Study on B+ Tree

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1 Introduction

1.1 What is a B+ Tree?

A B+ Tree is primarily utilized for implementing dynamic indexing on multiple levels. Compared to B-Tree, the B+ Tree stores the data pointers only at the leaf nodes of the Tree, which makes search more process more accurate and faster.

1.2 Motivation

B+ Tree are used to store the large amount of data which can not be stored in the main memory. Due to the fact that, size of main memory is always limited, the internal nodes (keys to access records) of the B+ tree are stored in the main memory whereas, leaf nodes are stored in the secondary memory.

1.3 History

The B tree was first described in the paper Organization and Maintenance of Large Ordered Indices. Acta Informatica 1: 173–189 (1972) by Rudolf Bayer and Edward M. McCreight. There is no single paper introducing the B+ tree concept. Instead, the notion of maintaining all data in leaf nodes is repeatedly brought up as an interesting variant. An early survey of B trees also covering B+ trees is Douglas Comer. Comer notes that the B+ tree was used in IBM's VSAM data access software and he refers to an IBM published article from 1973.

1.4 B+ tree in real life

The ReiserFS, NSS, XFS, JFS, ReFS, and BFS filesystems all use this type of tree for metadata indexing; BFS also uses B+ trees for storing directories. NTFS uses B+ trees for directory and security-related metadata indexing.

2 Problem Formulation

In computer science, a B+ tree is a type of tree data structure. It represents sorted data in a way that allows for efficient insertion and removal of elements. It is a dynamic, multilevel index with maximum and minimum bounds on the number of keys in each node.

A B+ tree is a variation on a B-tree. In a B+ tree, in contrast to a B tree, all data are saved in the leaves. Internal nodes contain only keys and tree pointers. All leaves are at the same lowest level. Leaf nodes are also linked together as a linked list to make range queries easy.

The maximum number of keys in a record is called the order of the B+ tree.

The minimum number of keys per record is 1/2 of the maximum number of keys. For example, if the order of a B+ tree is n, each node (except for the root) must have between n/2 and n keys.

The number of keys that may be indexed using a B+ tree is a function of the order of the tree and its height.

For a n-order B+ tree with a height of h:

- maximum number of keys is n^h.
- minimum number of keys is $2(n/2)^{h-1}$.

The leaves (the bottom-most index blocks) of the B+ tree are often linked to one another in a linked list; this makes range queries or an (ordered) iteration through the blocks simpler and more efficient (though the aforementioned upper bound can be achieved even without this addition). This does not substantially increase space consumption or maintenance on the tree. This illustrates one of the significant advantages of a B+tree over a B-tree; in a B-tree, since not all keys are present in the leaves, such an ordered linked list cannot be constructed. A B+tree is thus particularly useful as a database system index, where the data typically resides on disk, as it allows the B+tree to actually provide an efficient structure for housing the data itself (this is described in as index structure.
If a storage system has a block size of B bytes, and the keys to be stored have a size of k, arguably the most efficient B+ tree is one where $b = \frac{B}{k} - 1$. Although theoretically the one-off is unnecessary, in practice there is often a little extra space taken up by the index blocks (for example, the linked list references in the leaf blocks). Having an index block which is slightly larger than the storage system's actual block represents a significant performance decrease; therefore erring on the side of caution is preferable.
If nodes of the B+ tree are organized as arrays of elements, then it may take a considerable time to insert or delete an element as half of the array will need to be shifted on average. To overcome this problem, elements inside a node can be organized in a binary tree or a B+ tree instead of an array.
B+ trees can also be used for data stored in RAM. In this case a reasonable choice for block size would be the size of processor's cache line. Space efficiency of B+ trees can be improved by using some compression techniques. One possibility is to use delta encoding to compress keys stored into each block. For internal blocks, space saving can be achieved by either compressing keys or pointers. For string keys, space can be saved by using the following technique: Normally the $i - th$ entry of an internal block contains the first key of block $i + 1$.. Instead of storing the full key, we could store the shortest prefix of the first key of block $i + 1$ that is strictly greater (in lexicographic order) than last key of block i. There is also a simple way to compress pointers: if we suppose that some consecutive blocks $i, i + 1, ...i + k$ are stored contiguously, then it will suffice to store only a pointer to the first block and the count of consecutive blocks

