Theoretical Computer Science

Lab Session 4

February 22, 2023

Innopolis University

Agenda

- Recap: Pumping lemma
- Exercises

Pumping lemma

Given a regular language $\mathbf{L}$
there exists an integer (critical length) $\mathbf{m}$ such that
for any string $\mathbf{w} \in \mathbf{L}$ with length $|\ \mathbf{w}\ | \geqslant \mathbf{m}$
we can find a split $\mathbf{w} = \mathbf{x}\ \mathbf{y}\ \mathbf{z}$ such that:

- $|\mathbf{x}\ \mathbf{y}| \leqslant \mathbf{m}$
- $|\mathbf{y}| \geqslant 1$
- $\mathbf{x}\ \mathbf{y}^i\mathbf{z} \in \mathbf{L}$ for all $i \geq 0$

Pumping lemma: contrapositive

Given a language **L**. If we show that
for any integer $m \geq 1$
there exists a string $w \in L$ such that $|w| \geqslant m$
and **for all** possible splits $x, y, z \in \Sigma^*$ with

- $|xy| \leqslant m$
- $|y| \geqslant 1$
- $w = xyz$

there exists: $i \in \mathbb{N}$ such that $xy^iz \notin L$.
Then, applying the Pumping lemma for regular languages, one can deduce that L is not regular.

Exercises

Using Pumping lemma prove that L_1, L_2, L_3 and L_4 are not regular languages:

1. $L_1 = \{vv^R \mid v \in \Sigma_1^*\} \quad where \quad \Sigma_1 = \{a, b\}$
2. $L_2 = \{v|$ v has an equal number of a's and b's$\}$ over $\Sigma_2 = \{a, b\}$
3. $L_3 = \{a^{n!} \mid n \geq 0\} \quad over \quad \Sigma_3 = \{a\}$
4. $L_4 = \{a^n b^l c^{n+l} \mid n, l \geq 0\} \quad over \quad \Sigma_4 = \{a, b, c\}$

Solution 1: $L_1 = \{vv^R \mid v \in \Sigma_1^*\}$ *where* $\Sigma_1 = \{a, b\}$

Let's take an arbitrary integer $m \geq 1$.
Let $w = a^m b^m b^m a^m$
$|w| = 4m \geq m, w \in L$.
Split w in the form xyz: as $|xy| \leq m$ and $w = a^m b^m b^m a^m$,
$y = a^k, k \geq 1$.
Let's look at xy^2z. It will have the form $a^{m+k} b^m b^m a^m$.
As $k \geq 1$, $xy^2z \notin L$.
We have shown that for any m we can find $w \in L$, such that $|w| \geq m$ and for all $x, y, z \in \Sigma^*$ with $|xy| \leqslant m$ and $|y| \geqslant 1$ and $w = xyz$ there exists $i \in \mathbb{N}$ such that $xy^iz \notin L$.
So, applying Pumping lemma we can deduce that L is not regular.

Solution 2: $L_2 = \{v |$ v has equal number of a's and b's$\}$ over $\Sigma_2 = \{a, b\}$

Let's take an arbitrary integer $m \geq 1$.
Let $w = a^m b^m$
$|w| = 2m \geq m, w \in L$.
Split w in the form xyz: as $|xy| \leq m$ and $w = a^m b^m$,
$y = a^k, k \geq 1$.
Let's look at xy^2z. It will have the form $a^{m+k}b^m$.
As $k \geq 1$, $xy^2z \notin L$.
We have shown that for any m we can find $w \in L$, such that $|w| \geq m$ and for all $x, y, z \in \Sigma^*$ with $|xy| \leqslant m$ and $|y| \geqslant 1$ and $w = xyz$ there exists $i \in \mathbb{N}$ such that $xy^iz \notin L$.
So, applying Pumping lemma we can deduce that L is not regular.

Solution 3: $L_3 = \{a^{n!} \mid n \geq 0\}$ over $\Sigma_3 = \{a\}$

Let's take an arbitrary integer $m \geq 1$.
Let $w = a^{m!}$
$|w| = m! \geq m, w \in L$.
Split w in the form xyz: as $|xy| \leq m$ and $w = a^{m!}$,
$y = a^k, m \geq k \geq 1$.
Let's look at xy^2z. It will have the form $a^{m!+k}$.
As $k \geq 1, m! < m! + k$.
As $m \geq k$, $m! + k \leq m! + m$.
By algebra, $m! + m < (m+1)!$[1], as $(m+1)! = m! + m! * m$
So for $m > 1$ we get that $m! < m! + k < (m+1)!$, which means that there is no such $p \in \mathbb{N}$ that $(m! + k) = p!$, so $xy^2z \notin L$
For $m = 1, w = a$, so $y = a$, and the string $xy^3z = aaa \notin L$
Applying Pumping lemma we can deduce that L_3 is not regular.

[1] for $m > 1$

Solution 4: $L_4 = \{a^n b^l c^{n+l} \mid n, l \geq 0\}$ over $\Sigma_4 = \{a, b, c\}$

Let's take an arbitrary integer $m \geq 1$.
Let $w = a^m b^m c^{2m}$
$|w| = 4m \geq m, w \in L$.
Split w in the form xyz: as $|xy| \leq m$ and $w = a^m b^m c^{2m}$,
$y = a^k, k \geq 1$.
Let's look at xy^2z. It will have the form $a^{m+k} b^m c^{2m}$. As $k \geq 1$,
$xy^2z \notin L$, applying Pumping lemma we can deduce that L_4 is not regular.