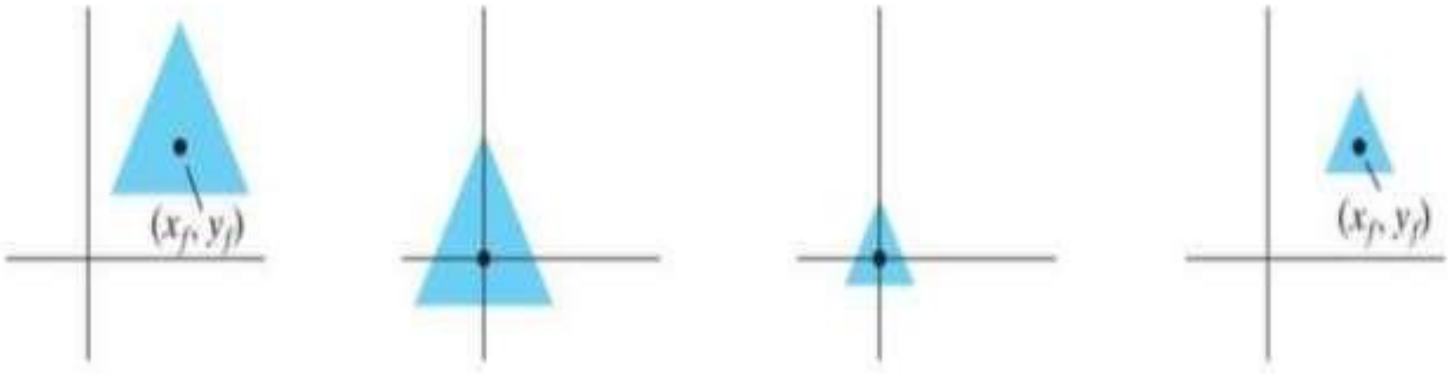
$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r \sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r \sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}(x_r, y_r) \cdot \mathbf{R}(\theta) \cdot \mathbf{T}(-x_r, -y_r) = \mathbf{R}(x_r, y_r, \theta)$$

Scaling with respect to a fixed reference point

- Transformation sequence to produce scaling with respect to a selected fixed position (xf, yf) using a scaling function that can only scale relative to the coordinate origin.

1. Translate object so that the fixed point coincides with the coordinate origin.
2. Scale the object with respect to the coordinate origin.
3. Use the inverse translation of step 1 to return the object to its original position.



(a) Original Position of Object and Fixed Point

(b) Translate Object so that Fixed Point (x_f, y_f) is at Origin

(c) Scale Object with Respect to Origin

(d) Translate Object so that the Fixed Point is Returned to Position (x_f, y_f)

- Concatenating the matrices for these three operations produces the required scaling matrix:

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}(x_f, y_f) \cdot \mathbf{S}(s_x, s_y) \cdot \mathbf{T}(-x_f, -y_f) = \mathbf{S}(x_f, y_f, s_x, s_y)$$

- **Perform a 45 degree rotation of a triangle ABC having the vertices at A(0,0), B(10,10) and C(50,20)**

i. About the origin

ii. About an arbitrary point P(-10,-10)

Answers:

i. Rotation about the origin

Given:

Triangle A(0,0), B(10,10) and C(50,20)

Rotation $\theta = 45°$

$P' = R * P$

$$P' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} Ax & Bx & Cx \\ Ay & By & Cy \end{bmatrix}$$

$$P' = \begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 0 & 10 & 50 \\ 0 & 10 & 20 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 0 & 10 & 50 \\ 0 & 10 & 20 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 0 & 21.21 \\ 0 & 14.14 & 49.49 \end{bmatrix}$$

After rotation about origin, the new points of triangle are

A(0,0), B(0,14), C(21,49)

