Theoretical Computer Science

Lab Session 1

February 1, 2023

Agenda

- Introduction (rules of the game!)
- Preliminaries - Sets
- Formal Languages
- Operations on Formal Languages

Rules of the Game

Laboratory Exercises: There are weekly laboratory exercises.

Assessment: Mid-term Exam (20%), Final Exam (30%),
2 Assignments (30%), 5 Quizzes (25%).

Group switching

Switching lab groups is allowed under the following conditions:

- The group size limit is 30 students.
- TA approval

Rules of the Game

Policy and Procedures on Cheating and Plagiarism

Tests and Exams policy: If two or more students are caught communicating for any reason during exams (including mid-terms) they will be asked to leave the room and their exam will be failed. Same will happen for unauthorized use of devices.

Rules of the Game

Policy and Procedures on Cheating and Plagiarism

Report policy: If a submitted report contains work other than student's one it is necessary to explicitly acknowledge the source. It is encouraged to refer and quote other works, but it has to be made clear which words and ideas are property and creation of the student, and which ones have come from others (which must not correspond to more than 30% of the work). If two or more reports show evidence of being produced by unauthorized cooperative work, i.e. copied from fellow students, they will be all failed without further investigation on who produced the results and who actually copied.

Extra Points

How to get Extra Points?

During the labs at week 2 and 3, we will select the **most active student from each group** for an extra task worth **5 points** on top of the final grade.

Details

Working on a small project for the TCS classes. The idea to create a modern documentation-style website for the labs. All materials are ported from the pdf's, so no dramatic changes.

Outcomes

Learn how to build and maintain modern static website. And this can be a good experience to have!

Preliminaries - Sets

Sets

A finite set can be described, at least in principle, by listing its elements: $A = \{1, 2, 4, 8\}$ says that A is the set whose elements are $1, 2, 4$, and 8.

For infinite (even for finite sets if they have more than just a few elements) sets ellipses $(\ldots)$ are sometimes used to describe how the elements might be listed: $B = \{0, 3, 6, 9, \ldots\}$

A more reliable way is to give the property that characterises their elements (also called set comprehension). Set $B = \{0, 3, 6, 9, \ldots\}$ can be described as

$B = \{x \mid x \text{ is a non-negative integer multiple of } 3\}$

It reads: "B is the set of all x such that x is a non-negative integer multiple of 3"

Sets (notation)

- For any set A, the statement that x is an element of A is written $x \in A$.
- $A \subseteq B$ means that A is a subset of B: every element of A is an element of B.
- $\emptyset$ denotes the empty set: the set with no elements.

To show that two sets A and B are the same, we must show that A and B have exactly the same elements, i.e. $A \subseteq B$ and $B \subseteq A$.

Sets (operations)

For two sets A and B, we can define their union $A \cup B$, their intersection $A \cap B$, and their difference $A \backslash B$ (sometimes denoted as $A - B$), as follows[1]:

$$\begin{aligned} A \cup B &= \{x \mid x \in A \vee x \in B\} \\ A \cap B &= \{x \mid x \in A \wedge x \in B\} \\ A \backslash B &= \{x \mid x \in A \wedge x \notin B\} \end{aligned}$$

[1] $\vee$ and $\wedge$ denote the logical 'or' and logical 'and' respectively.

Sets (Union of any number of sets) - Notation

If A_0, A_1, A_2, ... are sets, the union of these sets can be denoted as

$$\bigcup\{A_i \mid i \geq 0\} = \{x \mid x \in A_i \text{ for at least one } i \text{ with } i \geq 0\}$$

or

$$\bigcup_{i=0}^{\infty} A_i$$

Sets (Power Sets)

For a set A, the set of all subsets of A is called the power set. Can be denoted as $\mathcal{P}(A)$ or as 2^A.
Power set of set $\{a, b, c\}$ is

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

For a set A, the set $\mathcal{P}(A)$ has exactly 2^n elements, where n is the cardinality of A.

Languages

Notation and Terminology

Alphabet: a finite set of symbols, e.g. $\{a, b\}$, or $\{0, 1\}$. Normally denoted by Σ

String: a string over an alphabet (Σ) is a finite sequence of symbols in Σ.

length: for a string x, $|x|$ is the number of symbols of x.

empty string: is the null string over Σ. It is denoted as ϵ. By definition, $|\epsilon| = 0$

Set of all strings: the set of all strings over Σ is denoted by Σ^*, e.g. for the alphabet $A = \{a, b\}$
$A^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

Concatenation of strings

If x and y are two strings over an alphabet, the concatenation xy (sometimes denoted as $x \cdot y$) consists of the symbols of x followed by those of y:

$$x = ab$$
$$y = bab$$
$$xy = abbab$$

Concatenation is an associative operation: $(xy)z = x(yz)$ for all possible strings x, y, and z.