3 General Solution

3.1 Operations of B+ tree

There are 3 types of operations we can perform using B+ tree. They are:

- Search Operation
- Insert Operation
- Delete Operation

3.1.1 Search Operation

In B+ Tree, a search is one of the easiest procedures to execute and get fast and accurate results from it. The following search algorithm is applicable:

- To find the required record, you need to execute the binary search on the available records in the Tree.
- In case of an exact match with the search key, the corresponding record is returned to the user.
- In case the exact key is not located by the search in the parent, current, or leaf node, then a ”not found message” is displayed to the user.
- The search process can be re-run for better and more accurate results.

```
 1. Call the binary search method on the records in the B+ Tree.
 2. If the search parameters match the exact key
            The accurate result is returned and displayed to the user
          Else, if the node being searched is the current and the exact key is n
ot found by the algorithm
            Display the statement "Recordset cannot be found."
```

Figure 1: Search Operation Algorithm

Output:
The matched record set against the exact key is displayed to the user; otherwise, a failed attempt is shown to the user.

3.1.2 Insert Operation

The following algorithm is applicable for the insert operation:

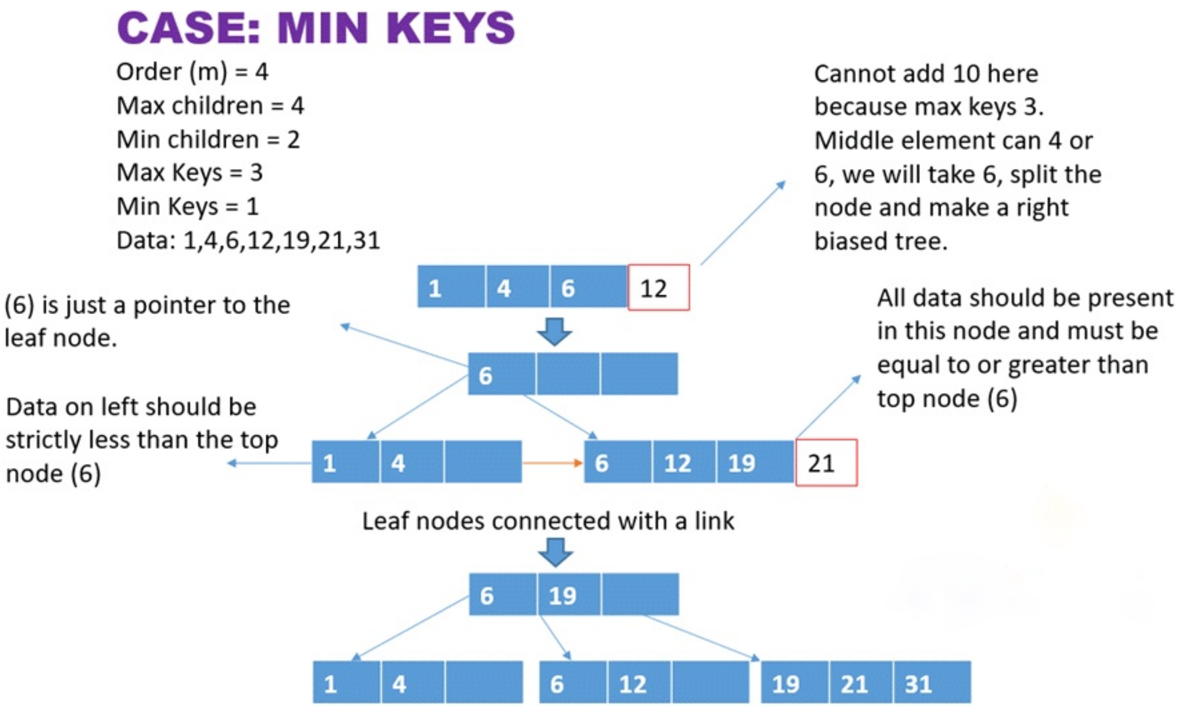
- 50 percent of the elements in the nodes are moved to a new leaf for storage.
- The parent of the new Leaf is linked accurately with the minimum key value and a new location in the Tree.
- Split the parent node into more locations in case it gets fully utilized.
- Now, for better results, the center key is associated with the top-level node of that Leaf.
- Until the top-level node is not found, keep on iterating the process explained in the above steps.

```
1.      Even inserting at-least 1 entry into the leaf container does not make it
full then add the record
     2. Else, divide the node into more locations to fit more records.
      a. Assign a new leaf and transfer 50 percent of the node elements to a new
placement in the tree
      b. The minimum key of the binary tree leaf and its new key address are ass
ociated with the top-level node.
      c. Divide the top-level node if it gets full of keys and addresses.
          i. Similarly,  insert a key in the center of the top-level node in the
hierarchy of the Tree.
     d. Continue to execute the above steps until a top-level node is found that
does not need to be divided anymore.

3) Build a new top-level root node of 1 Key and 2 indicators.
```

