Exercise

1. Decide the definiteness of the following ~~matrices~~ quadratic form using eigen values.

(a) $Q(X) = x_1^2 + 10x_1x_2 + x_2^2$ Ans: indefinite

(b) $Q(X) = 4x_1^2 - 4x_1x_2 + 4x_2^2$ Ans Decide?

(c) $-x_1^2 - 2x_1x_2 - x_2^2 = Q(X)$, Ans: Negative semi definite

2. Find a change of variable $X = PY$ that removes the cross product term x_1x_2 in the quadratic form. Also find formula and new quadratic form. Find the value of Y if $X = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in each case.

(a) $5x_1^2 - 4x_1x_2 + 5x_2^2$ (b) $x_1^2 + 10x_1x_2 + x_2^2$

(c) $4x_1^2 - 4x_1x_2 + 4x_2^2$ (d) $x_1^2 - 6x_1x_2 + 9x_2^2$

Ans: (a) $3y_1^2 + 7y_2^2$ (b) $6y_1^2 - 4y_2^2$

(c) $6y_1^2 + 2y_2^2$ (d) y_1^2 $(\lambda_1 = 10, \lambda_2 = 0)$

(3) Let A be the matrix of the quadratic form
$$Q(X) = X^TAX = 9x_1^2 + 7x_2^2 + 11x_3^2 - 8x_1x_2 + 8x_1x_3$$
Find the eigen values of A. Find the orthogonal matrix P that transform the variable $X = PY$ changes X^TAX into a quadratic form with no cross product term.

Find the new quadratic form.

Ans: $A = \begin{pmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 13 \end{pmatrix}$, $\lambda_1 = 3$, $\lambda_2 = 9$, $\lambda_3 = 15$

$$P = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

New quadratic form is $13y_1^2 + 9y_2^2 + 3y_3^2$.

4. Consider the quadratic form

(a) $Q(X) = f(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 - 4x_1x_3 + 2x_2x_3 - 4x_3^2$

Express $Q(X)$ as the sum/difference of squares.

Find the matrix form $\bar{\alpha} = AX$.

<u>Hint</u>: $Q(X) = (x_1 + x_2 - 2x_3)^2 - (x_2 - 3x_3)^2 + x_3^2$

$$\bar{\alpha} = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = AX.$$

(b) For $Q(X) = 2x_1^2 + 2x_2^2 + 2x_1x_2$.

find $\bar{\alpha} = AX$.

5. List the conditions under which the ~~quadratic form~~ conic section

$$Ax_1^2 + 2Bx_1x_2 + Cx_2^2 + Dx_1 + Ex_2 + F = 0$$

where A, B, C are not all zero.

represents circle, parabola, ellipse and hyperbola.

What conic does the following conic represent?

Find centre, length of axis and draw the shape of the graph (depending upon curve)

(a) $2x_1^2 + 3x_2^2 - 4x_1 + 5x_2 + 4 = 0$ (Ellipse)

(b) $3x_1^2 - 5x_1x_2 + 2x_2^2 + 5x_1 + 11x_2 - 8 = 0$ (Hyperbola)

(c) $3x^2 + 3y^2 + 2x + 4y - 1 = 0$ (circle)

(d) $x^2 + xy + y^2 + x + y + 1 = 0$ (Ellipse)

(e) $y^2 + 2y + 1 = x$ (parabola)

Theorem

Let A be a 3×3 square matrix with a quadratic form $Q(X) = X^TAX$ in three variables, $\forall X \in \mathbb{R}^3$. Then there exists a symmetric matrix $B \in \mathbb{R}^{3\times3}$ (The set of all 3×3 matrices) such that

$$X^TAX = X^TBX.$$

We can extend this theorem for $n\times n$ square matrix.

Verify this theorem for the matrix (i) $A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 7 \\ 8 & 1 & 3 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 3 \\ 0 & 3 & -1 \end{pmatrix}$

Optimization using quadratic forms

Theorem: Prove the following Theorem ~~Let~~ A be a $n\times n$ symmetric matrix whose eigen values are in ~~order~~ decending order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. ~~Then~~ Then quadratic form $Q(X) = X^TAX$, $x \in \mathbb{R}^n$, ~~contains~~ attains a maximum value and minimum value on the set of vectors under the contraint $\|X\| = 1$ ie $X^TX = x_1^2 + x_2^2 + \cdots + x_n = 1$, and has

~~Then~~

(i) Maximum value, occur at a vector corresponding to the eigen value λ_1

(ii) Minimum value, occur at a vector corresponding to the eigen value λ_n.

Use above ~~theorem~~ Application, solve the following problem.

1. Find Maximum and minimum value of the following quadratic form using eigen value under $\|x\|=1$

(a) $Q(X) = 2x_1x_2$ [Hint, $Q(X) = X^TAX$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ find $\lambda_1 = 1$, $\lambda_2 = -1$, $u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $u_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

(b) $Q(X) = X^TAX$, where $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

(c) $7x_1^2 + 3x_2^2 + 3x_1x_2 = Q(X)$

Ans; Max $= 6$ at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
Min $= 1$

(2) Let $A = \begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix}$

Solve the following optimization problem

$$\max Q(x) = x^TAx \text{ subject to } \|x\|^2 = 1.$$

(3) Maximize and minimize the product $f(x,y) = xy$ subject to the condition $x^2+y^2=1$.

Ans: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$

$u_1 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $u_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(4) Find the maximum and minimum value of quadratic form $Q(X) = 9x_1^2 + 4x_2^2 + 3x_3^2$, provided $x_1^2 + x_2^2 + x_3^2 = 1$.

Hint: $3 \le Q(X) \le 9$., max $= 9$, minimum $= 3$.

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