Theoretical Computer Science

Lab Session 3

February 15, 2023

Agenda

- Exercises on FST
- Operations on FSA (Exercises)

Exercises on FST

Finite State Transducer

Finite State Transducer

A Finite State Transducer (FST) is a tuple $\langle Q, I, \delta, q_0, F, O, \eta \rangle$ where

- Q, I, δ, q_0, F: just like acceptors;
- O is the output alphabet;
- $\eta : Q \times I \to O^*$.

Remark:

- the condition for acceptance remains the same as in acceptors;
- the translation is performed only on accepted strings.

FST: an example

Build a complete FST accepting the following language over the alphabet $A = \{0, 1\}$

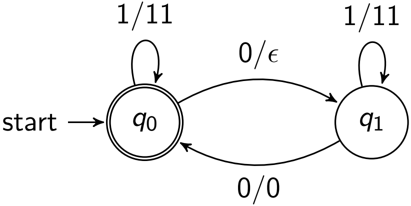
$$L = \{x \in A^* \mid \text{ the number of 0's is even}\}$$

The FST outputs the string obtained by removing every odd occurrence of 0 and doubling every occurrence of 1. Examples of inputs recognised by L and their respective outputs:

- input: 010010, output: 110110
- input: 00, output: 0
- input: 000100011, output: 011001111

FST: an example

$$L = \{x \in A^* \mid \text{ the number of 0's is even}\}$$



Exercises

Build complete FSAs over the languages given below:

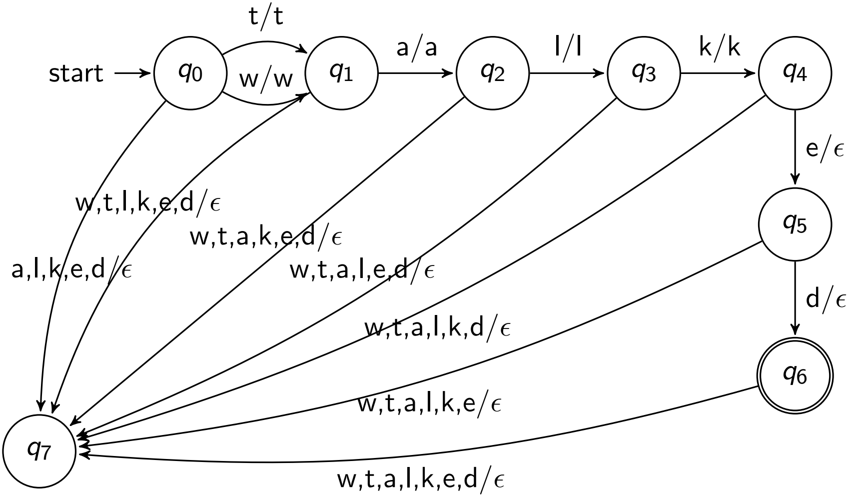
1. $A = \{w, t, a, l, k, e, d\}$ that accepts only the verb "walked" or "talked". The FST will translate the input verb to present form ex: walked to walk.
2. $A = \{a, b\}$ that accepts only strings ending with the letter b. The FST will translate the input string where every second symbol a in the input is erased.

Your Practice Work

Build complete FSAs over the languages given below:

3. $A = \{0, 1\}$ that accepts strings that are binary representation of integers divisible by 2. The FST will translate the input string into result of division by 2.
4. $A = \{0, 1\}$ that accepts strings that are binary representation of integers divisible by 3. The FST will translate the input string into result of division by 3.