Figure 2: Insert Operation Algorithm

The above B+ Tree sample example is explained in the steps below:



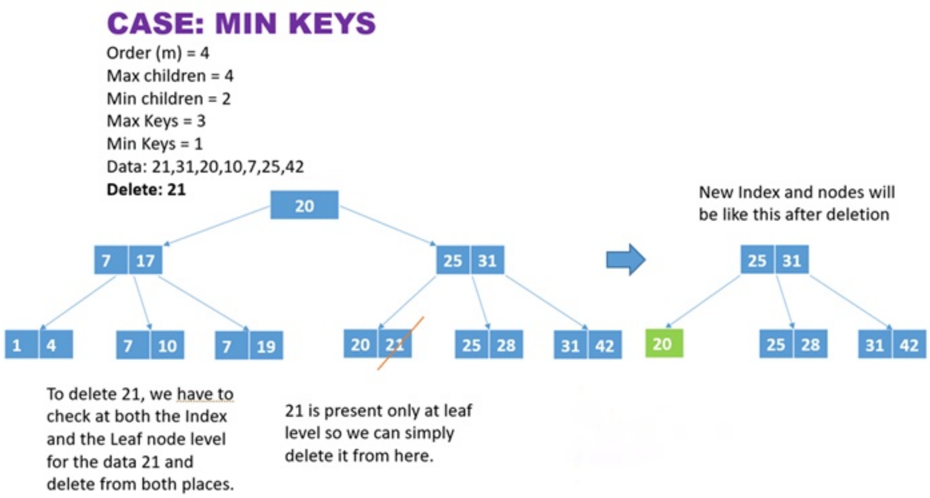
- Firstly, we have 3 nodes, and the first 3 elements, which are 1, 4, and 6, are added on appropriate locations in the nodes.
- The next value in the series of data is 12 that needs to be made part of the Tree.
- To achieve this, divide the node, add 6 as a pointer element.
- Now, a right-hierarchy of a tree is created, and remaining data values are adjusted accordingly by keeping in mind the applicable rules of equal to or greater than values against the key-value nodes on the right.

