Theoretical Computer Science

Tutorial Week 5

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Agenda

- **Myhill-Nerode criteria (Negative examples)**
- Pumping Lemma
- Pushdown automata

Myhill-Nerode criteria

For a language L over an alphabet A,

$$s_1 \not\equiv_L s_2 \Leftrightarrow (\exists t \in A^*)\,[(s_1 t \notin L \,\&\, s_2 t \in L) \vee (s_1 t \in L \,\&\, s_2 t \notin L)]$$

Myhill-Nerode theorem

A language L is regular iff $\equiv_L$ has a finite number of equivalent classes.

Myhill-Nerode method. Examples

Negative Example

$L_{01} = \{0^n 1^n \mid n \in \mathbb{N}\}$ is not regular.

$$L_{01} = \{\epsilon, 01, 0011, 000111, \ldots\}$$

Proof

For $m \neq k$,

$$0^m \not\equiv_{L_{01}} 0^k,$$

since $0^m 1^k \notin L_{01}, 0^k 1^k \in L_{01}$ (a distinguishing ext. is 1^k).
Therefore, there are infinity many equivalence classes!
So, L_{01} is not regular.

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- **Pumping Lemma**
- Pushdown automata

Pumping Lemma

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Suppose that a FSA with k states recognize L_{01}.

$$\underbrace{\epsilon,\ 0,\ 00,\ \ldots,\ 0^k}_{k+1}$$

$$\underbrace{q_0,\ q_{i_1},\ q_{i_2},\ \ldots,\ q_{i_k}}_{k+1}$$

Pumping Lemma

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$$q_{i_s} \xrightarrow{0} q_{i_{s+1}} \xrightarrow{0} q_{i_{s+2}} \xrightarrow{0} \cdots \xrightarrow{0} q_{i_s}$$

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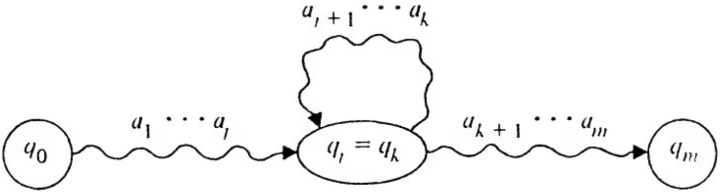
$$q_{i_s} \xrightarrow{0} q_{i_{s+1}} \xrightarrow{0} q_{i_{s+2}} \xrightarrow{0} \cdots \xrightarrow{0} q_{i_s}$$

Therefore, if the FSA recognizes 0^n1^n then it recognizes also $0^{n+t}1^n$ for some $t > 0$. It's a contrudiction.

Pumping lemma



Pumping lemma

If $L \subseteq \Sigma^*$ is a regular language then there exists $m \geq 1$ such that any $w \in L$ with $|w| \geq m$ can be represented as $w = xyz$ such that

- $y \neq \epsilon$,
- $|xy| \leq m$,
- $xy^i z \in L$ for any $i \geq 0$.

How Pumping lemma is useful?

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- **Can we use this theorem to prove that a set is regular?**
  No, because it gives only a necessary condition for a language to be regular (and not a sufficient condition).
- **We can use it to prove that a language is not regular. How?**

Pumping lemma

Pumping lemma

If $L \subseteq \Sigma^*$ is a regular language then there exists $m \geq 1$ such that any $w \in L$ with $|w| \geq m$ can be represented as $w = xyz$ such that

- $y \neq \epsilon$ and $|xy| \leq m$,
- $xy^iz \in L$ for any $i \geq 0$.

Corollary (Contrapositive)

If for any $m \geq 1$ there is $w \in L$ such that $|w| \geq m$ and for any representation $w = xyz$ with $y \neq \epsilon$ and $|xy| \leq m$

$$xy^iz \notin L \text{ for some } i \geq 0.$$

Then L is not a regular language.

Pumping lemma. Examples

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$L_{01} = \{0^n 1^n \mid n \in \mathbb{N}\}$ is not regular.

Proof

For $0^n 1^n = xyz \in L_{01}$ with $y \neq \epsilon$, we have 3 cases:

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For $0^n 1^n = xyz \in L_{01}$ with $y \neq \epsilon$, we have 3 cases:

1) $\underbrace{0^{n-p_1-p_2}}_{x}\underbrace{(0^{p_1})}_{y}\underbrace{0^{p_2}1^n}_{z}$ and $p_1 \neq 0$

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$$xy^2z =$$

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Then L is not a regular language.

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Example 2

$L_2 = \{a^n b a^n \mid n \in \mathbb{N}\}$ is not regular.

