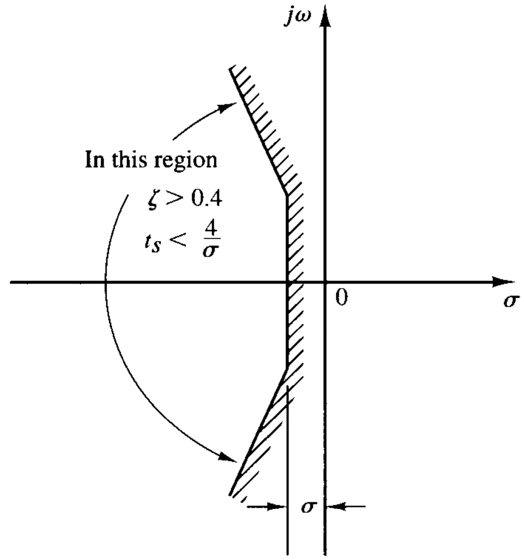

Figure 5–23
Region in the complex plane satisfying the conditions $\zeta > 0.4$ and $t_s < 4/\sigma$.

5–5 ROUTH'S STABILITY CRITERION

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system? In Section 5–4 it was stated that a control system is stable if and only if all closed-loop poles lie in the left-half s plane. Since most linear closed-loop systems have closed-loop transfer functions of the form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = \frac{B(s)}{A(s)}$$

where the a's and b's are constants and $m \leq n$, we must first factor the polynomial $A(s)$ in order to find the closed-loop poles. A simple criterion, known as Routh's stability criterion, enables us to determine the number of closed-loop poles that lie in the right-half s plane without having to factor the polynomial.

Routh's stability criterion. Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them. This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic equation.

The procedure in Routh's stability criterion is as follows:

1. Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0 \tag{5–6}$$

where the coefficients are real quantities. We assume that $a_n \neq 0$; that is, any zero root has been removed.

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots that are imaginary or that have positive real parts.

Therefore, in such a case, the system is not stable. If we are interested in only the absolute stability, there is no need to follow the procedure further. Note that all the coefficients must be positive. This is a necessary condition, as may be seen from the following argument: A polynomial in s having real coefficients can always be factored into linear and quadratic factors, such as $(s + a)$ and $(s^2 + bs + c)$, where a, b, and c are real. The linear factors yield real roots and the quadratic factors yield complex roots of the polynomial. The factor $(s^2 + bs + c)$ yields roots having negative real parts only if b and c are both positive. For all roots to have negative real parts, the constants a, b, c, and so on, in all factors must be positive. The product of any number of linear and quadratic factors containing only positive coefficients always yields a polynomial with positive coefficients. It is important to note that the condition that all the coefficients be positive is not sufficient to assure stability. The necessary but not sufficient condition for stability is that the coefficients of Equation (5–6) all be present and all have a positive sign. (If all a's are negative, they can be made positive by multiplying both sides of the equation by -1.)

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$
\begin{array}{lllllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdot & \cdot & \cdot \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
s^2 & e_1 & e_2 & & & & & \\
s^1 & f_1 & & & & & & \\
s^0 & g_1 & & & & & &
\end{array}
$$

The coefficients b_1, b_2, b_3, and so on, are evaluated as follows:

$$
b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}
$$

$$
b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}
$$

$$
b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}
$$

$$
\vdots
$$

The evaluation of the b's is continued until the remaining ones are all zero. The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$\vdots$$

and

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$\vdots$$

This process is continued until the nth row has been completed. The complete array of coefficients is triangular. Note that in developing the array an entire row may be divided or multiplied by a positive number in order to simplify the subsequent numerical calculation without altering the stability conclusion.

Routh's stability criterion states that the number of roots of Equation (5–6) with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array. It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed. The necessary and sufficient condition that all roots of Equation (5–6) lie in the left-half s plane is that all the coefficients of Equation (5–6) be positive and all terms in the first column of the array have positive signs.