3.1.3 Delete Operation

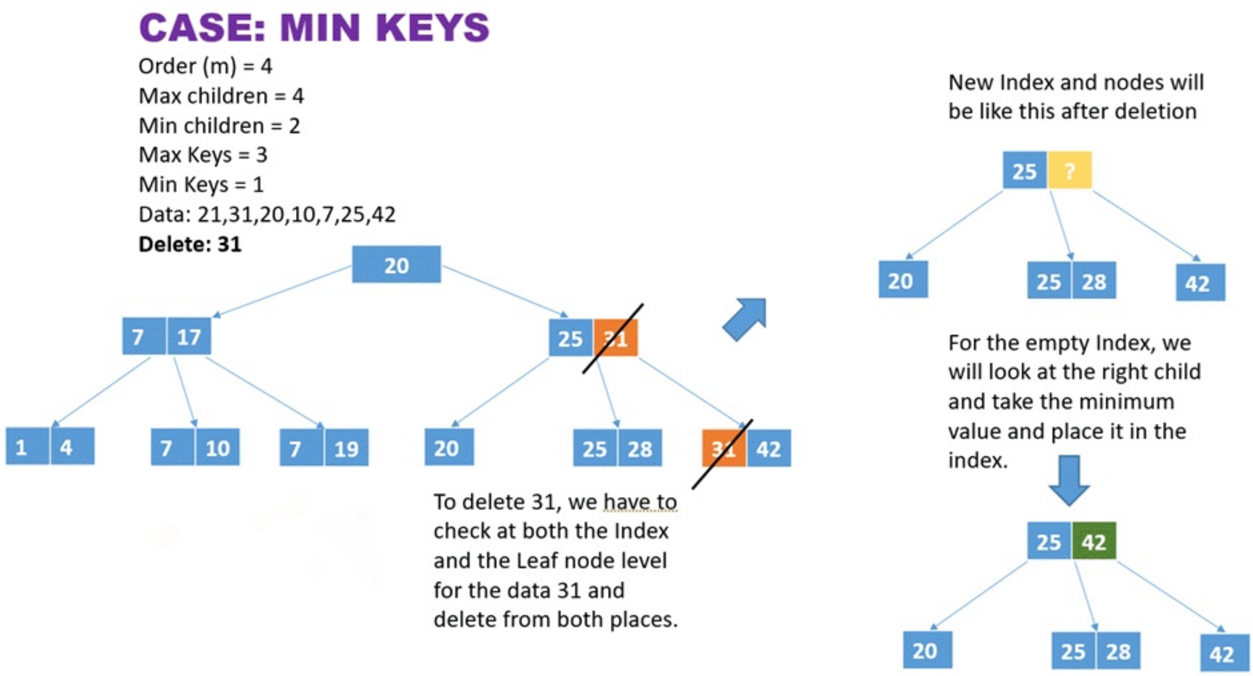
The complexity of the delete procedure in the B+ Tree surpasses that of the insert and search functionality.
The following algorithm is applicable for the insert operation:

- Firstly, we need to locate a leaf entry in the Tree that is holding the key and pointer. , delete the leaf entry from the Tree if the Leaf fulfills the exact conditions of record deletion.
- In case the leaf node only meets the satisfactory factor of being half full, then the operation is completed; otherwise, the Leaf node has minimum entries and cannot be deleted.
- The other linked nodes on the right and left can vacate any entries then move them to the Leaf. If these criteria is not fulfilled, then they should combine the leaf node and its linked node in the tree hierarchy.
- Upon merging of leaf node with its neighbors on the right or left, entries of values in the leaf node or linked neighbor pointing to the top-level node are deleted.

The example above illustrates the procedure to remove an element from the B+ Tree of a specific order.



- Firstly, the exact locations of the element to be deleted are identified in the Tree.
- Here the element to be deleted can only be accurately identified at the leaf level and not at the index placement. Hence, the element can be deleted without affecting the rules of deletion, which is the value of the bare-minimum key.