Constructing new Languages

Languages are sets.

- Operations on languages are ways of constructing new languages: for two languages L_1 and L_2 over the alphabet Σ, $L_1 \cup L_2$, $L_1 \cap L_2$, and $L_1 \backslash L_2$ are also languages over Σ.
- String operation of concatenation is also used to construct new languages: if L_1 and L_2 are both languages over Σ, the concatenation of L_1 and L_2 is the language

$$L_1 L_2 = \{xy \mid x \in L_1, y \in L_2\}$$

Example:

$$\{a, aa\}\{\epsilon, b, ab\} = \{a, ab, aab, aa, aab, aaab\}$$

Is this statement true?

$$L_1 L_2 = L_2 L_1$$

Exponential notation

The concatenation of k copies of a single symbol a, a single string s, or a single language L is defined as:
If $k = 0$, then

$$a^k = \{\epsilon\}$$

If $k > 0$, then

$$a^k = aa \ldots a$$

where there are k occurrences of a, similarly for s^k and L^k. In the case where L is simply the alphabet Σ,

$$\Sigma^k = \{x \in \Sigma^* \mid |x| = k\}$$

Example:

$$\Sigma = \{0, 1\}$$
$$\Sigma^2 = \{00, 01, 10, 11\}$$

Operations on Languages

Operations on Languages

- Union
- Intersection
- Set difference
- Complement: if L is a language over Σ,

$$\overline{L} = \Sigma^* \backslash L$$

- Concatenation: if L_1 and L_2 are both languages over Σ,

$$L_1 L_2 = \{xy \mid x \in L_1, y \in L_2\}$$

- Power of n

$$L^n = \{x_1 x_2 ... x_n \mid x_i \in L \text{ for all } 1 \leq i \leq n\}$$

- Kleene Star

$$L^* = \{x_1 x_2 ... x_n \mid n \in \mathbb{N}, x_1, x_2, ..., x_n \in L\} = \bigcup_{n \in \mathbb{N}} L^n$$

Exercises

Exercises (0)

What are the sets D and E?:

i $D = \{\{x\} \mid x \text{ is a non-negative integer such that } x \leq 4\}$

ii $E = \{3i + 5j \mid i \text{ and } j \text{ are non-negative integers}\}$

Are the following statements true?

iii $\{0, 1\} = \{1, 0\}$

iv $\{0, 1, 2, 1, 0\} = \{1, 1, 1, 1, 0, 2, 2\}$

Exercises (1)

Construct the power set for the following sets:

i $\{a, b\}$

ii $\{0, 1\} \cup \{1, 2\}$

iii $\{z\}$

iv $\{0, 1, 2, 3, 4\} \cap \{1, 3, 5, a\}$

v $\{0, 1, 2, 3\} \backslash \{1, 3, 5, a\}$

vi $\emptyset$

Determine the following languages over the alphabet $\Sigma = \{0, 1\}$

vii Σ^0

viii Σ^4

ix $\mathcal{P}(\Sigma)$

x $\mathcal{P}(\Sigma^*)$

Solutions (1)

Construct the power set for the following sets:

i $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

ii $\mathcal{P}(\{0, 1\} \cup \{1, 2\}) = \mathcal{P}(\{0, 1, 2\}) =$
$\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

iii $\mathcal{P}(\{z\}) = \{\emptyset, \{z\}\}$

iv $\mathcal{P}(\{0, 1, 2, 3, 4\} \cap \{1, 3, 5, a\}) =$
$\mathcal{P}(\{1, 3\}) = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$

v $\mathcal{P}(\{0, 1, 2, 3\} \backslash \{1, 3, 5, a\}) = \mathcal{P}(\{0, 2\}) = \{\emptyset, \{0\}, \{2\}, \{0, 2\}\}$

vi $\mathcal{P}(\emptyset) = \{\emptyset\}$

Solutions (1)

Determine the following languages over the alphabet $\Sigma = \{0, 1\}$

vii $\Sigma^0 = \{\epsilon\}$

viii

$$\Sigma^4 = \{0000, 0001, 0010, 0011,\\ 0100, 0101, 0110, 0111,\\ 1000, 1001, 1010, 1011,\\ 1100, 1101, 1110, 1111\}$$

ix $\mathcal{P}(\Sigma) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$

x $\mathcal{P}(\Sigma^*) =$ (Infinite set which includes empty set and all combinations of 0s and 1s)

Exercises (2)

Find a possible alphabet for the following languages[2]

i. The language $L = \{oh, ouch, ugh\}$
ii. The language $L = \{apple, pear, 4711\}$
iii. The language of all binary strings

Determine what the Kleene star operation produces over the following alphabets:

iv. $\Sigma = \{0, 1\}$
v. $\Sigma = \{a\}$
vi. $\Sigma = \emptyset$ (the empty alphabet)

[2] A word *foo* should be interpreted as a string of characters f, o, and o.

