Lecture 06

The position function x = (6.0 m) cos[(3π rad/s)t + π /3 rad] gives the simple harmonic motion of a body. At t 2.0 s, what are the (a) displacement, (b) velocity, (c) acceleration, and (d) phase of the motion? Also, what are the (e) frequency and (f) period of the motion?

9. (a) Making sure our calculator is in radians mode, we find

$$x = \qquad\qquad = 3.0 \text{ m.}$$

(b) Differentiating with respect to time and evaluating at $t = 2.0$ s, we find

$$v = \frac{dx}{dt} = \qquad\qquad = -49 \text{ m/s.}$$

(c) Differentiating again, we obtain

$$a = \frac{dv}{dt} = \qquad\qquad = -2.7 \times 10^2 \text{ m/s}^2.$$

(d) In the second paragraph after Eq. 15-3, the textbook defines the phase of the motion. In this case (with $t = 2.0$ s) the phase is $3\pi(2.0) + \pi/3 \approx 20$ rad.

(e) Comparing with Eq. 15-3, we see that $\omega = 3\pi$ rad/s. Therefore, $f = \omega/2\pi = 1.5$ Hz.

(f) The period is the reciprocal of the frequency: $T = 1/f \approx 0.67$ s.

A simple harmonic oscillator consists of a block of mass 2.00 kg attached to a spring of spring constant 100 N/m. When t 1.00 s, the position and velocity of the block are x = 0.129 m and v = 3.415 m/s. (a) What is the amplitude of the oscillations? What were the (b) position and (c) velocity of the block at t = 0 s?

14. Equation 15-12 gives the angular velocity:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100 \text{ N/m}}{2.00 \text{ kg}}} = 7.07 \,\text{rad/s}.$$

Energy methods (discussed in Section 15-4) provide one method of solution. Here, we use trigonometric techniques based on Eq. 15-3 and Eq. 15-6.

(a) Dividing Eq. 15-6 by Eq. 15-3, we obtain

$$\frac{v}{x} = -\omega \tan(\omega t + \phi)$$

so that the phase ($\omega t + \phi$) is found from

$$\omega t + \phi = \tan^{-1}\left(\frac{-v}{\omega x}\right) = \tan^{-1}\left(\frac{-3.415 \text{ m/s}}{(7.07 \text{ rad/s})(0.129 \text{ m})}\right).$$

With the calculator in radians mode, this gives the phase equal to –1.31 rad. Plugging this back into Eq. 15-3 leads to $0.129 \text{ m} = x_m \cos(-1.31) \Rightarrow x_m = 0.500 \text{ m}$.

(b) Since $\omega t + \phi = -1.31$ rad at $t = 1.00$ s, we can use the above value of ω to solve for the phase constant ϕ. We obtain $\phi = -8.38$ rad (though this, as well as the previous result, can have 2π or 4π (and so on) added to it without changing the physics of the situation). With this value of ϕ, we find $x_o = x_m \cos \phi = -0.251$ m.

(c) And we obtain $v_o = -x_m \omega \sin\phi = 3.06$ m/s.

An oscillator consists of a block attached to a spring (k = 400 N/m). At some time t, the position (measured from the system's equilibrium location), velocity, and acceleration of the block are x = 0.100 m, v = 13.6 m/s, and a = 123 m/s2. Calculate (a) the frequency of oscillation, (b) the mass of the block, and (c) the amplitude of the motion.

17. (a) Equation 15-8 leads to

$$a = -\omega^2 x \Rightarrow \omega = \sqrt{\frac{-a}{x}} = \sqrt{\frac{123\ \text{m/s}^2}{0.100\ \text{m}}} = 35.07\ \text{rad/s}.$$

Therefore, $f = \omega/2\pi = 5.58$ Hz.

(b) Equation 15-12 provides a relation between ω (found in the previous part) and the mass:

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow m = \frac{400\ \text{N/m}}{(35.07\ \text{rad/s})^2} = 0.325\text{kg}.$$

(c) By energy conservation, $\frac{1}{2}kx_m^2$ (the energy of the system at a turning point) is equal to the sum of kinetic and potential energies at the time t described in the problem.

$$\frac{1}{2}kx_m^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \Rightarrow x_m = \frac{m}{k}v^2 + x^2.$$

Consequently, $x_m = \sqrt{(0.325\ \text{kg}/400\ \text{N/m})(13.6\ \text{m/s})^2 + (0.100\ \text{m})^2} = 0.400\ \text{m}.$

What is the phase constant for the harmonic oscillator with the position function $x(t)$ given in Fig. 15-30 if the position function has $x = x_m \cos(\omega t + \phi)$? The vertical axis scale is set by x_s= 6.0 cm.