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Let $w = xyz \in L_2$ and $y \neq \epsilon$

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Let $w = xyz \in L_2$ and $y \neq \epsilon$

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$$w = \underbrace{a^{n-p_1}}_{x}\underbrace{a^{p_1}}_{y}\underbrace{ba^n}_{z}, \text{ or } w = \underbrace{a^n b}_{x}\underbrace{a^{p_2}}_{y}\underbrace{a^{n-p_2}}_{z}$$

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Pumping lemma. For practice

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Corollary

If for any $m \geq 1$ there is $w \in L$ such that $|w| \geq m$ and for any representation $w = xyz$ with $y \neq \epsilon$ and $|xy| \leq m$

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Negative Example 1

$L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Proof

For each m, let $w = a^m b^m$. If $w = xyz$ and $|xy| \leq m$, then $w = \underbrace{a^{m-p_1-p_2}}_{x}\underbrace{(a^{p_1})}_{y}\underbrace{a^{p_2} b^m}_{z}$ and hence $xy^2 z \notin L_1$

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If for any $m \geq 1$ there is $w \in L$ such that $|w| \geq m$ and for any representation $w = xyz$ with $y \neq \epsilon$ and $|xy| \leq m$

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Negative Example 2

$L_2 = \{a^nba^n \mid n \in \mathbb{N}\}$ is not regular.

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For each m, let $w = a^mba^m$. If $w = xyz$ and $|xy| \leq m$, then $w = \underbrace{a^{m-p_1}}_{x}\underbrace{a^{p_1}}_{y}\underbrace{ba^m}_{z}$ and hence $xy^2z \notin L_2$

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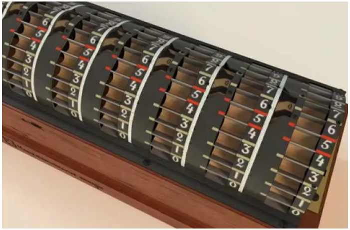
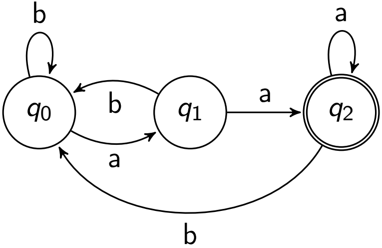
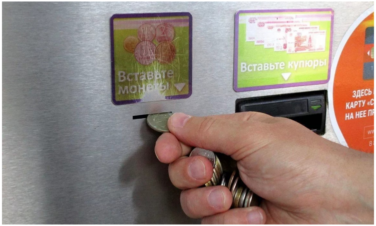
FSA

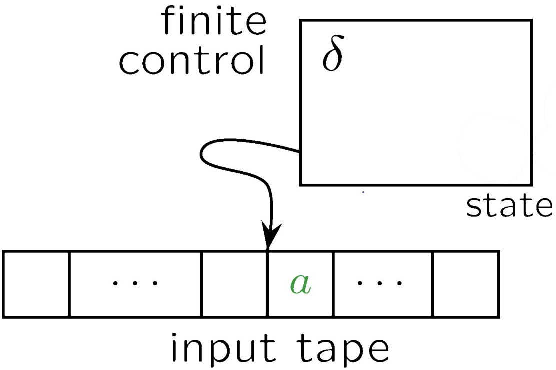




Pushdown Automata (PDA)



Finite State Automata



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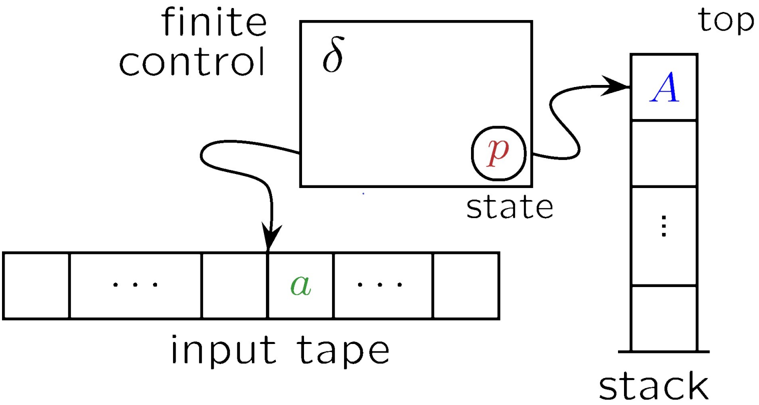

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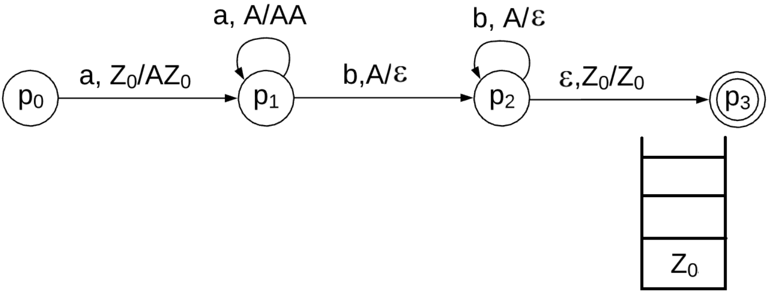
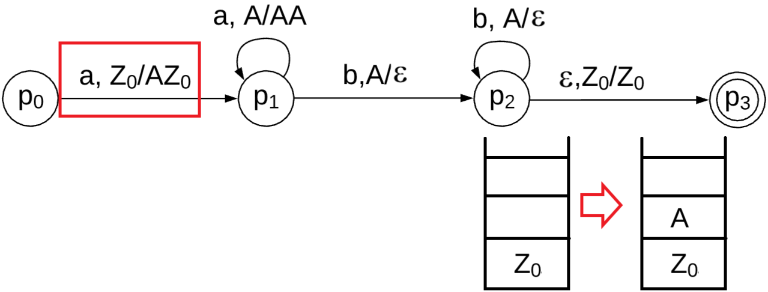


$\Gamma = \{Z_0, A\}$, Z_0 is the initial stack symbol.