Solution (1)



start
q_0
t/t
w/w
q_1
a/a
q_2
l/l
q_3
k/k
q_4
e/ϵ
q_5
d/ϵ
q_6
q_7
w,t,l,k,e,d/ϵ
a,l,k,e,d/ϵ
w,t,a,k,e,d/ϵ
w,t,a,l,e,d/ϵ
w,t,a,l,k,d/ϵ
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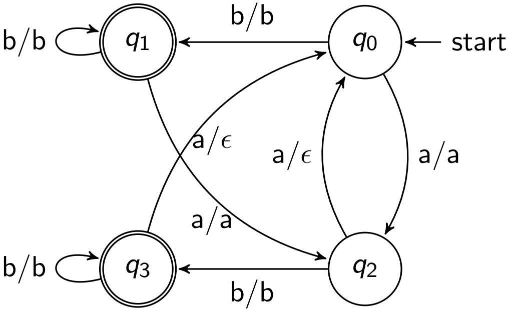
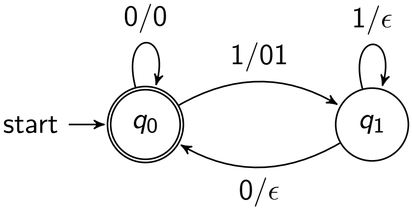
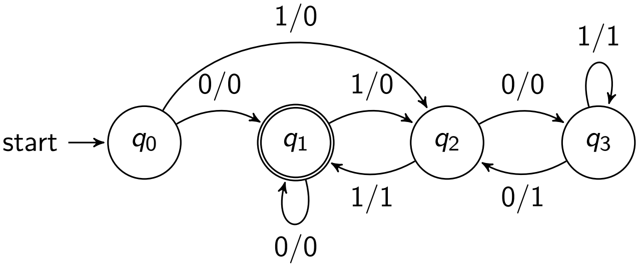
Solution (2)



Solution (3)



Solution (4)



Operations on FSA

Intersection (Formally)

Suppose $M_1 = (Q_1, A, \delta_1, q_0^1, F_1)$ and $M_2 = (Q_2, A, \delta_2, q_0^2, F_2)$ are finite automata accepting L_1 and L_2, respectively. Let M be the complete FSA $M = (Q, A, \delta, q_0, F)$, where

$$Q = Q_1 \times Q_2$$
$$q_0 = (q_0^1, q_0^2)$$

the transition function δ is defined by the formula

$$\delta((q, p), a) = (\delta_1(q, a), \delta_2(p, a))$$

for every $q \in Q_1$, every $p \in Q_2$, and every $a \in A$. And the set of final states is defined as

$$F = \{(q, p) \mid q \in F_1 \land p \in F_2\}$$

M accepts the language $L_1 \cap L_2$.

Intersection (Example)

Let M_1 be a complete FSA defined as

$$\begin{array}{ll} M_1 = \langle & \\ \quad \{q_0, q_1\}, & \text{set of states} \\ \quad \{a\}, & \text{input alphabet} \\ \quad \{((q_0, a), q_1), ((q_1, a), q_0)\}, & \text{partial transition function} \\ \quad q_0, & \text{initial state} \\ \quad \{q_1\} & \text{set of final states} \\ \rangle & \end{array}$$

Intersection (Example)

Let M_1 be a complete FSA defined as

$$\begin{aligned} M_1 = \langle &\{q_0, q_1\}, \{a\}, \\ &\{((q_0, a), q_1), ((q_1, a), q_0)\}, \\ &q_0, \{q_1\}\rangle \end{aligned}$$

and M_2 be a complete FSA defined as

$$\begin{aligned} M_2 = \langle &\{p_0\}, \{a\}, \\ &\{((p_0, a), p_0)\}, \\ &p_0, \{p_0\}\rangle \end{aligned}$$

then

$$\begin{aligned} (M_1 \cap M_2) = \langle &\{(q_0, p_0), (q_1, p_0)\}, \{a\}, \\ &\Big\{\Big(((q_0, p_0), a), (q_1, p_0)\Big), \Big(((q_1, p_0), a), (q_0, p_0)\Big)\Big\}, \\ &(q_0, p_0), \{(q_1, p_0)\}\rangle \end{aligned}$$

Intersection (Example)

