Probabilistic Ranking Principle

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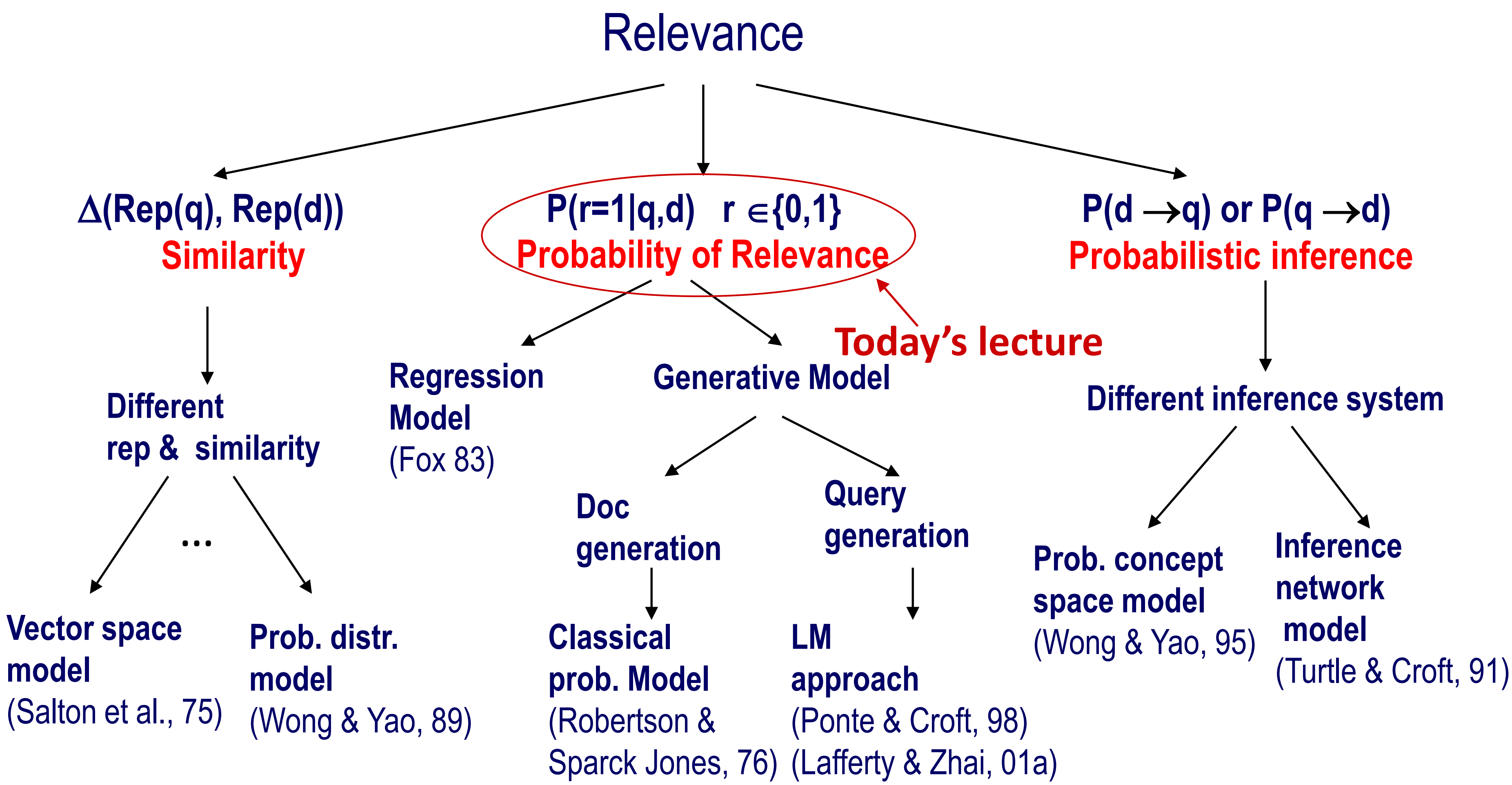
Notion of relevance