Pushdown automata



aabb

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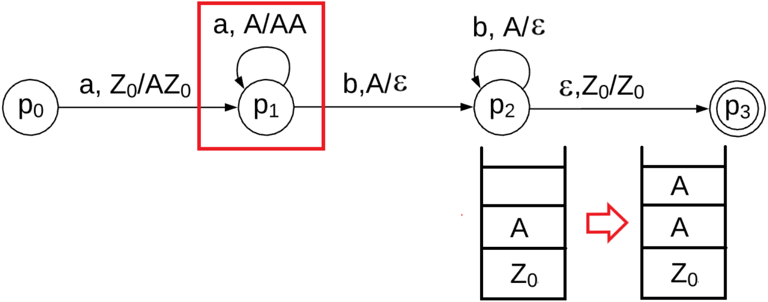

$aabb$

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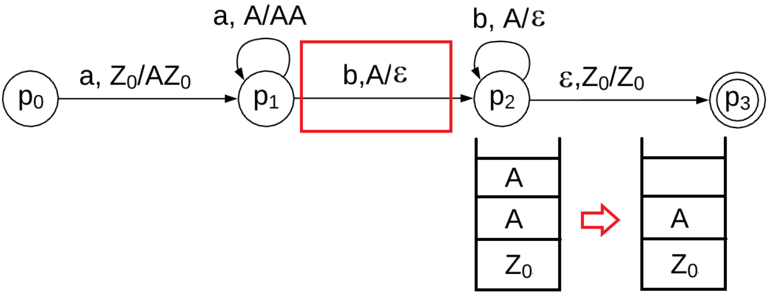

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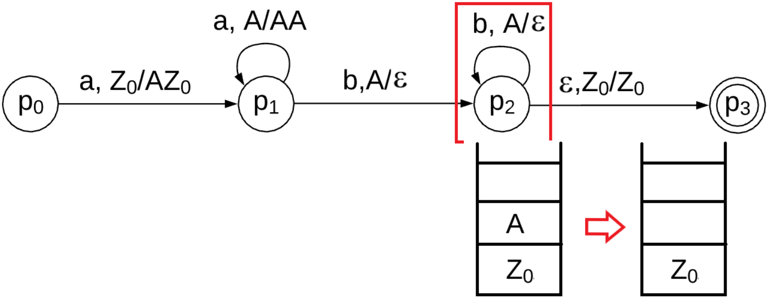
Pushdown automata



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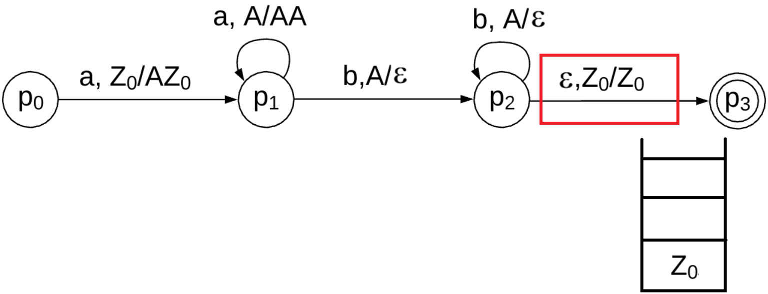
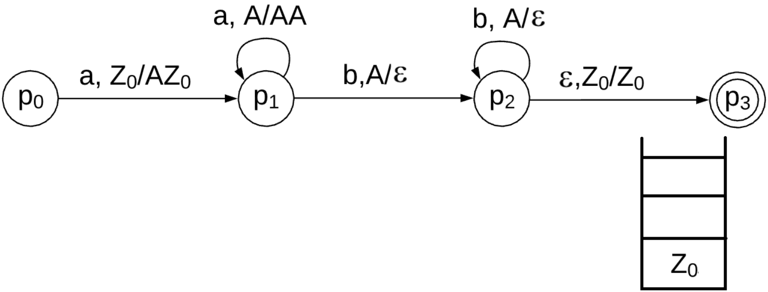


$aabb \in L_1$

PDA. Examples

Example 1

$L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular, but is recognized by a PDA.



PDA. Examples

Example 2

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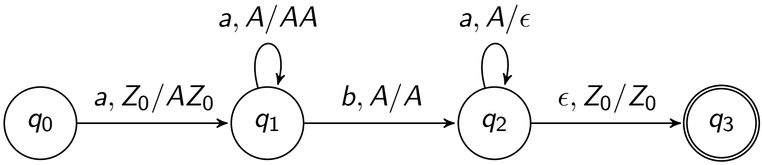

Thank you for your attention!