- In the above example, we have to delete 31 from the Tree.
- We need to locate the instances of 31 in Index and Leaf.
- We can see that 31 is available in both Index and Leaf node level. Hence, we delete it from both instances.
- But, we have to fill the index pointing to 42. We will now look at the right child under 25 and take the minimum value and place it as an index. So, 42 being the only value present, it will become the index.

```
1) Start at the root and go up to leaf node containing the key K
2) Find the node n on the path from the root to the leaf node containing K
    A. If n is root, remove K
         a. if root has more than one key, done
         b. if root has only K
            i) if any of its child nodes can lend a node
               Borrow key from the child and adjust child links
            ii) Otherwise merge the children nodes. It will be a new root
         c. If n is an internal node, remove K
            i) If n has at least ceil(m/2) keys, done!
            ii) If n has less than ceil(m/2) keys,
                If a sibling can lend a key,
                Borrow key from the sibling and adjust keys in n and the parent
node
                    Adjust child links
                Else
                    Merge n with its sibling
                    Adjust child links
         d. If n is a leaf node, remove K
            i) If n has at least ceil(M/2) elements, done!
                In case the smallest key is deleted, push up the next key
            ii) If n has less than ceil(m/2) elements
            If the sibling can lend a key
                Borrow key from a sibling and adjust keys in n and its parent no
de
            Else
                Merge n and its sibling
                Adjust keys in the parent node
```

Figure 3: Delete Operation Algorithm

Output:
The Key "K" is deleted, and keys are borrowed from siblings for adjusting values in n and its parent nodes if needed.

4 Example

We will see an example of Insertion in B+ tree.

- Each node except root can have a maximum of **M** children and at least **ceil(M/2)** children.
- Each node can contain a maximum of **M** – 1 keys and a minimum of **ceil(M/2) – 1** keys.
- The root has at least two children and atleast one search key.
- While insertion overflow of the node occurs when it contains more than **M** – 1 search key values.

Here M is the order of B+ tree.

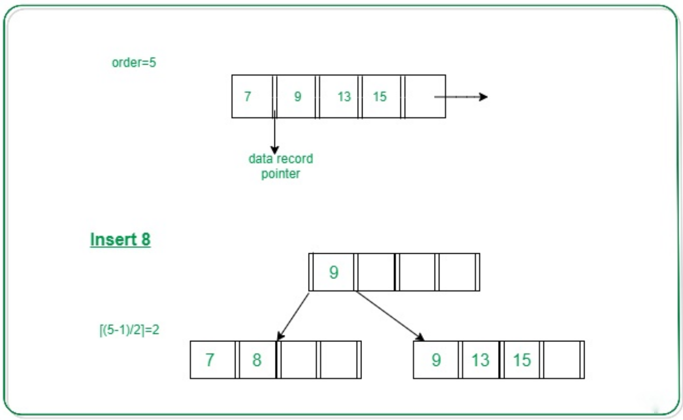
Steps for insertion in B+ Tree

Step 1: Every element is inserted into a leaf node. So, go to the appropriate leaf node.
Step 2: Insert the key into the leaf node in increasing order only if there is no overflow. If there is an overflow go ahead with the following steps mentioned below to deal with overflow while maintaining the B+ Tree properties.

Case 1:

1. Split the leaf node into two nodes.
2. First node contains ceil((m-1)/2) values. environment.
3. Second node contains the remaining values.
4. Copy the smallest search key value from second node to the parent node.(Right biased)

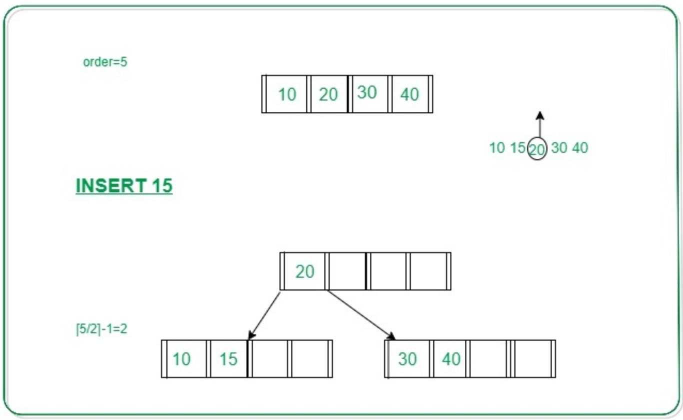
Below is the illustration of inserting 8 into B+ Tree of order of 5:



Case 2:

1. Split the leaf node into two nodes.
2. First node contains ceil(m/2)-1 values.
3. Move the smallest among remaining to the parent.
4. Second node contains the remaining keys.