Rotation about arbitrary point

Arbitrary Point (-10,-10)

$$P' = T(x_r, y_r) \cdot [R(\theta) \cdot \{T(-x_r, -y_r) \cdot P\}]$$
$$P' = \{T(x_r, y_r) \cdot R(\theta) \cdot T(-x_r, -y_r)\} \cdot P$$
$$P' = \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$
$$\boxed{P' = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r \sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \cdot P}$$

$$P' = \begin{bmatrix} \cos 45 & -\sin 45 & -10(1-\cos 45) - 10\sin 45 \\ 0 & 14.14 & 49.49 \end{bmatrix} \begin{bmatrix} 0 & 10 & 50 \\ 0 & 10 & 20 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} \cos 45 & -\sin 45 & -10(1-\cos 45) - 10\sin 45 \\ \sin 45 & \cos 45 & -10(1-\cos 45) + 10\sin 45 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 10 & 50 \\ 0 & 10 & 20 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.707 & -0.707 & -10(1-0.707) - 10 * 0.707 \\ 0.707 & 0.707 & -10(1-0.707) + 10 * 0.707 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 10 & 50 \\ 0 & 10 & 20 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.707 & -0.707 & -10 \\ 0.707 & 0.707 & 4.14 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 10 & 50 \\ 0 & 10 & 20 \\ 0 & 0 & 1 \end{bmatrix}$$

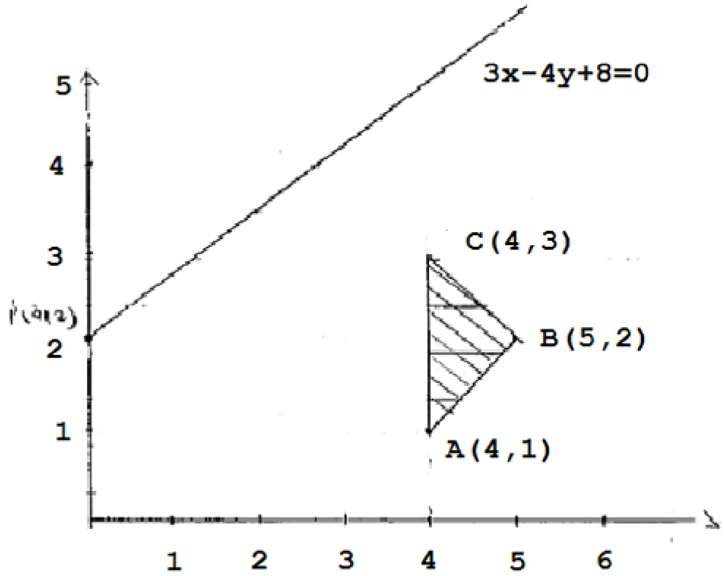
$$P' = \begin{bmatrix} 0 & 0 & 11.21 \\ 0 & 14.14 & 53.63 \\ 0 & 0 & 1 \end{bmatrix}$$

- After rotation about arbitrary point, the new points are A(0,0), B(0,14), C(11,54)

Q) Reflect a triangle ABC about the line 3x-4y+8=0. The position vector of the coordinate ABC is given as A(4,1), B(5,2) and C(4,3).

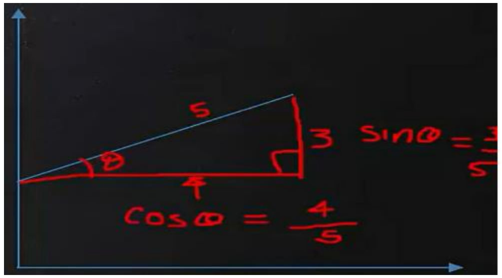
$$y = mx + c,\ m = \frac{3}{4} \text{ and } c = 2$$

$$\theta = \tan^{-1} m \qquad \theta = -36.87^{o}$$



The composite transformation matrix

$$[T] = [Tr][R][M][R]^{-1}[Tr]^{-1}$$



The final matrix P =

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos -\theta & -\sin -\theta & 0 \\ \sin -\theta & \cos -\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} P$$