Basic concepts in probability

- Random experiment
  - An experiment with uncertain outcome (e.g., tossing a coin, picking a word from text)
- Sample space (S)
  - All possible outcomes of an experiment, e.g., tossing 2 fair coins, S={HH, HT, TH, TT}
- Event (E)
  - $E \subseteq S$, E happens iff outcome is in S, e.g., E={HH} (all heads), E={HH,TT} (same face)
  - Impossible event ({}), certain event (S)
- Probability of event
  - $0 \leq P(E) \leq 1$

Essential probability concepts

- Probability of events
  - Mutually exclusive events
    - $P(A \cup B) = P(A) + P(B)$
  - General events
    - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - Independent events
    - $P(A \cap B) = P(A)P(B)$

Joint probability, or simply as $P(A, B)$

Essential probability concepts

- Conditional probability
  - $P(B|A) = P(A,B)/P(A)$
  - Bayes' Rule: $P(B|A) = P(A|B)P(B)/P(A)$
  - For independent events, $P(B|A) = P(B)$
- Total probability
  - If $A_1, \dots, A_n$ form a non-overlapping partition of S
    - $P(B \cap S) = P(B \cap A_1) + \cdots + P(B \cap A_n)$
    - $P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)} \propto P(B|A_i)P(A_i)$
    - This allows us to compute $P(A_i|B)$ based on $P(B|A_i)$

Interpretation of Bayes' rule

Hypothesis space: $H = \{H_1, \dots, H_n\}$, Evidence: E

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{P(E)}$$

If we want to pick the most likely hypothesis H*, we can drop $P(E)$

Posterior probability of H_i **Prior probability of H_i**

$$P(H_i \mid E) \propto P(E \mid H_i)P(H_i)$$

Likelihood of data/evidence given H_i

Theoretical justification of ranking

- As stated by William Cooper

  *"If a reference retrieval system's response to each request is a ranking of the documents in the collections in order of decreasing **probability of usefulness** to the user who submitted the request, where the probabilities are estimated as accurately as possible on the basis of whatever data made available to the system for this purpose, then the overall effectiveness of the system to its users will be the **best** that is obtainable on the basis of that data."*

  – Rank by probability of relevance leads to optimal retrieval performance

Justification

- From decision theory
  - Two types of loss
    - Loss(retrieved|non-relevant)=a_1
    - Loss(not retrieved|relevant)=a_2
  - $\phi(d_i, q)$: probability of d_i being relevant to q
  - Expected loss regarding to the decision of including d_i in the final results
    - Retrieve: $\left(1 - \phi(d_i, q)\right)a_1$
    - Not retrieve: $\phi(d_i, q)a_2$

Your decision criterion?

Justification

- From decision theory
  - We make decision by
    - If $\left(1-\phi(d_i,q)\right)a_1 < \phi(d_i,q)a_2$, retrieve d_i
    - Otherwise, not retrieve d_i
  - Check if $\phi(d_i,q) > \frac{a_1}{a_1+a_2}$
  - Rank documents by descending order of $\phi(d_i,q)$ would minimize the loss

According to PRP, what we need is

- A relevance measure function F(q,d)
  - For all q, d_1, d_2,
    $F(q,d_1) > F(q,d_2)$ iff. $p(Rel|q,d_1) > p(Rel|q,d_2)$
  - Assumptions
    - Independent relevance
    - Sequential browsing

Most existing research on IR models so far has fallen into this line of thinking… (Limitations?)

Probability of relevance

- Three random variables
  - Query Q
  - Document D
  - Relevance $R \in \{0,1\}$
- Goal: rank D based on $P(R=1|Q,D)$
  - Compute $P(R=1|Q,D)$
  - Actually, one only needs to compare $P(R=1|Q,D_1)$ with $P(R=1|Q,D_2)$, i.e., rank documents
- Several different ways to define $P(R=1|Q,D)$

Conditional models for P(R=1|Q,D)

