Questions & Answers

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Q1: If i and j are recurrent communicating states, why $\sum_{1\le\nu<\infty} f_{ij}^{(\nu)} = 1$?

A Recall that $(f_{ij}^{(\nu)})_{1\le\nu<\infty}$ is the density of the random variable $T_j \mathbb{1}_{\{T_j<+\infty\}}$ where T_j is first return time to j starting from i, namely with respect to the probability measure $\mathbb{P}_i$, i.e.

$$T_j = \min\{\, n \ge 1 \mid X_n = j \,\}, \qquad f_{ij}^{(\nu)} = \mathbb{P}_i\{\, T_j = \nu \,\}$$

(with the agreement $\min\emptyset = +\infty$). If $i = j$, then $\sum_{1\le\nu<\infty} f_{jj}^{(\nu)} = \mathbb{P}_j\{\, T_j < +\infty \,\} = 1$ by definition of recurrent state. If $i \ne j$ then one can find integers $n, m > 0$ such that $p_{ij}^{(n)} = \mathbb{P}_i\{\, X_n = j \,\} > 0$ and $p_{ji}^{(m)} = \mathbb{P}_j\{\, X_m = i \,\} > 0$ because i and j communicate.

$$\begin{aligned}
\mathbb{P}_j\{\, T_j < +\infty \,\} &= \sum_{k\in I} \mathbb{P}_i\{\, T_j < +\infty, X_n = k \,\} \\
&= \sum_{k\in I} \mathbb{P}_i\{\, T_j < +\infty \mid X_n = k \,\}\, \mathbb{P}_i\{\, X_n = k \,\} \\
&= \sum_{k\in I} \mathbb{P}_k\{\, T_j < +\infty \,\}\, p_{ik}^{(n)} \\
&= \sum_{k,h\in I} \mathbb{P}_h\{\, T_j < +\infty \,\}\, p_{ik}^{(n)} p_{kh}^{(m)} \\
&= \sum_{h\in I} \mathbb{P}_h\{\, T_j < +\infty \,\}\, p_{ih}^{(n+m)}
\end{aligned}$$

If $\mathbb{P}_i\{\, T_j < +\infty \,\} < 1$ then, since $p_{ii}^{(n+m)} > 0$, we find

$$\mathbb{P}_j\{\, T_j < +\infty \,\} < \sum_{h\in I} p_{ih}^{(n+m)} = 1$$

contradicting the assumption j recurrent.

Q2: How can one compute the density of the first return time T_0 to 0 of the symmetric random walk on $\mathbb{Z}$ knowing the function $s \mapsto F_{00}(s)$?

A Recall that $F_{00} :]-1,1[\to \mathbb{R}$ is defined by

$$F_{00}(s) = \sum_{n>0} \mathbb{P}_0 \{ T_0 = n \} s^n \tag{1}$$

and we computed

$$F_{00}(s) = 1 - \sqrt{1-s^2}.$$

However, in order to compute explicitly $\mathbb{P}_0 \{ T_0 = n \}$, rather than differentiating n times F_{00}, it is more convenient to consider the change of variables $t = s^2$ and note that

$$1 - \sqrt{1-t} = \sum_{k \geq 1} \frac{2^{-2k}}{2k-1} \binom{2k}{k} t^k, \qquad |t| < 1 \tag{2}$$

Indeed

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(1 - \sqrt{1-t}\right) = \frac{1}{2}(1-t)^{-1/2}$$

and, iterating for $k \geq 2$

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left(1 - \sqrt{1-t}\right) &= \frac{1}{2}\left(1 - \frac{1}{2}\right) \cdots \left(k - 1 - \frac{1}{2}\right)(1-t)^{-(k-1/2)} \\ &= \frac{(2k-3)\cdots 3 \cdot 1}{2^k}(1-t)^{-(k-1/2)}, \end{aligned}$$

and, at $t = 0$, (recall that $(2k)!! = 2k(2k-2)\cdots 2 = 2^k k!$)

$$\left. \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left(1 - \sqrt{1-t}\right) \right|_{t=0} = \frac{(2k-2)!}{2^k (2k-2)!!} = \frac{(2k)!}{2^k (2k-1)(2k)!!} = \frac{2^{-2k}\, k!}{2k-1} \binom{2k}{k}.$$

Recalling that

$$1 - \sqrt{1-t} = \sum_{k \geq 1} \left. \frac{\mathrm{d}^k}{\mathrm{d}t^k} \left(1 - \sqrt{1-t}\right) \right|_{t=0} \frac{t^k}{k!}$$

equation (2) is proved. Turning back to F_{00}

$$F_{00}(s) = 1 - \sqrt{1-s^2} = \sum_{n \geq 1} \frac{2^{-2n}}{2n-1} \binom{2n}{n} s^{2n}, \qquad |s| < 1$$

and so, comparing with (1), we find $\mathbb{P}_0 \{ T_0 = n \} = 0$ for n odd and for $n \geq 1$

$$\mathbb{P}_0 \{ T_0 = 2n \} = \frac{2^{-2n}}{2n-1} \binom{2n}{n}.$$