3D TRANSFORMATION

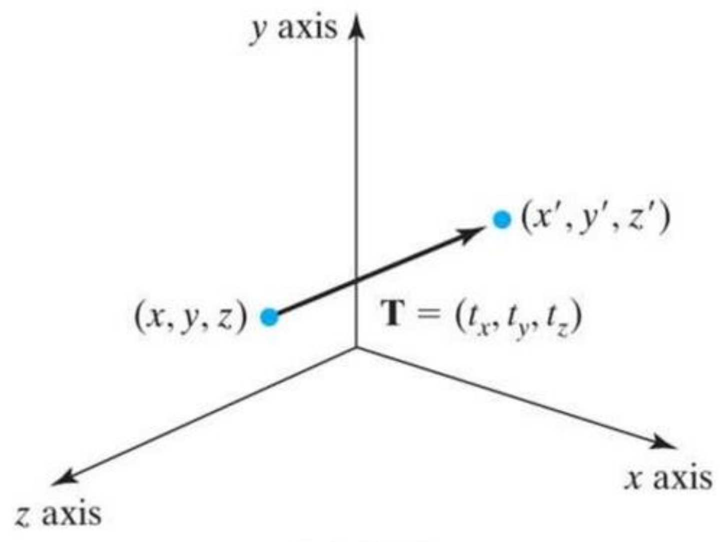
3D Transformation

- Geometric transformation and object modelling in 3D are extended from 2D methods by including considerations for the Z coordinate

Translation

- A point is translated from position P(x, y, z) to position P'(x', y', z')

$$x' = x + t_x$$
$$y' = y + t_y$$
$$z' = z + t_z$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation

- Obtain rotations around other axes through cyclic permutation of coordinate parameters

$$\boldsymbol{x} \rightarrow \boldsymbol{y} \rightarrow \boldsymbol{z}$$

Z –axis Rotation

- Positive rotation angles produce anticlockwise rotations about a coordinate axis

- *Z-axis Rotation*

$$\boxed{\begin{aligned} x' &= xCos\theta - ySin\theta \\ y' &= xSin\theta + yCos\theta \\ z' &= z \end{aligned}} \Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} Cos\theta & -Sin\theta & 0 & 0 \\ Sin\theta & Cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

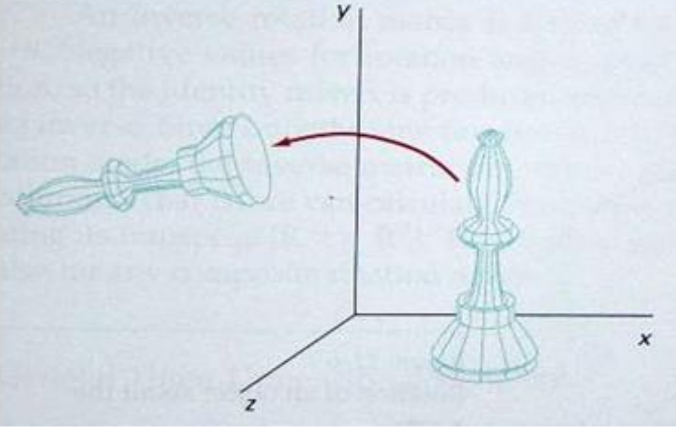


Figure 11-4
Rotation of an object about the z axis.

X axis Rotation

$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$
$$x' = x$$

$$\Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & Cos\theta & -Sin\theta & 0 \\ 0 & Sin\theta & Cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

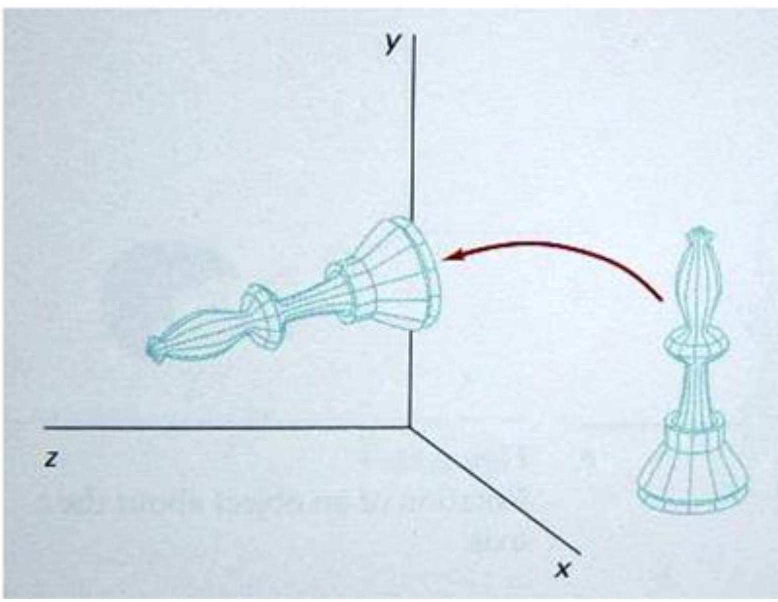


Figure 11-6
Rotation of an object about the *x* axis.