- Basic idea: relevance depends on how well a query matches a document
  - $P(R=1|Q,D)=g(Rep(Q,D)|\theta)$ ← **a functional form**
    - Rep(Q,D): feature representation of query-doc pair
      - E.g., #matched terms, highest IDF of a matched term, docLen
  - Using training data (with known relevance judgments) to estimate parameter θ
  - Apply the model to rank new documents
- Will be covered with more details in learning-to-rank

Generative models for P(R=1|Q,D)

- Basic idea
  - Compute Odd(R=1|Q,D) using Bayes' rule

$$Odd(R=1|Q,D)=\frac{P(R=1|Q,D)}{P(R=0|Q,D)}=\frac{P(Q,D|R=1)}{P(Q,D|R=0)}\boxed{\frac{P(R=1)}{P(R=0)}} \leftarrow \text{Ignored for ranking}$$

- Assumption
  - Relevance is a binary variable
- Variants
  - Document "generation"
    - P(Q,D|R)=P(D|Q,R)P(Q|R)
  - Query "generation"
    - P(Q,D|R)=P(Q|D,R)P(D|R)

Document generation model

$$Odd(R=1|Q,D) \propto \frac{P(Q,D|R=1)}{P(Q,D|R=0)}$$

	information	retrieval	retrieved	is	helpful	for	you	everyone
Doc1	1	1	0	1	1	1	0	1
Doc2	1	0	1	1	1	1	1	0

Assume independent attributes of $A_1 \ldots A_k$ ….(why?)

Let $D=d_1 \ldots d_k$, where $d_k \in \{0,1\}$ is the value of attribute A_k (Similarly $Q=q_1 \ldots q_k$)

$$Odd(R=1|Q,D) \propto \prod_{i=1}^{k} \frac{P(A_i=d_i|Q,R=1)}{P(A_i=d_i|Q,R=0)}$$

Terms occur in doc

Terms do not occur in doc

$$= \boxed{\prod_{i=1,d_i=1}^{k} \frac{P(A_i=1|Q,R=1)}{P(A_i=1|Q,R=0)}} \boxed{\prod_{i=1,d_i=0}^{k} \frac{P(A_i=0|Q,R=1)}{P(A_i=0|Q,R=0)}}$$

document	relevant(R=1)	nonrelevant(R=0)
term present A_i=1	p_i	u_i
term absent A_i=0	$1-p_i$	$1-u_i$

Document generation model

$$Odd(R=1|Q,D) \propto \prod_{i=1}^{k} \frac{P(A_i = d_i \,|\, Q, R=1)}{P(A_i = d_i \,|\, Q, R=0)}$$

$$= \prod_{i=1, d_i=1}^{k} \frac{P(A_i = 1 \,|\, Q, R=1)}{P(A_i = 1 \,|\, Q, R=0)} \prod_{i=1, d_i=0}^{k} \frac{P(A_i = 0 \,|\, Q, R=1)}{P(A_i = 0 \,|\, Q, R=0)}$$

Terms occur in doc

Terms do not occur in doc

Important tricks

Assumption: terms not occurring in the query are equally likely to occur in relevant and nonrelevant documents, i.e., $p_t = u_t$

document	relevant(R=1)	nonrelevant(R=0)
term present $A_i=1$	p_i	u_i
term absent $A_i=0$	$1-p_i$	$1-u_i$

Robertson-Sparck Jones Model

(Robertson & Sparck Jones 76)

$$\log O(R=1|Q,D) \overset{Rank}{\approx} \sum_{i=1,d_i=q_i=1}^{k} \log \frac{p_i(1-u_i)}{u_i(1-p_i)} = \sum_{i=1,d_i=q_i=1}^{k} \log \frac{p_i}{1-p_i} + \log \frac{1-u_i}{u_i}$$ (RSJ model)

Two parameters for each term A_i:

p_i = P(A_i=1|Q,R=1): prob. that term A_i occurs in a relevant doc

u_i = P(A_i=1|Q,R=0): prob. that term A_i occurs in a non-relevant doc

How to estimate these parameters?