Let M_1 be a complete FSA defined as

$$\begin{aligned} M_1 = \langle &\{q_0, q_1\}, \{a\}, \\ &\{((q_0, a), q_1), ((q_1, a), q_0)\}, \\ &q_0, \{q_1\}\rangle \end{aligned}$$

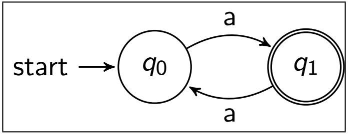
and M_2 be a complete FSA defined as

$$\begin{aligned} M_2 = \langle &\{p_0\}, \{a\}, \\ &\{((p_0, a), p_0)\}, \\ &p_0, \{p_0\}\rangle \end{aligned}$$

then

$$\begin{aligned} (M_1 \cap M_2) = \langle &\{(q_0, p_0), (q_1, p_0)\}, \{a\}, \\ &\Big\{\Big(((q_0, p_0), a), (q_1, p_0)\Big), \Big(((q_1, p_0), a), (q_0, p_0)\Big)\Big\}, \\ &(q_0, p_0), \{(q_1, p_0)\}\rangle \end{aligned}$$

Intersection (Example — Graphically)



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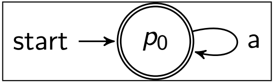

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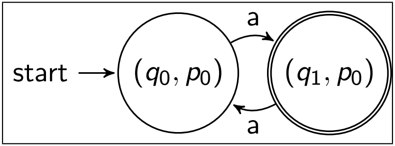

Union (Formally)

Suppose $M_1 = (Q_1, A, \delta_1, q_0^1, F_1)$ and $M_2 = (Q_2, A, \delta_2, q_0^2, F_2)$ are finite automata accepting L_1 and L_2, respectively. Let M be the complete FSA $M = (Q, A, \delta, q_0, F)$, where

$$Q = Q_1 \times Q_2$$
$$q_0 = (q_0^1, q_0^2)$$

the transition function δ is defined by the formula

$$\delta((q, p), a) = (\delta_1(q, a), \delta_2(p, a))$$

for every $q \in Q_1$, every $p \in Q_2$, and every $a \in A$. And the set of final states is defined as

$$F = \{(q, p) \mid q \in F_1 \vee p \in F_2\}$$

M accepts the language $L_1 \cup L_2$.

Difference (Formally)

Suppose $M_1 = (Q_1, A, \delta_1, q_0^1, F_1)$ and $M_2 = (Q_2, A, \delta_2, q_0^2, F_2)$ are finite automata accepting L_1 and L_2, respectively. Let M be the FSA $M = (Q, A, \delta, q_0, F)$, where

$$Q = Q_1 \times Q_2$$
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the transition function δ is defined by the formula

$$\delta((q, p), a) = (\delta_1(q, a), \delta_2(p, a))$$

for every $q \in Q_1$, every $p \in Q_2$, and every $a \in A$. And the set of final states is defined as

$$F = \{(q, p) \mid q \in F_1 \wedge p \notin F_2\}$$

M accepts the language $L_1 \setminus L_2$.

Complement (Formally)

Suppose $M = (Q, A, \delta, q_0, F)$ is a complete finite state automaton accepting L. A complement M^c is a complete FSA $M^c = (Q, A, \delta, q_0, F^c)$, where

$$F^c = Q \backslash F$$

M^c accepts the language L.