Solution (2)

Find a possible alphabet for the following languages[3]

i The language $L = \{oh, ouch, ugh\}$: $\Sigma = \{o, h, u, c, g\}$

ii The language $L = \{apple, pear, 4711\}$:
$\Sigma = \{a, p, l, e, r, 4, 7, 1\}$

iii The language of all binary strings: $\Sigma = \{0, 1\}$

Determine what the Kleene star operation produces over the following alphabets:

iv $\Sigma = \{0, 1\}$: All binary strings

v $\Sigma = \{a\}$: All strings which contain nothing but a's

vi $\Sigma = \emptyset$ (the empty alphabet): the language that contains only the empty string

[3] A word *foo* should be interpreted as a string of characters f, o, and o.

Exercises (3)

State the alphabet Σ for the following languages:

i $L = \Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$

ii $L = \Sigma^* = \{\epsilon, a, aa, aaa, aaaa, \ldots\}$

Assuming that $\Sigma = \{0, 1\}$, construct complement languages for the following:

iii $\overline{\{010, 101, 11\}}$

iv $\overline{\Sigma^* \backslash \{110\}}$

State the following languages explicitly

v $\mathcal{P}(\{a, b\}) \backslash \mathcal{P}(\{a, c\})$

vi $\{x \mid x, y \in \mathbb{N} \wedge \exists y : y < 10 \wedge (y + 2 = x)\}$ ($\mathbb{N}$ is the set of all non-negative integers)

Solution (3)

State the alphabet Σ for the following languages:

i $L = \Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$
$\Sigma = \{0, 1\}$

ii $L = \Sigma^* = \{\epsilon, a, aa, aaa, aaaa, \ldots\}$
$\Sigma = \{a\}$

Assuming that $\Sigma = \{0, 1\}$, construct complement languages for the following:

iii $\overline{\{010, 101, 11\}}$
$L =$
$\{\epsilon, 0, 1, 00, 01, 10, 000, 001, 011, 100, 110, 111, 0000, 0001, \ldots\}$

iv $\overline{\Sigma^* \setminus \{110\}}$
$L = \{110\}$

Solution (3)

State the following languages explicitly

v $\mathcal{P}(\{a, b\}) \backslash \mathcal{P}(\{a, c\})$
  $L = \{\{b\}, \{a, b\}\}$

vi $\{x \mid x, y \in \mathbb{N} \wedge \exists y : y < 10 \wedge (y + 2 = x)\}$ ($\mathbb{N}$ is the set of all non-negative integers)
  $L = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Exercises on Operations on Languages

Exercises (4)

1. Let $L = \{a^i, i \geq 0\}$ be a language over $\Sigma = \{a, b\}$. Find $\overline{L}$ and L^*
2. Let L_1, L_2 be languages over $\Sigma = \{a, b\}$. Find $L_1 L_2$
   a) $L_1 = \{\epsilon, a, aa\}$, $L_2 = \{aa, aaa\}$
   b) $L_1 = \{a, a^2, a^4\}$, $L_2 = \{b^0, b^2, b^3\}$
3. Let $L = \{0, 01, 001\}$. Find L^2.
4. Describe in plain English the following languages over $\Sigma = \{a, b\}$:
   a) $L = \{a, b\}^*$
   b) $L = \{a\}^* \cup \{b\}^*$
   c) $L = \{a\}^* \cap \{b\}^*$
   d) $L = \{aa\}^* \backslash \{aaaa\}^*$
5. Write out in full the strings $0^5, 0^3 1^3, (010)^2, (01)^3 0, 1^0$

Solutions (4)

1. Let $L = \{a^i, i \geq 0\}$, $\Sigma = \{a, b\}$.
   $\overline{L} =$ all nonempty strings containing at least one b
   $L^* = \{a^i, i \geq 0\}$
2. Let L_1, L_2 be languages over $\Sigma = \{a, b\}$. Find $L_1 L_2$
   a) $\{\epsilon, a, aa\}\{aa, aaa\} = \{aa, aaa, aaaa, aaaaa\}$
   b) $\{a, a^2, a^4\}\{b^0, b^2, b^3\} = \{a, aa, aaaa\}\{\epsilon, bb, bbb\} =$
   $\{a, aa, aaaa, abb, aabb, aaaabb, abbb, aabbb, aaaabbb\}$
3. $\{0, 01, 001\}^2 =$
   $\{00, 001, 0001, 010, 0101, 01001, 0010, 00101, 001001\}$
4. a) $L = \{a, b\}^*$ - all strings of a's and b's, including empty string
   b) $L = \{a\}^* \cup \{b\}^*$ - empty string and strings of only a's or only b's
   c) $L = \{a\}^* \cap \{b\}^*$ - empty string
   d) $L = \{aa\}^* \backslash \{aaaa\}^*$ - strings of even number of a's which is not a multiple of 4
5. $00000, 000111, 010010, 0101010, \epsilon$