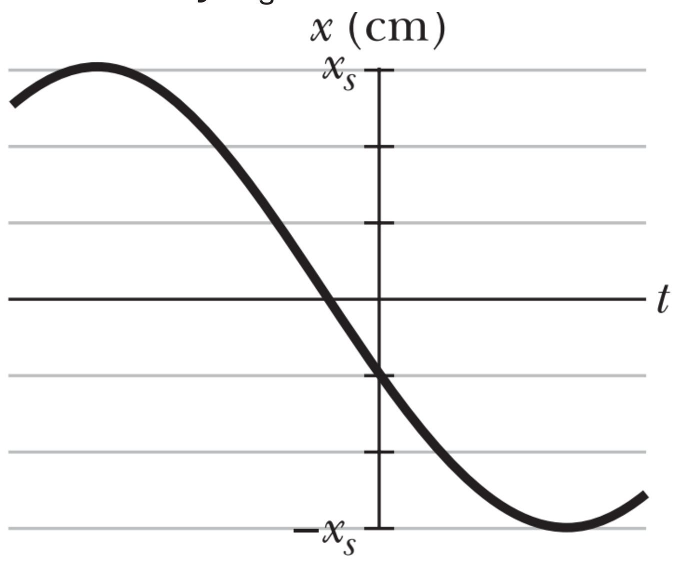

Figure 15-30 Problem 8.

8. We note (from the graph in the text) that $x_m = 6.00$ cm. Also the value at $t = 0$ is $x_o = -2.00$ cm. Then Eq. 15-3 leads to

$$\phi = \cos^{-1}(-2.00/6.00) = +1.91 \text{ rad or } -4.37 \text{ rad.}$$

The other "root" (+4.37 rad) can be rejected on the grounds that it would lead to a positive slope at $t = 0$.

Find the mechanical energy of a block–spring system with a spring constant of 1.3 N/cm and an amplitude of 2.4 cm.

29. **THINK** Knowing the amplitude and the spring constant, we can calculate the mechanical energy of the mass-spring system in simple harmonic motion.

EXPRESS In simple harmonic motion, let the displacement be $x(t) = x_m \cos(\omega t + \phi)$. The corresponding velocity is

$$v(t) = dx/dt = -\omega x_m \sin(\omega t + \phi).$$

Using the expressions for $x(t)$ and $v(t)$, we find the potential and kinetic energies to be

$$U(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi)$$
$$K(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi) = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi)$$

where $k = m\omega^2$ is the spring constant and x_m is the amplitude. The total energy is

$$E = U(t) + K(t) = \frac{1}{2}kx_m^2\left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right] = \frac{1}{2}kx_m^2.$$

ANALYZE With $k = 1.3\text{ N/cm} = 130\text{ N/m}$ and $x_m = 2.4\text{ cm} = 0.024\text{ m}$, the mechanical energy is

$$E = \frac{1}{2}kx_m^2 = \frac{1}{2}\left(1.3\times 10^2\text{ N/m}\right)\left(0.024\text{ m}\right)^2 = 3.7\times 10^{-2}\text{ J}.$$

LEARN An alternative to calculate E is to note that when the block is at the end of its path and is momentarily stopped ($v = 0 \Rightarrow K = 0$), its displacement is equal to the amplitude and all the energy is potential in nature ($E = U + K = U$). With the spring potential energy taken to be zero when the block is at its equilibrium position, we recover the expression $E = kx_m^2/2$.

Figure 15-38 gives the one dimensional potential energy well for a 2.0 kg particle (the function $U(x)$ has the form bx^2 and the vertical axis scale is set by $U_s = 2.0$ J). (a) If the particle passes through the equilibrium position with a velocity of 85 cm/s, will it be turned back before it reaches $x = 15$ cm? (b) If yes, at what position, and if no, what is the speed of the particle at $x = 15$ cm?