Operations on FSAs (Summary)

Suppose $M_1 = (Q_1, A, \delta_1, q_0^1, F_1)$ and $M_2 = (Q_2, A, \delta_2, q_0^2, F_2)$ are finite automata accepting L_1 and L_2, respectively.
Let M be the FSA $M = (Q, A, \delta, q_0, F)$, where

$$\begin{aligned} Q &= Q_1 \times Q_2 \\ q_0 &= (q_0^1, q_0^2) \\ \delta((q,p),a) &= (\delta_1(q,a), \delta_2(p,a)) \end{aligned}$$

The set of final states will be defined as

$$\begin{aligned} \text{Difference } L_1 \setminus L_2 : F &= \{(q,p) \mid q \in F_1 \wedge p \notin F_2\} \\ \text{Union } L_1 \cup L_2 \text{:} \; F &= \{(q,p) \mid q \in F_1 \vee p \in F_2\} \\ \text{Intersection } L_1 \cap L_2 \text{:} \; F &= \{(q,p) \mid q \in F_1 \wedge p \in F_2\} \end{aligned}$$

Exercises (Part 1)

Let $A = \{0, 1\}$ be the alphabet.

1. Build a complete FSA M_1 that recognises the language:
   $L_1 = \{x \in A^* \mid\ x \text{ has an even number of 1's}\}$;
2. Build a complete FSA M_2 that recognises the language:
   $L_2 = \{x \in A^* \mid\ x \text{ has an odd number of 0's}\}$;

Exercises (Part 1)

Let $A = \{0, 1\}$ be the alphabet.

1. Build a complete FSA M_1 that recognises the language: $L_1 = \{x \in A^* \mid x \text{ has an even number of 1's}\}$;
2. Build a complete FSA M_2 that recognises the language: $L_2 = \{x \in A^* \mid x \text{ has an odd number of 0's}\}$;
3. Build a complete FSA that accepts when either M_1 or M_2 accepts.
4. Build a complete FSA that accepts when both M_1 and M_2 accept.
5. Build a complete FSA that accepts when M_1 accepts and M_2 rejects.
6. Build a complement for M_1.

Part 1 Solution (1)

Let $A = \{0, 1\}$ be the alphabet.

1. Build a complete FSA M_1 that recognises the language:
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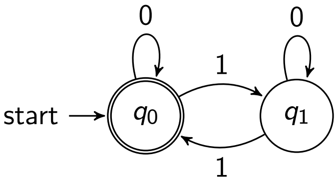
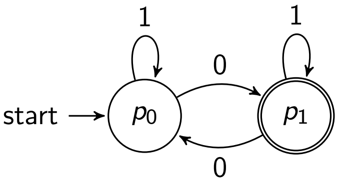


Part 1 Solution (2)

Let $A = \{0, 1\}$ be the alphabet.

2. Build a complete FSA M_2 that recognises the language:
   $L_2 = \{x \in A^* \mid\ x \text{ has an odd number of 0's}\}$;

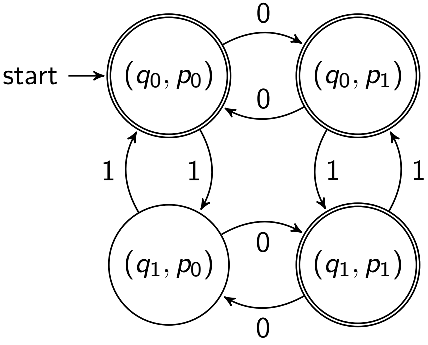


Part 1 Solution (3)

Let $A = \{0, 1\}$ be the alphabet.

3. Build a complete FSA that accepts when either M_1 or M_2 accepts.

Build a complete FSA for $L_1 \cup L_2$:

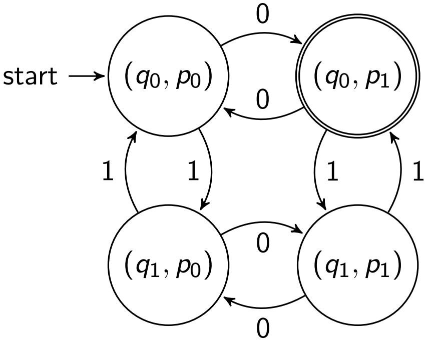


Part 1 Solution (4)

Let $A = \{0, 1\}$ be the alphabet.