Y axis Rotation

- *Y-axis Rotation*

$$\boxed{\begin{aligned} z' &= z\cos\theta - x\sin\theta \\ x' &= z\sin\theta + x\cos\theta \\ y' &= y \end{aligned}} \Longleftrightarrow \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

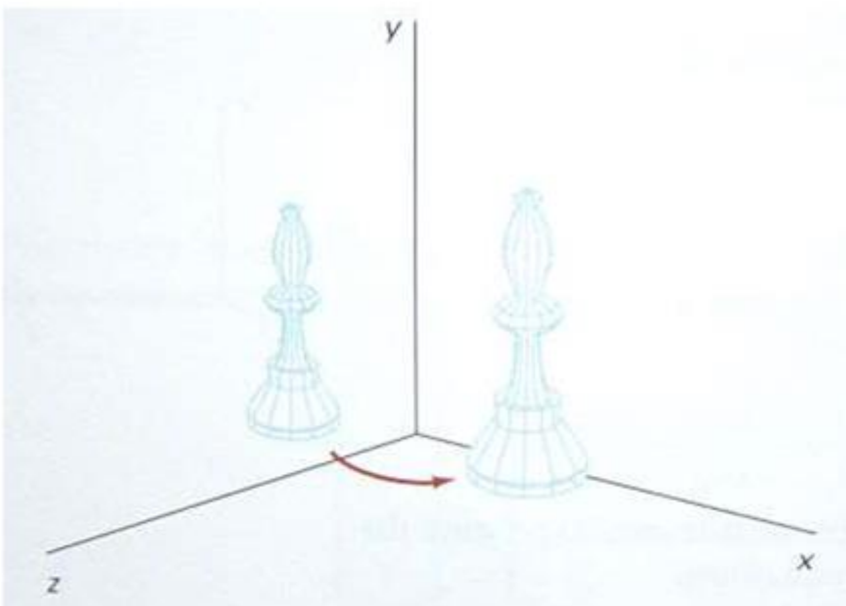


Figure 11-7
Rotation of an object about the *y* axis.