Suppose we have relevance judgments,

$$\hat{p}_i = \frac{\#(rel.\ doc\ with\ A_i) + 0.5}{\#(rel.doc) + 1} \qquad \hat{u}_i = \frac{\#(nonrel.\ doc\ with\ A_i) + 0.5}{\#(nonrel.doc) + 1}$$

- "+0.5" and "+1" can be justified by Bayesian estimation as priors

Per-query estimation!

Parameter estimation

- General setting:
  - Given a (hypothesized & probabilistic) model that governs the random experiment
  - The model gives a probability of any data $p(D|\theta)$ that depends on the parameter θ
  - Now, given actual sample data X={x_1,…,x_n}, what can we say about the value of θ?
- Intuitively, take our best guess of θ -- "best" means "best explaining/fitting the data"
- Generally an optimization problem

Maximum likelihood vs. Bayesian

- Maximum likelihood estimation
  - "Best" means "data likelihood reaches maximum"

$$\hat{\boldsymbol{\theta}} = \mathbf{argmax}_{\boldsymbol{\theta}} \mathbf{P}(\mathbf{X}|\boldsymbol{\theta})$$

  - Issue: small sample size

ML: Frequentist's point of view

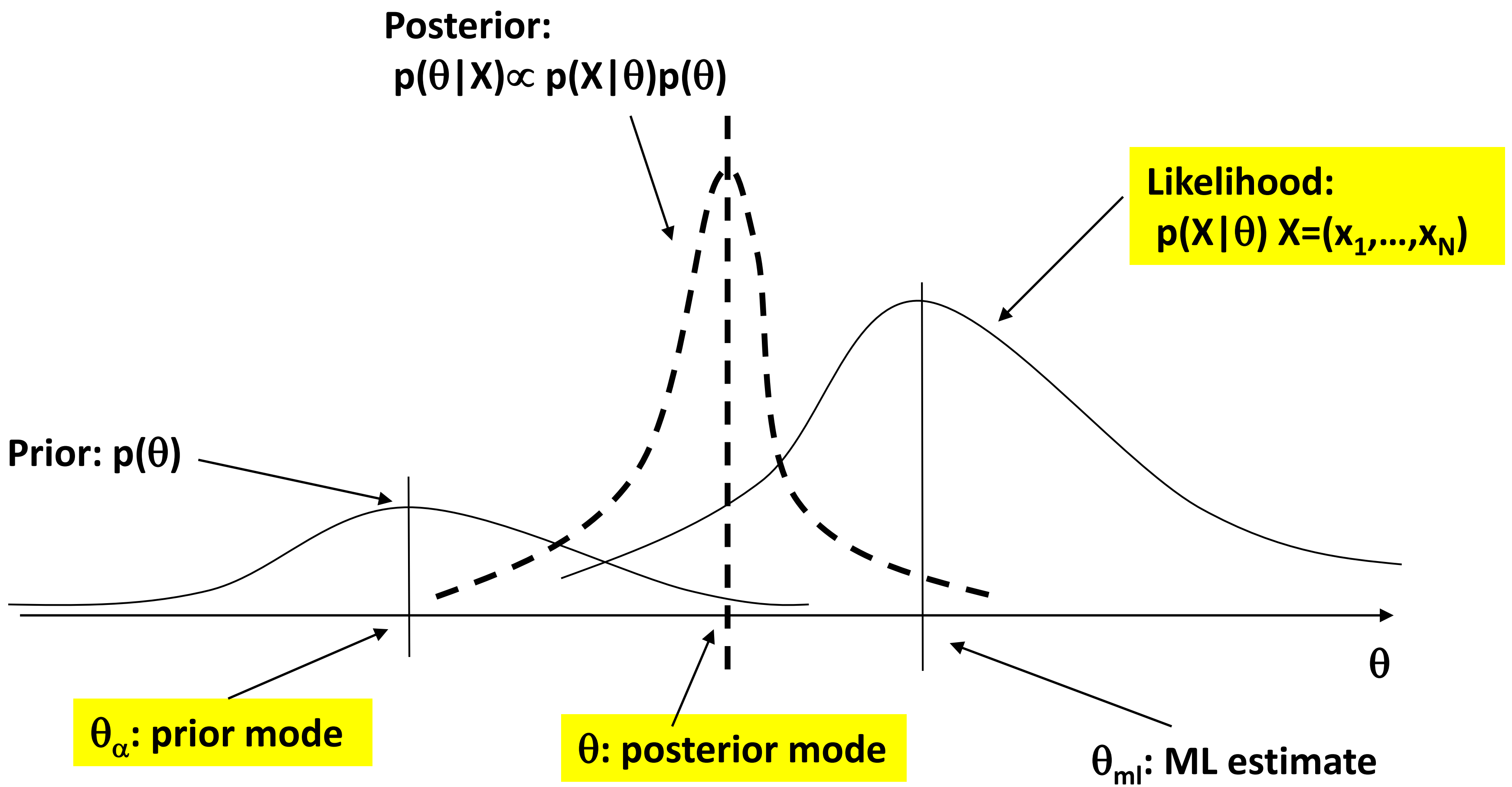
- Bayesian estimation
  - "Best" means being consistent with our "prior" knowledge and explaining data well