4. Build a complete FSA that accepts when both M_1 and M_2 accepts.

Build a complete FSA for $L_1 \cap L_2$:

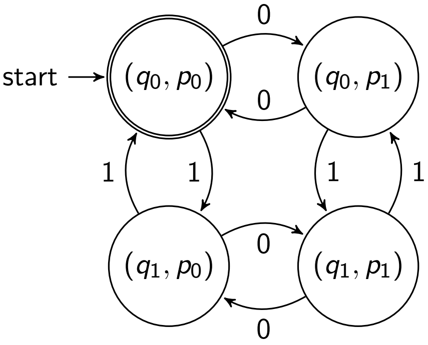


Part 1 Solution (5)

Let $A = \{0, 1\}$ be the alphabet.

5. Build a complete FSA that accepts when M_1 accepts and M_2 rejects

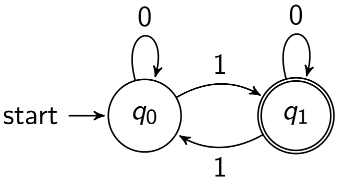
Build a complete FSA for $L_1 \backslash L_2$:



Part 1 Solution (6)

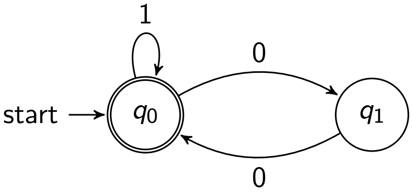
Let $A = \{0, 1\}$ be the alphabet.

6. Build a complement of M_1



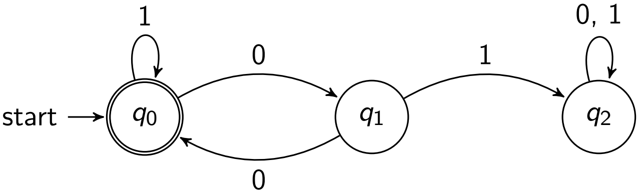
Exercises (Part 2)

Construct a complement for the following FSA

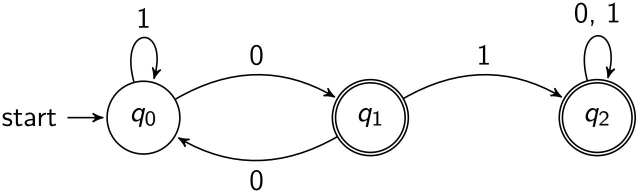


Solution (Part 2)

First, we have to complete the FSA



The complement:



Exercises (Part 3)

Let $A = \{0, 1\}$ be the alphabet.

1. Build a complete FSA M_a that recognises the language:
   $L_a = \{x \in A^* \mid x$ is the binary representation of an integer, and it is divisible by $2\}$[1];
2. Build a complete FSA M_b that recognises the language:
   $L_b = \{x \in A^* \mid x$ is the binary representation of an integer, and it is divisible by $3\}$;

[1] For simplicity assume that ϵ is a part of L_a and L_b

Exercises (Part 3)

Let $A = \{0, 1\}$ be the alphabet.

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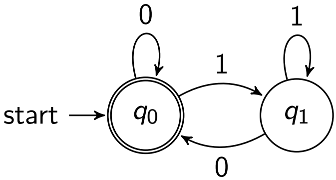
   $L_b = \{x \in A^* \mid x$ is the binary representation of an integer, and it is divisible by $3\}$;
3. Build a complete FSA that accepts when both M_a and M_b accept.
4. Build a complete FSA that accepts when either M_a or M_b accepts.
5. Build a complete FSA that accepts when M_a accepts and M_b rejects.

[1] For simplicity assume that ϵ is a part of L_a and L_b

Part 3 Solution (1)

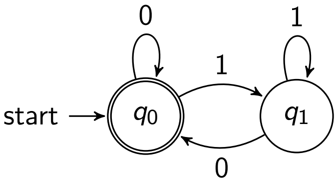
Let $A = \{0, 1\}$ be the alphabet.

