Models of Computation Assignment 1

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Q1 Propositional Logic: Island Puzzle

Task A

Propositional Letters

$$A : \text{A is a knight}$$

$$B : \text{B is a knight}$$

$$C : \text{C is a knight}$$

$$M_K : \text{Mimic is a knight}$$

$$M_A : \text{A is the mimic}$$

$$M_B : \text{B is the mimic}$$

$$M_C : \text{C is the mimic}$$

Using these propositional letters we will form our propositional formulas based on each of the statements made:

$$A \rightarrow (M_C \vee C) \qquad \text{C is either the mimic or a knight, or both.}$$

$$B \rightarrow \neg(M_A \wedge \neg C) \quad \text{It is not the case that both A is the mimic and C is a knave}$$

$$C \rightarrow (B \rightarrow \neg M_K) \qquad \text{If B is a knight, then the mimic is a knave}$$

Task B

Assume Mimic is a knave:

Assume A is the Mimic:

$\therefore \{A \mapsto 0, M_A \mapsto 1, M_K \mapsto 0\}$ and A's hence the following is true

- C is NEITHER a mimic or a knight or both
- That is $\{M_C \mapsto 0, C \mapsto 0\}$
- $\therefore C$'s statement is false and they are a knave

And so far $\{A \mapsto 0, M_A \mapsto 1, M_K, \mapsto 0, M_C \mapsto 0, C \mapsto 0\}$

So if A is the mimic and a knave and C is also a knave, we know that B too must be a knave because their statement would be a lie in this case and otherwise we get a contradiction.

$\therefore B \mapsto 0$

So for C's statement to be a lie the Mimic must be a knave which it is making it "not the case that if B is a knight, then the mimic is a knave"

So if all of A, B and C are lying, their statements hold and in this case `A is the mimic`

Q2 Propositional Logic: Validity and Satisfiability

1.

Simplify to a logically equivalent formula

$$\neg P \land (P \to \neg P)$$

$$\equiv \neg P \land (\neg P \lor \neg P)$$

$$\equiv \neg P \land \neg P$$

$$\equiv \neg P$$

Under the truth assignment $\{P \mapsto 1\}$, the equation comes out to 0, and under $\{P \mapsto 0\}$ and the formula comes out to 1.

Hence, $\neg P \land (P \to \neg P)$ is `contingent`

2.

Simplify to a logically equivalent formula

$$(P \lor (P \land (Q \to Q))) \land (\neg P \lor \neg(\neg Q \to \neg Q))$$

$$\equiv (P \lor (P \land \top)) \land (\neg P \lor \neg \top)$$

$$\equiv (P \lor (P \land \top)) \land (\neg P \lor \bot)$$

$$\equiv (P \lor (P \land \top)) \land (\neg P \lor \bot)$$

Consider the following:

$(P \land \top) \equiv P$ and $(\neg P \lor \bot) \equiv \neg P$

Knowing this we can now put the formula in CNF:

$$(P \lor P) \land \neg P$$

$$\equiv (P \land \neg P)$$

Then using refutation resolution we get:

$$P \qquad \neg P$$

$$\bot$$

Hence, $(P \lor (P \land (Q \to Q))) \land (\neg P \lor \neg(\neg Q \to \neg Q))$ is `unsatisfiable`

3.

Formula:

$$\neg((Q \lor P) \to P) \lor (P \leftrightarrow Q) \lor (P \land \neg Q)$$

`Truth table` for the formula:

P	Q	$(Q \lor P)$	$((Q \lor P) \to P)$	$\neg((Q \lor P) \to P)$	$(P \leftrightarrow Q)$	$(\neg Q)$	$(P \land \neg Q)$	**Formula Result**
0	0	0	1	0	1	1	0	1
0	1	1	0	1	0	0	0	1
1	0	1	1	0	0	1	1	1
1	1	1	1	0	1	0	0	1

According to our truth table the formula is valid

Let us prove using resolution refutation:

Eliminate the implication: $(Q \lor P) \to P \equiv \neg(Q \lor P) \lor P$

Deal with the biimplication: $(P \leftrightarrow Q)$

$$\begin{aligned} &(P \leftrightarrow Q) \\ &\equiv (P \land Q) \lor (\neg P \land \neg Q) \end{aligned}$$

To prove that the formula is valid using refutation we will assume the the contrary (that it is unsatisfiable) and negate it :

$$\begin{aligned} &\neg(\neg(\neg(Q \lor P) \lor P) \lor ((P \land Q) \lor (\neg P \land \neg Q)) \lor (P \land \neg Q)) \\ &\equiv (\neg(Q \lor P) \lor P) \land \neg((P \land Q) \lor (\neg P \land \neg Q)) \land \neg(P \land \neg Q) \\ &\equiv ((\neg Q \land \neg P) \lor P) \land (\neg(P \land Q) \land \neg(\neg P \land \neg Q)) \land (\neg P \lor Q) \\ &\equiv ((P \lor \neg Q) \land (P \lor \neg P)) \land ((\neg P \lor \neg Q) \land (P \lor Q)) \land (\neg P \lor Q) \\ &\equiv ((P \lor \neg Q) \land \top) \land (\neg P \lor \neg Q) \land (P \lor Q) \land (\neg P \lor Q) \\ &\equiv (P \lor \neg Q) \land (\neg P \lor \neg Q) \land (P \lor Q) \land (\neg P \lor Q) \end{aligned}$$