Exercises (5)

Perform operations on the languages over $\Sigma = \{0, 1\}$:
$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$L_2 = \{0, 1\}^*$,
$L_3 = \{w \mid w \in \Sigma^*, |w| = 1\}$,
$L_4 = \{w \mid w \in \Sigma^*, |w| = 2\}$,
$L_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\}$

1. $L_1 \cup L_2, \quad L_3 \cup L_4$
2. $L_1 \cap L_2, \quad L_1 \cap L_3, \quad L_1 \cap L_4, \quad L_1 \cap L_5, \quad L_3 \cap L_4$
3. $L_1 \backslash L_2, \quad L_1 \backslash L_3, \quad L_3 \backslash L_4, \quad L_4 \backslash L_5, \quad L_5 \backslash L_4$
4. $\overline{L_1}, \quad \overline{L_2}, \quad \overline{L_3}, \quad \overline{L_5 \backslash L_4}$
5. $L_1 L_2, \quad L_3 L_4, \quad L_4 L_3$
6. $L_2^*, \quad L_3^*, \quad L_4^*$

Solution (5)

$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$\mathrm{L}_2 = \{0, 1\}^* =$
$\{\epsilon, 0, 1, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ...\}$,
$\mathrm{L}_3 = \{w \mid w \in \Sigma^*, |w| = 1\} = \{0, 1\}$,
$\mathrm{L}_4 = \{w \mid w \in \Sigma^*, |w| = 2\} = \{00, 11, 01, 10\}$,
$\mathrm{L}_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\} =$
$\{0,1,00,11,01,10,000,111,010,011,100,101,...\} = L_2 \setminus \{\epsilon\}$,

Solution (5)

$L_1 = \{0, 1, 00, 11, 000, 111, ...\}, \quad L_2 = \{0, 1\}^*,$
$\mathrm{L}_3 = \{0, 1\}, \quad L_4 = \{00, 11, 01, 10\}, \quad L_5 = L_2 \backslash \{\epsilon\},$

1. $L_1 \cup L_2 = L_2,$
   $L_3 \cup L_4 = \{0, 1, 00, 11, 01, 10\}$
2. $L_1 \cap L_2 = L_1$
   $L_1 \cap L_3 = L_3$
   $L_1 \cap L_4 = \{00, 11\}$
   $L_1 \cap L_5 = L_1$
   $L_3 \cap L_4 = \emptyset$

Solution (5)

$$L_1 = \{0, 1, 00, 11, 000, 111, ...\}, \quad L_2 = \{0, 1\}^*,$$
$$L_3 = \{0, 1\}, \quad L_4 = \{00, 11, 01, 10\}, \quad L_5 = L_2 \backslash \{\epsilon\},$$

3. $L_1 \backslash L_2 = \emptyset$
   $L_1 \backslash L_3 = \{00, 11, 000, 111, ...\}$
   $L_3 \backslash L_4 = L_3 = \{0, 1\}$
   $L_4 \backslash L_5 = \emptyset$
   $L_5 \backslash L_4 = \{w \mid w \in \Sigma^*, |w| = 1 \text{ or } |w| \geq 3\}$
4. $\overline{L_1} = \{\epsilon, 01, 10, 010, 011, 100, 101, ...\}$
   $\overline{L_2} = \emptyset$
   $\overline{L_3 =} \{\epsilon, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ..\}$
   $\overline{L_5 \backslash L_4} = L_4 \cup \{\epsilon\}$

Solution (5)

$L_1 = \{0, 1, 00, 11, 000, 111, ...\}, \quad L_2 = \{0, 1\}^*,$
$\mathsf{L}_3 = \{0, 1\}, \quad L_4 = \{00, 11, 01, 10\}, \quad L_5 = L_2 \backslash \{\epsilon\},$

5. $L_1 L_2 = L_2 \backslash \{\epsilon\}$
   $L_3 L_4 = \{000, 011, 001, 010, 100, 111, 101, 110\}$
   $L_4 L_3 = L_3 L_4 = \{000, 001, 110, 111, 010, 011, 100, 101\}$
6. $L_2^* = L_2$
   $L_3^* = L_2$
   $L_4^* = \{w \mid w \in \Sigma^*, |w| = 2k, k \in \mathbb{N}\}$