FFT Butterfly: Intuition and Math

YESSSS 🔥 Let’s completely clean this up and rebuild it. If you don’t understand the butterfly, you don’t understand the FFT.

We are going to slow down completely. No code. No speed. Just pure intuition and beautifully formatted math.

🌊 Step 1: Start From the Original DFT

This is the original Discrete Fourier Transform. It is just one big sum:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

To make it cleaner, we use the "twiddle factor" notation, where $W_N = e^{-j2\pi/N}$. So the original sum is simply:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

Nothing fancy yet. Just a standard formula.

🧠 Step 2: Split the Signal (Even & Odd)

The core trick of the FFT is to split that big sum into **EVEN** and **ODD** indices. Every integer n is either even ($n = 2m$) or odd ($n = 2m + 1$).

Instead of summing n from 0 to $N - 1$, we sum m from 0 to $N/2 - 1$. We just substitute this into our formula:

Even Terms:

$$\sum_{m=0}^{N/2-1} x[2m] W_N^{k(2m)}$$

Odd Terms:

$$\sum_{m=0}^{N/2-1} x[2m+1]W_N^{k(2m+1)}$$

Putting them together, we just rewrote the exact same sum, separated into two parts:

$$X[k] = \sum_{m=0}^{N/2-1} x[2m]W_N^{2km} + \sum_{m=0}^{N/2-1} x[2m+1]W_N^{k(2m+1)}$$

🚀 Step 3: The Magic Simplification

Now we look at the twiddle factor for the even terms: W_N^{2km}.

If we expand it out:

$$W_N^{2km} = e^{-j2\pi(2km)/N} = e^{-j2\pi km/(N/2)}$$

But look closely at that last term! That is exactly the definition of a twiddle factor for a sequence of half the size: $W_{N/2}^{km}$. 🔥 🔥 🔥 **This is the key reason the FFT works!**

We can rewrite our split formula as:

$$X[k] = \underbrace{\sum_{m=0}^{N/2-1} x[2m]W_{N/2}^{km}}_{E[k]} + W_N^k \underbrace{\sum_{m=0}^{N/2-1} x[2m+1]W_{N/2}^{km}}_{O[k]}$$

- $E[k]$ is simply a smaller DFT of the even samples.
- $O[k]$ is another smaller DFT of the odd samples.

🦋 Step 4: The Butterfly Emerges

Because we only calculated $N/2$ values for $E[k]$ and $O[k]$, we still need the upper half of the spectrum for our full N-point DFT. We need to evaluate the formula at $k + N/2$.

Here is the deep insight: the twiddle factor has a special periodicity:

$$W_N^{N/2} = e^{-j2\pi(N/2)/N} = e^{-j\pi} = -1$$

So, when we plug in $k + N/2$, that exponential property flips the sign of the odd part! This gives us our final, elegant pair of equations:

$$X[k] = E[k] + W_N^k O[k]$$

$$X[k + N/2] = E[k] - W_N^k O[k]$$

THIS PAIR is called the butterfly.

Why Is It Called a Butterfly?

If you draw out the signal flow:

- You take two inputs ($E[k]$ and $O[k]$).
- You rotate the odd part (multiply by W_N^k).
- You add them to get one output.
- You subtract them to get the other output.
- The crossing lines in the computation graph look exactly like butterfly wings!

🎯 Why Do We Compute Both?

If you only computed the first branch ($E[k] + W_N^k O[k]$), you would only get the bottom half of your frequency spectrum!

We need an N-point DFT, which means N independent equations. Since E and O are only $N/2$ points long, each pair of $(E[k], O[k])$ contains enough mathematical information to generate *two* distinct frequency bins. The `+` branch builds the lower frequencies, and the `-` branch builds the upper frequencies. Without the minus branch, you lose exactly half of your frequency information.

🌟 Concrete Example (N=4)

Let's ground this in reality with a 4-point signal: x_0, x_1, x_2, x_3.

1. **Split:** * Even: $x_0, x_2 \to$ gives us $E[0], E[1]$
   - Odd: $x_1, x_3 \to$ gives us $O[0], O[1]$
2. **Combine (The Butterfly):**
   - For $k = 0$ ($W_4^0 = 1$):
     - $$X[0] = E[0] + O[0]$$
     - $$X[2] = E[0] - O[0]$$

- For $k = 1$ ($W_4^1 = -j$):
  - $$X[1] = E[1] - jO[1]$$
  - $$X[3] = E[1] + jO[1]$$

Instead of doing 16 complex multiplications (for an N^2 standard DFT), we broke it down into smaller, simpler operations, drastically reducing the total work to $N \log_2 N$.

💡 The Deep "Why"

Why does this decomposition work at all?

It works because the complex exponentials in the Fourier basis are **eigenfunctions of time shifts**. Evaluating the second half of the spectrum ($k + N/2$) naturally creates a $(-1)^n$ term.

- $(-1)^n$ is $+1$ for even samples.
- $(-1)^n$ is -1 for odd samples.

This creates a natural, mathematical separation between even/odd samples in the time domain, and lower/upper halves in the frequency domain.

Now I have a question to test your intuition: If instead of using complex exponentials we used a cosine-only basis (like a real Fourier series or DCT), do you think this clean halving trick would still work mathematically?