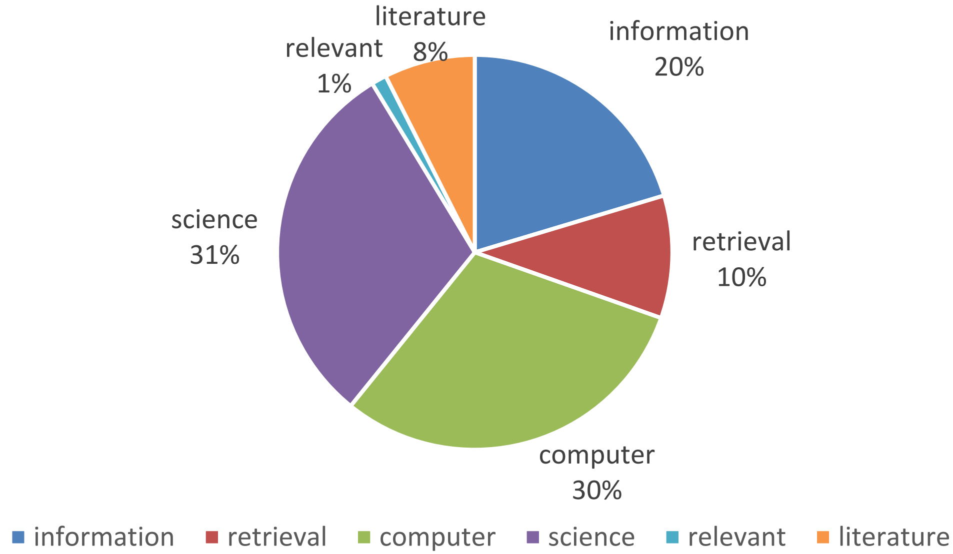
$$\hat{\boldsymbol{\theta}} = \mathbf{argmax}_{\boldsymbol{\theta}} \boldsymbol{P}(\boldsymbol{\theta}|\boldsymbol{X}) = \mathbf{argmax}_{\boldsymbol{\theta}} \mathbf{P}(\mathbf{X}|\boldsymbol{\theta})\mathbf{P}(\boldsymbol{\theta})$$

  - A.k.a, Maximum a Posterior estimation
  - Issue: how to define prior?

MAP: Bayesian's point of view

Illustration of Bayesian estimation





- Maximum likelihood estimator: $\hat{\theta} = argmax_{\theta} p(W|\theta)$

$$p(W|\theta) = \binom{N}{c(w_1),\dots,c(w_N)} \prod_{i=1}^{N} \theta_i^{c(w_i)} \propto \prod_{i=1}^{N} \theta_i^{c(w_i)} \Rightarrow \log p(W|\theta) = \sum_{i=1}^{N} c(w_i) \log \theta_i$$

$$\Rightarrow L(W,\theta) = \sum_{i=1}^{N} c(w_i) \log \theta_i + \lambda \left(\sum_{i=1}^{N} \theta_i - 1 \right)$$