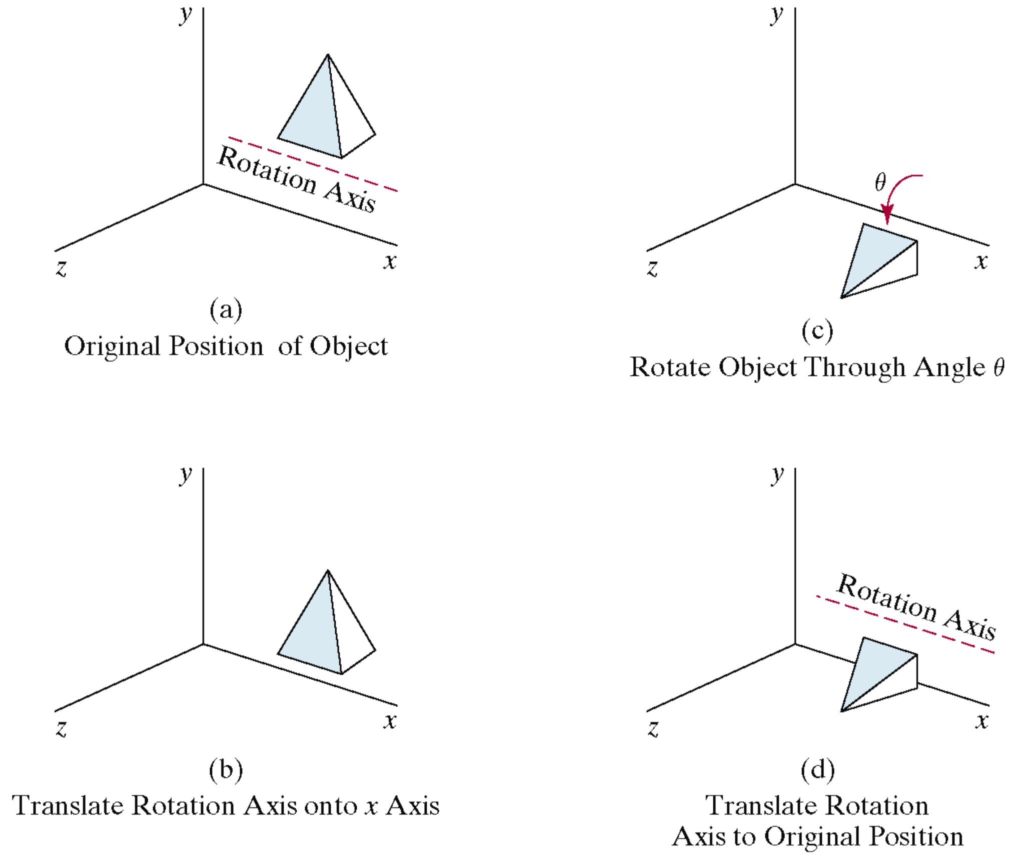
Rotation of an object about an axis that is parallel to one of the coordinate axes

1. Translate the object so that the rotation axis coincides with the parallel coordinate axis.
2. Perform the specified rotation about that ails.
3. Translate the object so that the rotation axis is moved back to its original position.

Any coordinate position P on the object is transformed with the sequence shown as