1. Build a complete FSA M_a that recognises the language: $L_a = \{x \in A^* \mid x$ is the binary representation of an integer, and it is divisible by $2\}$;



Solution (1) — Another representation of a complete FSA

Graphical Representation - State Transition Diagram



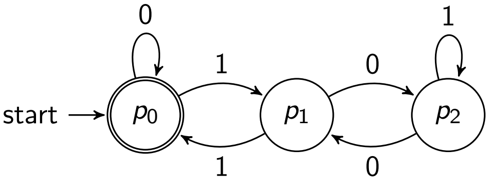
Graphical Representation — State Transition Table

	0	1
$\rightarrow^* q_0$	q_0	q_1
q_1	q_0	q_1

Part 3 Solution (2)

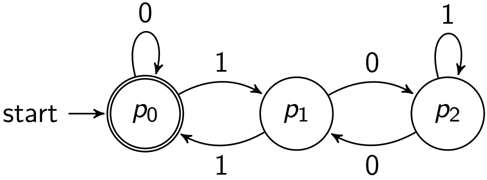
Let $A = \{0, 1\}$ be the alphabet.

2. Build a complete FSA M_b that recognises the language:
   $L_b = \{x \in A^* \mid x$ is the binary representation of an integer, and it is divisible by 3$\}$



Solution (2) – Another representation of a complete FSA

Graphical Representation - State Transition Diagram



Graphical Representation — State Transition Table

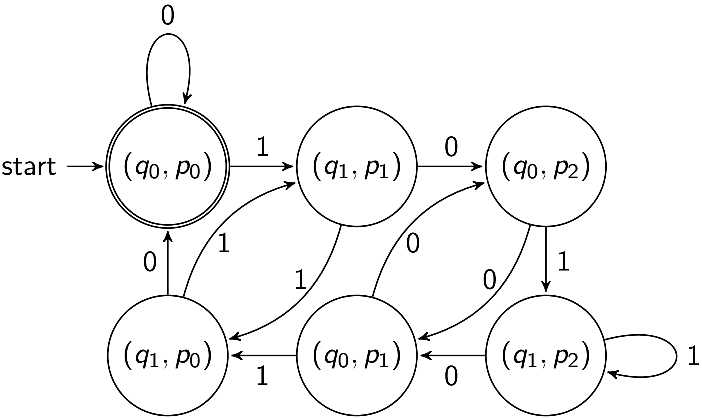
	0	1
$\rightarrow^* p_0$	p_0	p_1
p_1	p_2	p_0
p_2	p_1	p_2

Part 3 Solution (3)

Let $A = \{0, 1\}$ be the alphabet.

3. Build a complete FSA that accepts when both M_a and M_b accepts.

Build and complete FSA for $L_a \cap L_b$:

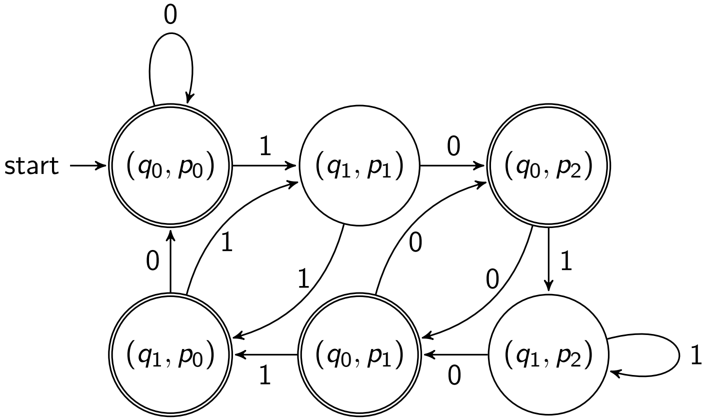


Part 3 Solution (4)

Let $A = \{0, 1\}$ be the alphabet.