Now resolving the CNF formula:

$$(P \lor \neg Q) \qquad (\neg P \lor \neg Q) \qquad (P \lor Q) \qquad (\neg P \lor Q)$$

$$(\neg Q \lor \neg Q) \qquad\qquad (Q \lor Q)$$

$$\neg Q \qquad\qquad Q$$

$$\bot$$

Therefore because the resolution of the negation of the formula: $\neg((Q \lor P) \to P) \lor (P \leftrightarrow Q) \lor (P \land \neg Q)$ is unsatisfiable, $\neg((Q \lor P) \to P) \lor (P \leftrightarrow Q) \lor (P \land \neg Q)$ is `valid`

4.

Formula:

$$(R \leftrightarrow P) \to ((R \to Q) \leftrightarrow (P \leftrightarrow Q))$$

`Truth table` for the formula:

P	Q	R	$(R \leftrightarrow P)$	$(P \leftrightarrow Q)$	$(R \rightarrow Q)$	$((R \rightarrow Q) \leftrightarrow (P \leftrightarrow Q))$	$(R \leftrightarrow P) \rightarrow ((R \rightarrow Q) \leftrightarrow (P \leftrightarrow Q))$
0	0	0	1	1	1	1	1
0	0	1	0	1	0	0	1
0	1	0	1	0	1	0	0
0	1	1	0	0	1	0	1
1	0	0	0	0	1	0	1
1	0	1	1	0	0	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

According to our truth table the formula is contingent

Let us now validate this with two different truth assignments to show both case

Consider the truth assignment $\{P \mapsto 0, Q \mapsto 1, R \mapsto 0\}$ to the formula:

$$\begin{aligned}&(0 \leftrightarrow 0) \rightarrow ((0 \rightarrow 1) \leftrightarrow (0 \leftrightarrow 1))\\&= 1 \rightarrow (1 \leftrightarrow 0)\\&= 1 \rightarrow 0\\&= 0\end{aligned}$$

Consider the truth assignment $\{P \mapsto 0, Q \mapsto 0, R \mapsto 0\}$ to the formula:

$$\begin{aligned}&(0 \leftrightarrow 0) \rightarrow ((0 \rightarrow 0) \leftrightarrow (0 \leftrightarrow 0))\\&= 1 \rightarrow (1 \leftrightarrow 1)\\&= 1 \rightarrow 1\\&= 1\end{aligned}$$

Q3 Predicate Logic: Translation and Semantics

Task A

1. `Iron is heavier than oxygen`

$$\begin{array}{ll} i: & \text{Iron (Const.)} \\ o: & \text{Oxygen (Const.)} \\ H(a,b) & \text{"a" heavier than "b" (Pred.)} \\ \hline H(i,o) & \text{Iron is heavier than oxygen} \end{array}$$

2. `All actinides are radioactive`

$$\begin{array}{ll} A(x) & \text{x is an actinide} \\ R(x) & \text{x is radioactive} \\ \forall x & \text{All x} \\ \hline \forall x(A(x) \rightarrow R(x)) & \text{All actinides are radioactive} \end{array}$$

3. `Some, but not all, lanthanides are radioactive`

$$\begin{array}{ll} x, y & \text{variables within the universe of discourse} \\ L(a) & \text{"a" is a lanthanide} \\ \exists x(L(x) \wedge R(x)) \wedge \exists y(L(y) \wedge R(y)) & \text{Some, but not all, lanthanides are radioactive} \\ \hline \end{array}$$

4. `Actinides are heavier than lanthanides`

$$\begin{array}{ll} \forall x \forall y((A(x) \wedge L(y)) \rightarrow H(x, y)) & \text{Actinides are heavier than lanthanides} \\ \hline \end{array}$$

5. `Both lanthanides and actinides are heavier than iron and oxygen`

$$\begin{array}{ll} \forall x((L(x) \vee A(x)) \rightarrow (H(x, i) \wedge H(x, o))) & \text{Both lanthanides and actinides are heavier than iron and oxygen} \\ \hline \end{array}$$

6. `At least three isotopes of lanthanides are radioactive, but the only lanthanide without any non-radioactive isotopes is promethium`

$$\begin{array}{ll} P(x) & \text{x is prometium} \\ I(x) & \text{x is an isotope} \\ x_n & n\text{th element} \\ \exists x_1 \exists x_2 \exists_3((I(x_1) \wedge I(x_2) \wedge I(x_3)) \wedge (R(x_1) \wedge R(x_2) \wedge R(x_3))) & \end{array}$$

Task B

To prove that the universe of every model of the formula;

$$\forall x \forall y(P(x, y) \rightarrow \neg P(y, x)) \wedge \forall x \exists y(P(x, y))$$

has at least 3 distict elements that is $U = \{a, b, c\}$, we shall assume for the sake of contradiction that the universe has 1 $U = \{a\}$, or 2 $U = \{a, b\}$ distinct elements.