EXAMPLE 5–1 Let us apply Routh's stability criterion to the following third-order polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the coefficients are positive numbers. The array of coefficients becomes

$$\begin{array}{lcc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

The condition that all roots have negative real parts is given by

$$a_1 a_2 > a_0 a_3$$

EXAMPLE 5–2 Consider the following polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Let us follow the procedure just presented and construct the array of coefficients. (The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these. If any coefficients are missing, they may be replaced by zeros in the array.)

$$
\begin{array}{llll||llll}
s^4 & 1 & 3 & 5 & s^4 & 1 & 3 & 5 \\
s^3 & 2 & 4 & 0 & s^3 & \cancel{2} & \cancel{4} & \cancel{0} \\
 & & & & & 1 & 2 & 0 \\
s^2 & 1 & 5 & & s^2 & 1 & 5 & \\
s^1 & -6 & & & s^1 & -3 & & \\
s_0 & 5 & & & s^0 & 5 & &
\end{array}
$$

The second row is divided by 2.

In this example, the number of changes in sign of the coefficients in the first column is 2. This means that there are two roots with positive real parts. Note that the result is unchanged when the coefficients of any row are multiplied or divided by a positive number in order to simplify the computation.

Special cases. If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ϵ and the rest of the array is evaluated. For example, consider the following equation:

$$s^3 + 2s^2 + s + 2 = 0 \tag{5–7}$$

The array of coefficients is

$$
\begin{array}{lll}
s^3 & 1 & 1 \\
s^2 & 2 & 2 \\
s^1 & 0 \approx \epsilon & \\
s^0 & 2 &
\end{array}
$$

If the sign of the coefficient above the zero (ϵ) is the same as that below it, it indicates that there are a pair of imaginary roots. Actually, Equation (5–7) has two roots at $s = \pm j$.

If, however, the sign of the coefficient above the zero (ϵ) is opposite that below it, it indicates that there is one sign change. For example, for the equation

$$s^3 - 3s + 2 = (s - 1)^2(s + 2) = 0$$

the array of coefficients is

$$
\begin{array}{llll}
 & s^3 & 1 & -3 \\
\text{One sign change:} & s^2 & 0 \approx \epsilon & 2 \\
\text{One sign change:} & s^1 & -3 - \dfrac{2}{\epsilon} & \\
 & s^0 & 2 &
\end{array}
$$

There are two sign changes of the coefficients in the first column. This agrees with the correct result indicated by the factored form of the polynomial equation.

If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s plane, that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots. In such a case, the evaluation of the rest of the array can be continued by forming an auxiliary polynomial with the coefficients of the last row and by using the coefficients of the derivative of this polynomial in the next row. Such roots with equal magnitudes and lying radially opposite in the s plane can be found by solving the auxiliary polynomial, which is always even. For a $2n$-degree auxiliary polynomial, there are n pairs of equal and opposite roots. For example, consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficients is

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \quad \leftarrow \text{Auxiliary polynomial } P(s) \\ s^3 & 0 & 0 & \end{array}$$

The terms in the s^3 row are all zero. The auxiliary polynomial is then formed from the coefficients of the s^4 row. The auxiliary polynomial $P(s)$ is

$$P(s) = 2s^4 + 48s^2 - 50$$

which indicates that there are two pairs of roots of equal magnitude and opposite sign. These pairs are obtained by solving the auxiliary polynomial equation $P(s) = 0$. The derivative of $P(s)$ with respect to s is

$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

The terms in the s^3 row are replaced by the coefficients of the last equation, that is, 8 and 96. The array of coefficients then becomes

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 8 & 96 & \quad \leftarrow \text{Coefficients of } dP(s)/ds \\ s^2 & 24 & -50 & \\ s^1 & 112.7 & 0 & \\ s^0 & -50 & & \end{array}$$

We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part. By solving for roots of the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0$$

we obtain

$$s^2 = 1, \qquad s^2 = -25$$

or

$$s = \pm 1, \qquad s = \pm j5$$

These two pairs of roots are a part of the roots of the original equation. As a matter of fact, the original equation can be written in factored form as follows:

$$(s + 1)(s - 1)(s + j5)(s - j5)(s + 2) = 0$$

Clearly, the original equation has one root with a positive real part.