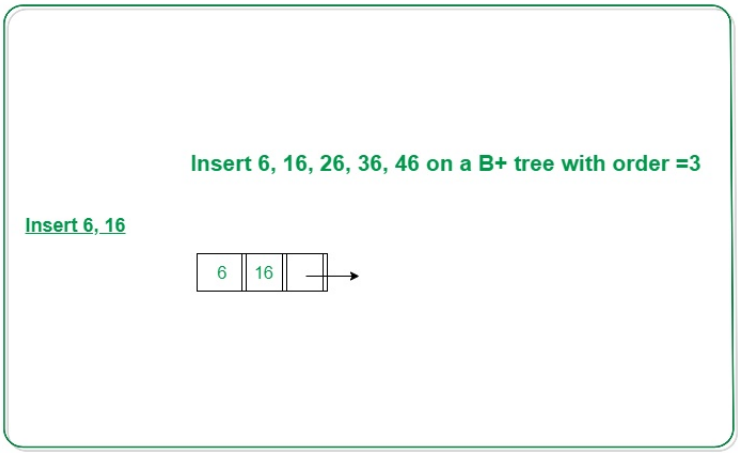
Below is the illustration of inserting 15 into B+ Tree of order of 5:



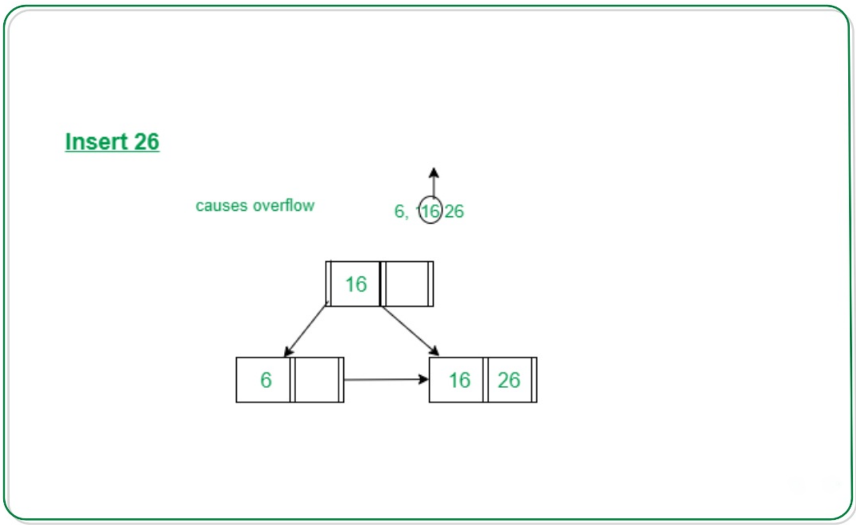
Example to illustrate insertion on a B+ tree

Problem: Insert the following key values 6, 16, 26, 36, 46 on a B+ tree with order = 3.
Solution:

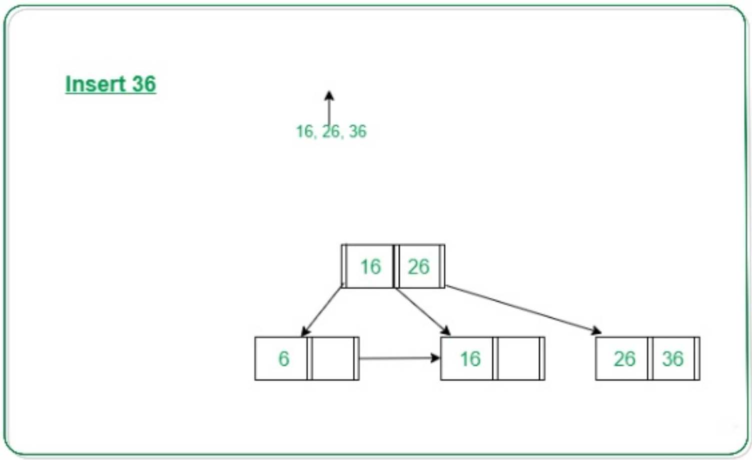
Step 1: The order is 3 so at maximum in a node so there can be only 2 search key values. As insertion happens on a leaf node only in a B+ tree so insert search key value 6 and 16 in increasing order in the node. Below is the illustration of the same:



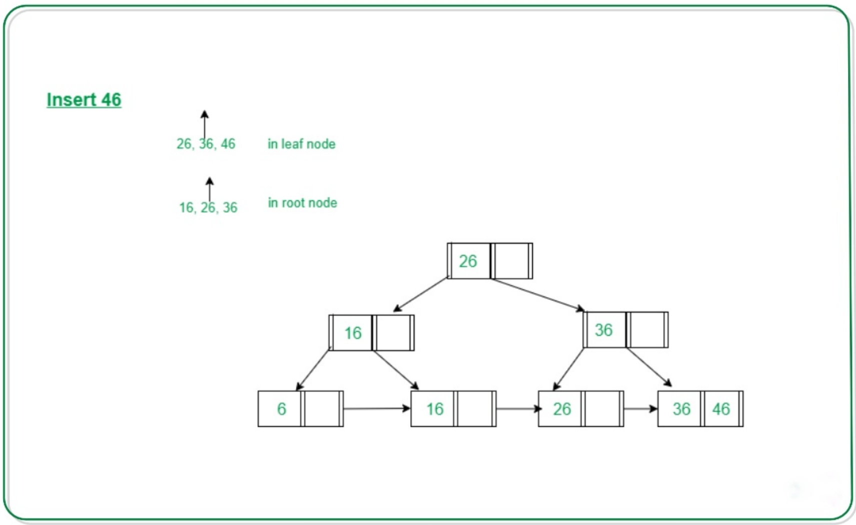
Step 2: We cannot insert 26 in the same node as it causes an overflow in the leaf node, We have to split the leaf node according to the rules. First part contains ceil((3-1)/2) values i.e., only 6. The second node contains the remaining values i.e., 16 and 26. Then also copy the smallest search key value from the second node to the parent node i.e., 16 to the parent node. Below is the illustration of the same:



Step 3: Now the next value is 36 that is to be inserted after 26 but in that node, it causes an overflow again in that leaf node. Again follow the above steps to split the node. First part contains ceil((3-1)/2) values i.e., only 16. The second node contains the remaining values i.e., 26 and 36. Then also copy the smallest search key value from the second node to the parent node i.e., 26 to the parent node. Below is the illustration of the same: The illustration is shown in the diagram below.



Step 4: Now the next value is 36 that is to be inserted after 26 but in that node, it causes an overflow again in that leaf node. Again follow the above steps to split the node. First part contains ceil((3-1)/2) values i.e., only 16. The second node contains the remaining values i.e., 26 and 36. Then also copy the smallest search key value from the second node to the parent node i.e., 26 to the parent node. Below is the illustration of the same: The illustration is shown in the diagram below.



5 Complexity Analysis

Table 1: Time complexity in big O notation

Algorithm	Average	Worst Case
Space	O(n)	O(n)
Search	O(log n)	O(log n + log L)
Insert	O(log n)	O(M*log n + log L)
Delete	O(log n)	O(M*log n + log L)