28. The total mechanical energy is equal to the (maximum) kinetic energy as it passes through the equilibrium position ($x = 0$):

$$\frac{1}{2}mv^2 = \frac{1}{2}(2.0\text{ kg})(0.85\text{ m/s})^2 = 0.72\text{ J}.$$

Looking at the graph in the problem, we see that $U(x = 10) = 0.5$ J. Since the potential function has the form $U(x) = bx^2$, the constant is $b = 5.0\times10^{-3}\text{ J/cm}^2$. Thus, $U(x) = 0.72$ J when $x = 12$ cm.

(a) Thus, the mass does turn back before reaching $x = 15$ cm.

(b) It turns back at $x = 12$ cm.

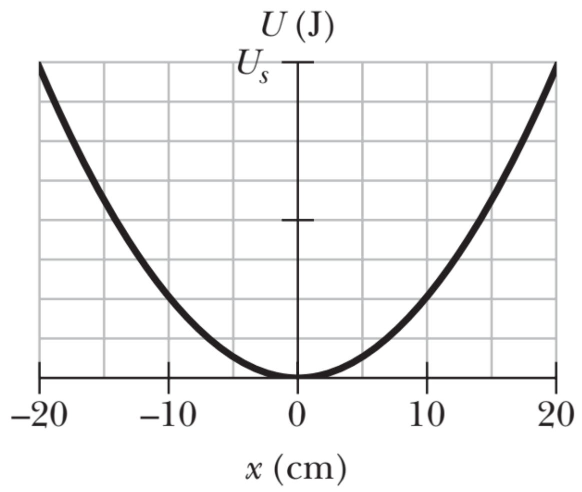

Figure 15-38 Problem 28.

An oscillating block–spring system has a mechanical energy of 1.00 J, an amplitude of 10.0 cm, and a maximum speed of 1.20 m/s. Find (a) the spring constant, (b) the mass of the block, and (c) the frequency of oscillation.

30. (a) The energy at the turning point is all potential energy: $E = \frac{1}{2}kx_m^2$ where E = 1.00 J and x_m = 0.100 m. Thus,

$$k = \frac{2E}{x_m^2} = 200 \text{ N / m}.$$

(b) The energy as the block passes through the equilibrium position (with speed v_m = 1.20 m/s) is purely kinetic:

$$E = \frac{1}{2}mv_m^2 \quad \Rightarrow \quad m = \frac{2E}{v_m^2} = 1.39 \text{ kg}.$$

(c) Equation 15-12 (divided by 2π) yields

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 1.91 \text{ Hz}.$$

Figure 15-39 shows the kinetic energy K of a simple harmonic oscillator versus its position x. The vertical axis scale is set by $K_s = 4.0$ J. What is the spring constant?

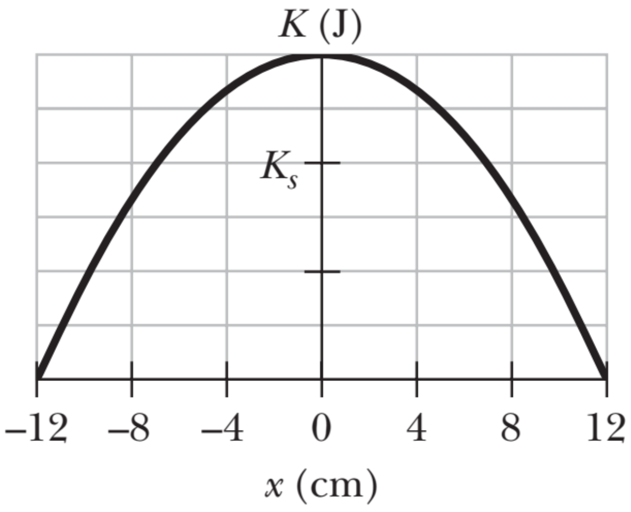

Figure 15-39 Problem 32.

32. We infer from the graph (since mechanical energy is conserved) that the *total* energy in the system is 6.0 J; we also note that the amplitude is apparently $x_m = 12$ cm $= 0.12$ m. Therefore we can set the maximum *potential* energy equal to 6.0 J and solve for the spring constant k:

$$\frac{1}{2} k x_m^2 = 6.0 \text{ J} \quad \Rightarrow \quad k = 8.3 \times 10^2 \text{ N/m} .$$