$$P' = T \cdot Rx(\theta) \cdot T^{-1}.\ P$$

Rotation axis parallel to the coordinate axis



$$T^{-1} \cdot Rx(\theta) \cdot T$$

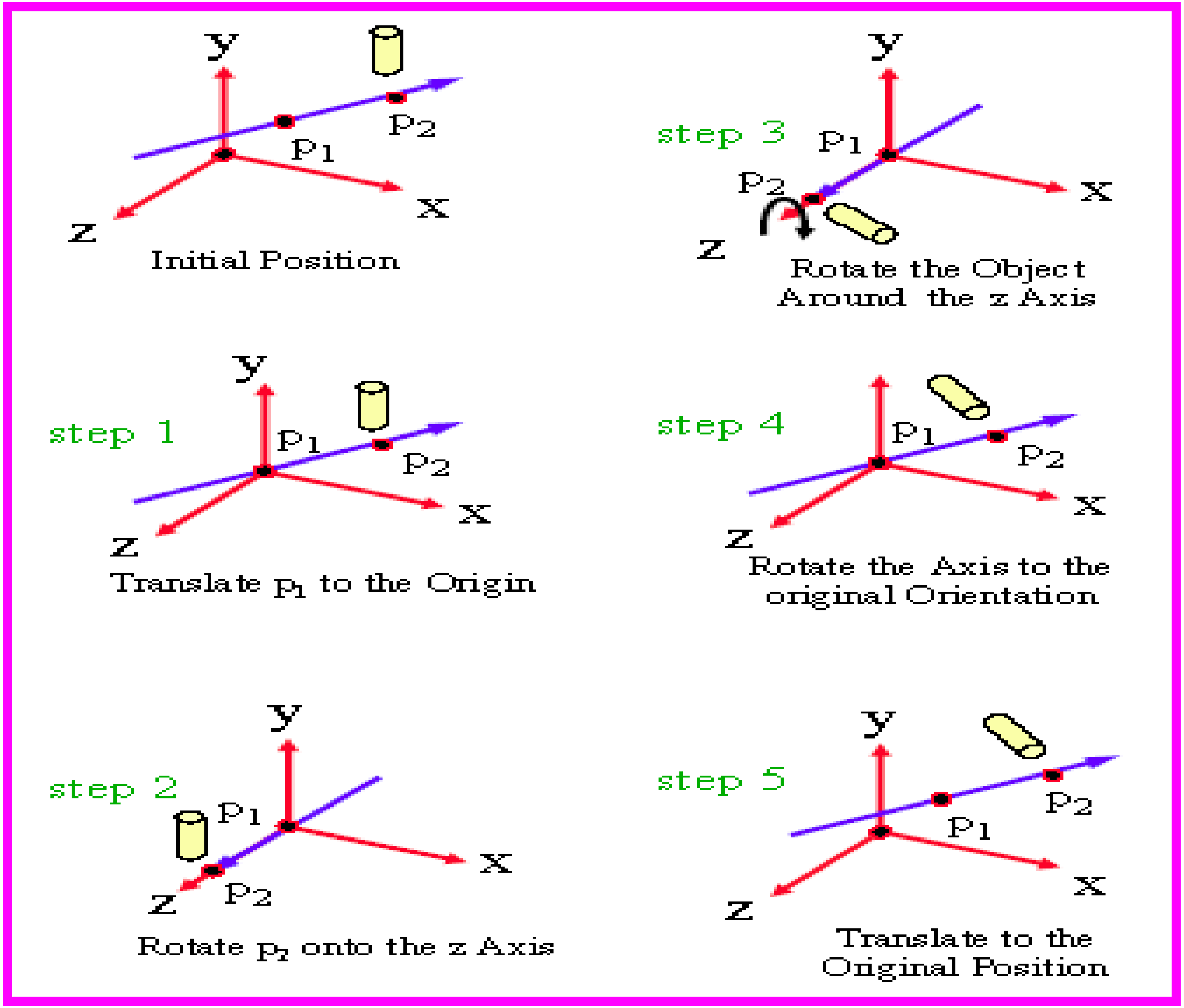
Rotation about an arbitrary axis

- When the object is rotated about an axis that is not parallel to any one of co-ordinate axis, i.e., x, y, z. Then additional transformations are required.

Rotation about an arbitrary axis

Following steps are required

1. Translate the object so that rotation axis pass through **coordinate origin**
2. Rotate object so that axis of object coincides with **any of coordinate axis**.
3. **Perform rotation** about co-ordinate axis with whom coinciding is done.
4. **Apply inverse rotation** to bring rotation axis back to its original position.
5. **Apply inverse translation** to bring rotation axis back to the original position.


y
p2
p1
x
z
Initial Position
step 3
y
p1
p2
x
z
Rotate the Object
Around the z Axis
step 1
y
p1
p2
x
z
Translate p1 to the Origin
step 4
p1
p2
x
z
Rotate the Axis to the
original Orientation
step 2
y
p1
x
z
p2
Rotate p2 onto the z Axis
step 5
y
p2
p1
x
z
Translate to the
Original Position

Scaling

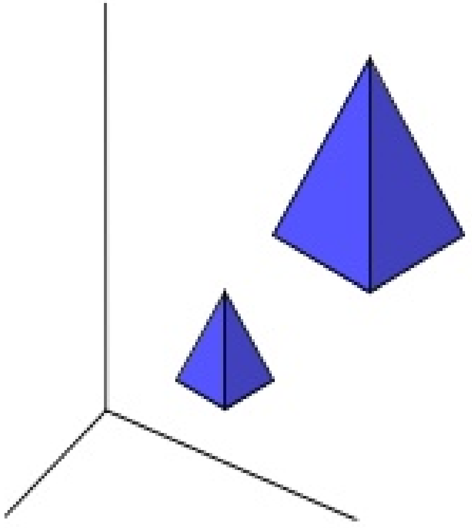
- The scaling transformation of a point P(x,y,z) relative to coordinate origin can be written as