Universe with 1 Element

Let $U = \{a\}$

If we obvserve the formula $\forall x \exists y(P(x, y))$ for a we need some y such that $P(a, y)$ holds,

However, since a is the only element in U, y must also be a.

$\therefore$ we need $P(a, a)$ to hold.

For $\forall x \forall y(P(x, y) \rightarrow \neg P(y, x))$ to be true we require $P(a, a) \rightarrow \neg P(a, a)$ to hold.

However this is a contradiction as it implies that if $P(a, a)$ holds, then $P(a, a)$ doesn't hold.

Universe with 2 Elements

Let $U = \{a, b\}$

The formula $\forall x \exists y(P(x, y))$ requires that $P(a, b)$ and $P(b, a)$ hold for the elements a and b respectively.

For the constraint $\forall x \forall y(P(x,y) \rightarrow \neg P(y,x))$, if $P(a,b)$ is true then $P(b,a)$ must false and vis versa.

However, $\forall x \exists y(P(x,y))$ requires both $P(a,b)$ and $P(b,a)$ to be true which is a contradiction.

Q4 Predicate Logic: Red-Black Trees

To prove that it is not a red black tree we must first negate the 2nd constraint:

$$\neg \forall x(R(x) \rightarrow (\neg R(l(x)) \wedge \neg R(r(x))))$$

$$\equiv \exists x \neg(\neg R(x) \vee (\neg R(l(x)) \wedge \neg R(r(x))))$$

$$\equiv \exists x(R(x) \wedge \neg(\neg R(l(x)) \wedge \neg R(r(x))))$$

$$\equiv \exists x(R(x) \wedge (R(l(x)) \vee R(r(x))))$$

> This translates to:
>
> "There exists an x such that x is a root node and it's left or right child is also red"

To apply resolution we then convert the second constraint to CNF:

$$R(x) \rightarrow (\neg R(l(x)) \wedge \neg R(r(x)))$$

$$\equiv \neg R(x) \vee (\neg R(l(x)) \wedge \neg R(r(x)))$$

$$\equiv (\neg R(x) \vee \neg R(l(x))) \wedge (\neg R(x) \vee \neg R(r(x)))$$

Now we have our clauses so we perform refutation:

$R(x)$ $\qquad$ $(\neg R(x) \vee \neg R(l(x)))$ $\qquad$ $(R(l(x)) \vee R(r(x)))$ $\qquad$ $(\neg R(x) \vee \neg R(r(x$

$\neg R(l(x))$ $\qquad$ $(R(l(x)) \vee \neg R(x))$

$\neg R(x)$

At this point we have derived $\neg R(x)$, however this contradicts our assumption $R(x)$.

$\therefore$ since the assumption was that $R(x) \wedge (R(l(x)) \vee R(r(x)))$ and that was shown to lead to $\neg R(x)$ when resolved against $(\neg R(x) \vee \neg R(l(x))) \wedge (\neg R(x) \vee \neg R(r(x)))$, it must be that tree in question is not a red-black tree.

Q5 Informal Proof: Palindromes

Issues

- The theory doesn't explicitly show how the reverse of the concatenation ss and how it results in ss.
- The theory fails to explicitly apply the definition of reversal correctly.
- The conclusion is uncrear, confusing and doesn't provide a clear and logical progression.

Corrected Proof

`Theorem:` Let s be a palindrome. Then their concatenation ss is also a palindrome

`Proof:` Let $s = a_1 \ldots a_n$ be a string of length n such that s is a palindrome. By definition, the palindrome s is equal to its reverse, so $s = a_n \ldots a_1$.

As s is a string and ss is their concatenation, it is also a string and is given by:

$$ss = a_1 \ldots a_n \ a_1 \ldots a_n$$

We want to reverse ss to show it is equal to ss:

$$\text{reverse}(ss) = \text{reverse}(s) \ \text{reverse}(s)$$

$$= \text{reverse}(a_1 \ldots a_n) \ \text{reverse}(a_1 \ldots a_n)$$

We then apply the definition of reversal from definition `4.c`, that is "the reverse of a nonempty string $a_1 \ldots a_n$ of length n is the reverse $b_1 \ldots b_n$" which gives us for $i = 1 \ldots i = n$:

$$= b_1 \ldots b_n \ b_1 \ldots b_n$$

In the definition $b_i = a_{n-i+1}$ for all positive integers, so we now have:

$$= a_n \ldots a_1 \ a_n \ldots a_1$$

$$\text{so}$$

$$\text{reverse}(ss) == a_n \ldots a_1 \ a_n \ldots a_1$$

$\therefore$ as per our original definition of the reverse of s and our derived s

$$a_n \ldots a_1 \ a_n \ldots a_1 = \text{reverse}(a_1 \ldots a_n) \ \text{reverse}(a_1 \ldots a_n) = ss$$

$\therefore$, ss is also a palindrome