Using Lagrange multiplier approach, we'll tune $\boldsymbol{\theta_i}$ to maximize $\boldsymbol{L(W,\theta)}$

$$\Rightarrow \frac{\partial L}{\partial \theta_i} = \frac{c(w_i)}{\theta_i} + \lambda \quad \rightarrow \quad \theta_i = -\frac{c(w_i)}{\lambda}$$

Set partial derivatives to zero

$\Rightarrow$ Since $\sum_{i=1}^{N} \theta_i = 1$ we have $\lambda = -\sum_{i=1}^{N} c(w_i)$

Requirement from probability

$$\Rightarrow \theta_i = \frac{c(w_i)}{\sum_{i=1}^{N} c(w_i)}$$

ML estimate

Robertson-Sparck Jones Model

(Robertson & Sparck Jones 76)

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- "+0.5" and "+1" can be justified by Bayesian estimation as priors

Per-query estimation!

RSJ Model without relevance info

(Croft & Harper 79)

	information	retrieval	retrieved	is	helpful	for	you	everyone
Doc1	1	1	0	1	1	1	0	1
Doc2	1	0	1	1	1	1	1	0

el)

Suppose we do not have relevance judgments,

- We will assume p_i to be a constant
- Estimate u_i by assuming all documents to be non-relevant

$$\log O(R=1|Q,D) \overset{Rank}{\approx} \sum_{i=1, d_i=q_i=1}^{k} c + \log \frac{N - n_i + 0.5}{n_i + 0.5}$$

Reminder:

$$IDF = 1 + \log \frac{N}{n_i}$$

IDF weighted Boolean model?

N: # documents in collection

n_i: # documents in which term A_i occurs

RSJ Model: summary

- The most important classic probabilistic IR model
- Use only term presence/absence, thus also referred to as Binary Independence Model
  - Essentially Naïve Bayes for doc ranking
  - Designed for short catalog records
- When without relevance judgments, the model parameters must be estimated in an ad-hoc way
- Performance isn't as good as tuned VS models

Improving RSJ: adding TF

Let D=$d_1 \dots d_k$, where d_k is the frequency count of term A_k

$$\frac{P(R=1|Q,D)}{P(R=0|Q,D)} \propto \prod_{i=1}^{k} \frac{P(A_i=d_i|Q,R=1)}{P(A_i=d_i|Q,R=0)}$$

$$= \prod_{i=1,d_i\geq 1}^{k} \frac{P(A_i=d_i|Q,R=1)}{P(A_i=d_i|Q,R=0)} \prod_{i=1,d_i=0}^{k} \frac{P(A_i=0|Q,R=1)}{P(A_i=0|Q,R=0)}$$

$$\propto \prod_{i=1,d_i\geq 1}^{k} \frac{P(A_i=d_i|Q,R=1)P(A_i=0|Q,R=0)}{P(A_i=d_i|Q,R=0)P(A_i=0|Q,R=1)}$$

2-Poisson mixture model for TF

Eliteness: if the term is about the concept asked in the query

$$p(A_i=f|Q,R) = p(E_i|Q,R)p(A_i=f|E) + P(\overline{E}_i|Q,R)p(A_i=f|\overline{E})$$

$$= p(E_i|Q,R)\frac{\mu_E{}^f}{f!}e^{-\mu_E} + P(\overline{E}_i|Q,R)\frac{\mu_{\overline{E}}{}^f}{f!}e^{-\mu_{\overline{E}}}$$

Many more parameters to estimate! (how many exactly?)
Compound with document length!