$$x' = x \cdot s_x$$

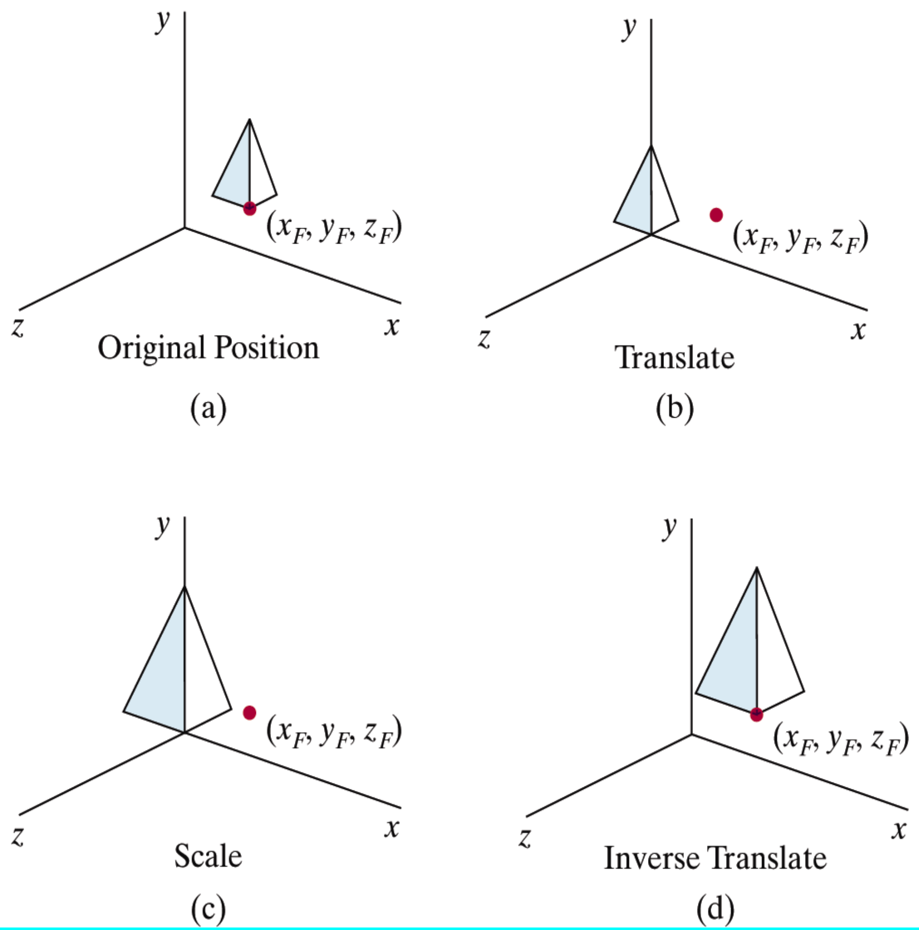
$$y' = y \cdot s_y$$

$$z' = z \cdot s_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Scaling about a fixed point



Scaling

- The matrix representation for an arbitrary fixed-point (x_f, y_f, z_f) can be expressed as:

$$\mathbf{T}(x_f, y_f, z_f) \cdot \mathbf{S}(s_x, s_y, s_z) \cdot \mathbf{T}(-x_f, -y_f, -z_f) = \begin{bmatrix} s_x & 0 & 0 & (1 - s_x)x_f \\ 0 & s_y & 0 & (1 - s_y)y_f \\ 0 & 0 & s_z & (1 - s_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Reflection

- 3D reflections can be performed wrt :-

  a reflection axis or a reflection plane
- **Reflection About an axis:** equivalent to 180˚rotation about that axis
- **Reflection About a plane:** equivalent to 180˚rotations in 4D space

- Reflection relative to X-Y plane

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Reflection Matrix

(Reflection Relative to YZ plane)

$$\begin{bmatrix} X_{new} \\ Y_{new} \\ Z_{new} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \\ Z_{old} \\ 1 \end{bmatrix}$$

3D Reflection Matrix

(Reflection Relative to XZ plane)

Composite 3D Transformations

- We can represent any sequence of transformations as a single matrix.
- – No special cases when transforming a point – matrix • vector.
- – Composite transformations – matrix • matrix.

- 3D Same way as in 2D:
- Multiply matrices
- Rightmost term in matrix product is the first transformation to be applied

- Composite transformations:
- – Rotate about an arbitrary point – translate, rotate, translate
- – Scale about an arbitrary point – translate, scale, translate
- – Change coordinate systems – translate, rotate, scale
- But the order of operations important