4. Build a complete FSA that accepts when either M_a or M_b accept.

Build a complete FSA for $L_a \cup L_b$:

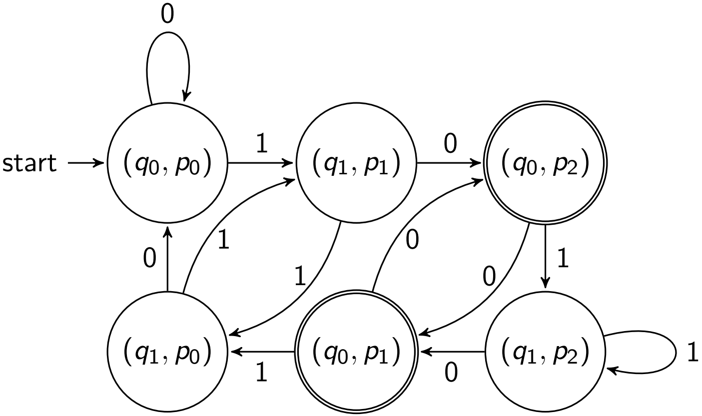


Part 3 Solution (5)

Let $A = \{0, 1\}$ be the alphabet.

5. Build a complete FSA that accepts when M_a accepts and M_b rejects.

Build and complete FSA for $L_a \backslash L_b$:

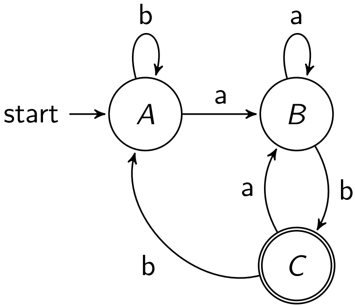


Exercises (Part 4)

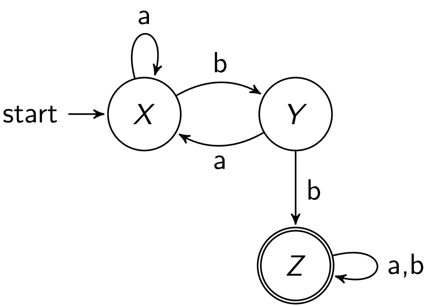
Let M_1 and M_2 be the complete FSAs depicted below, accepting languages L_1 and L_2, respectively. Draw complete FSAs accepting the following languages.

- i $L_1 \cup L_2$
- ii $L_1 \cap L_2$
- iii $L_1 \setminus L_2$

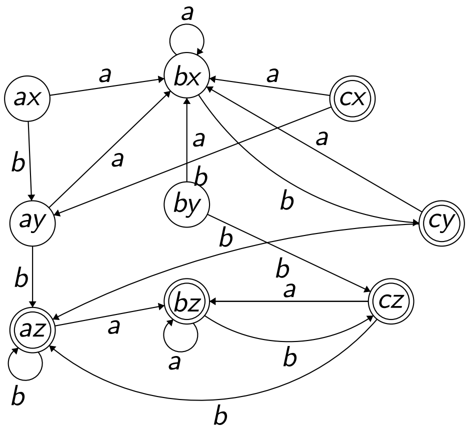
M_1



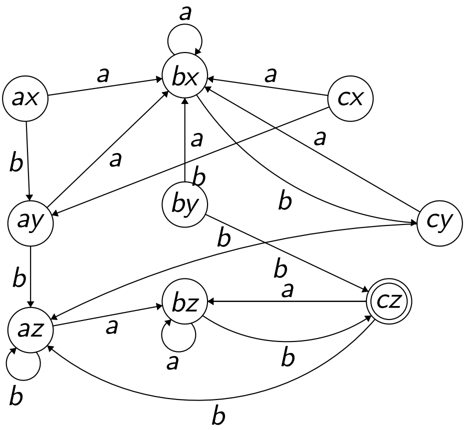
M_2



Solutions (Part 4): $L_1 \cup L_2$



Solutions (Part 4): $L_1 \cap L_2$



Solutions (Part 4): $L_1 \backslash L_2$