BM25/Okapi approximation

(Robertson et al. 94)

- Idea: Approximate p(R=1|Q,D) with a simpler function that approximates 2-Possion mixture model

$$\frac{P(R=1|Q,D)}{P(R=0|Q,D)} \propto \prod_{i=1,d_i\geq 1}^{k} \frac{P(A_i=d_i|Q,R=1)P(A_i=0|Q,R=0)}{P(A_i=d_i|Q,R=0)P(A_i=0|Q,R=1)}$$

- Observations:
  - log O(R=1|Q,D) is a sum of term weights occurring in both query and document
  - Term weight W_i= 0, if TF_i=0
  - W_i increases monotonically with TF_i
  - W_i has an asymptotic limit
- The simple function is $W_i = \frac{TF_i(k_1+1)}{K+TF_i}\log\frac{p_i(1-u_i)}{u_i(1-p_i)}$

Adding doc. length

- Incorporating doc length
  - Motivation: the 2-Poisson model assumes equal document length
  - Implementation: penalize long doc
    - $W_i = \frac{TF_i(k_1+1)}{K+TF_i} \log \frac{p_i(1-u_i)}{u_i(1-p_i)}$

    where $K = k_1((1-b) + b \times \frac{|d|}{avg|d|})$

Pivoted document length normalization

Adding query TF

- Incorporating query TF
  - Motivation
    - Natural symmetry between document and query
  - Implementation: a similar TF transformation as in document TF

$$W_i = \frac{QTF_i(k_s+1)}{k_s+QTF_i}\frac{TF_i(k_1+1)}{K+TF_i}\log\frac{p_i(1-u_i)}{u_i(1-p_i)}$$

- The final formula is called BM25, achieving top TREC performance

BM: best match

The BM25 formula

$$\sum_{T \in Q} w^{(1)} \frac{(k_1+1)tf}{K+tf} \frac{(k_3+1)\,qtf}{k_3+qtf} \qquad (1)$$

"Okapi TF/BM25 TF"

where

Q is a query, containing terms T
$w^{(1)}$ is the Robertson/Sparck Jones weight [5] of T in Q

$$\log \frac{(r+0.5)/(R-r+0.5)}{(n-r+0.5)/(N-n-R+r+0.5)} \qquad (2)$$

becomes IDF when no relevance info is available

N is the number of items (documents) in the collection
n is the number of documents containing the term
R is the number of documents known to be relevant to a specific topic
r is the number of relevant documents containing the term
K is $k_1((1-b)+b.dl/avdl)$
k_1, b and k_3 are parameters which depend on the on the nature of the queries and possibly on the database; k_1 and b default to 1.2 and 0.75 respectively, but smaller values of b are sometimes advantageous; in long queries k_3 is often set to 7 or 1000 (effectively infinite)
tf is the frequency of occurrence of the term within a specific document
qtf is the frequency of the term within the topic from which Q was derived
dl and $avdl$ are respectively the document length and average document length measured in some suitable unit.

The BM25 formula

- A closer look

TF-IDF component for document

TF component for query

$$rel(q,D)=\sum_{i=1}^{n}\boxed{IDF(q_i)\frac{tf_i(k_1+1)}{tf_i+k_1(1-b+b\frac{|D|}{avg|D|})}}\boxed{\frac{qtf_i(k_2+1)}{k_2+qtf_i}}$$

  - b is usually set to [0.75, 1.2]
  - k_1 is usually set to [1.2, 2.0]
  - k_2 is usually set to (0, 1000]

Vector space model with TF-IDF schema!

Extensions of “Doc Generation” models

- Capture term dependence [Rijsbergen & Harper 78]
- Alternative ways to incorporate TF [Croft 83, Kalt96]
- Feature/term selection for feedback [Okapi’s TREC reports]
- Estimate of the relevance model based on pseudo feedback [Lavrenko & Croft 01]

to be covered later

Query generation models

$$O(R=1|Q,D) \propto \frac{P(Q,D|R=1)}{P(Q,D|R=0)}$$

Query likelihood $p(q|\theta_d)$ **Document prior**

Assuming uniform document prior, we have

$$O(R=1|Q,D) \propto P(Q|D,R=1)$$

Now, the question is how to compute $P(Q|D,R=1)$?

Generally involves two steps:

(1) estimate a language model based on D

(2) compute the query likelihood according to the estimated model

Language models, we will cover it in next lecture!

What you should know

- Essential concepts in probability
- Justification of ranking by relevance
- Derivation of RSJ model
- Maximum likelihood estimation
- BM25 formula