Relative stability analysis. Routh's stability criterion provides the answer to the question of absolute stability. This, in many practical cases, is not sufficient. We usually require information about the relative stability of the system. A useful approach for examining relative stability is to shift the s-plane axis and apply Routh's stability criterion. That is, we substitute

$$s = \hat{s} - \sigma \qquad (\sigma = \text{constant})$$

into the characteristic equation of the system, write the polynomial in terms of $\hat{s}$, and apply Routh's stability criterion to the new polynomial in $\hat{s}$. The number of changes of sign in the first column of the array developed for the polynomial in $\hat{s}$ is equal to the number of roots that are located to the right of the vertical line $s = -\sigma$. Thus, this test reveals the number of roots that lie to the right of the vertical line $s = -\sigma$.

Application of Routh's stability criterion to control system analysis. Routh's stability criterion is of limited usefulness in linear control system analysis mainly because it does not suggest how to improve relative stability or how to stabilize an unstable system. It is possible, however, to determine the effects of changing one or two parameters of a system by examining the values that cause instability. In the following, we shall consider the problem of determining the stability range of a parameter value.

Consider the system shown in Figure 5–24. Let us determine the range of K for stability. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

The array of coefficients becomes

$$\begin{array}{llll} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & \frac{7}{3} & K & \\ s^1 & 2 - \frac{9}{7}K & & \\ s^0 & K & & \end{array}$$

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

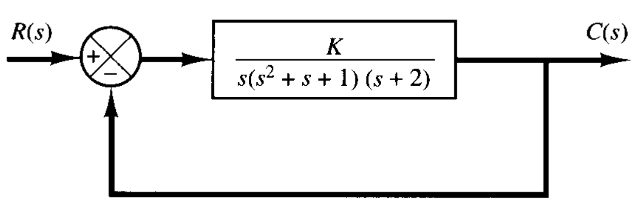

Figure 5–24
Control system.

When $K = \frac{14}{9}$, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude.

5–6 PNEUMATIC CONTROLLERS

As the most versatile medium for transmitting signals and power, fluids, either as liquids or gases, have wide usage in industry. Liquids and gases can be distinguished basically by their relative incompressibilities and the fact that a liquid may have a free surface, whereas a gas expands to fill its vessel. In the engineering field the term *pneumatic* describes fluid systems that use air or gases and *hydraulic* applies to those using oil.

Pneumatic systems are extensively used in the automation of production machinery and in the field of automatic controllers. For instance, pneumatic circuits that convert the energy of compressed air into mechanical energy enjoy wide usage, and various types of pneumatic controllers are found in industry.

Since pneumatic systems and hydraulic systems are often compared, in what follows we shall give a brief comparison of these two kinds of systems.

Comparison between pneumatic systems and hydraulic systems. The fluid generally found in pneumatic systems is air; in hydraulic systems it is oil. And it is primarily the different properties of the fluids involved that characterize the differences between the two systems. These differences can be listed as follows:

1. Air and gases are compressible, whereas oil is incompressible.
2. Air lacks lubricating property and always contains water vapor. Oil functions as a hydraulic fluid as well as a lubricator.
3. The normal operating pressure of pneumatic systems is very much lower than that of hydraulic systems.
4. Output powers of pneumatic systems are considerably less than those of hydraulic systems.
5. Accuracy of pneumatic actuators is poor at low velocities, whereas accuracy of hydraulic actuators may be made satisfactory at all velocities.
6. In pneumatic systems, external leakage is permissible to a certain extent, but internal leakage must be avoided because the effective pressure difference is rather small. In hydraulic systems internal leakage is permissible to a certain extent, but external leakage must be avoided.
7. No return pipes are required in pneumatic systems when air is used, whereas they are always needed in hydraulic systems.
8. Normal operating temperature for pneumatic systems is 5° to 60°C (41° to 140°F). The pneumatic system, however, can be operated in the 0° to 200°C (32° to 392°F) range. Pneumatic systems are insensitive to temperature changes, in contrast to hydraulic systems, in which fluid friction due to viscosity depends greatly on temperature. Normal operating temperature for hydraulic systems is 20° to 70°C(68° to 158°F).
9. Pneumatic systems are fire- and explosion-proof, whereas hydraulic systems are not.

In what follows we begin with a mathematical modeling of pneumatic systems. Then we shall present pneumatic proportional controllers. We shall illustrate the fact that