System of Linear Equations (Problem Sheet)

EXERCISE A

1. Using Gauss Elimination method, show that the following system of equations are consistent and unique solution:

   a) $x + 2y = 5, 5x - 3y = -1$ Ans: (1, 2)

   b) $3x - 2y = 5, 4x - y = 10.$ Ans: (3, 2)

   c) $x+3y - 2z = 0, 2x-3y + z = 1, 4x - 3y + z = 3.$ Ans: (1, 1, 2)

   d) $x+3y-z=-2, 3x+ 2y - z = 3, -6x - 4y - 2z=18.$ Ans: (1, –3, – 6).

   e) $x_1 - 2x_2 + 3x_3 = 10, 2x_1 + 3x_2 - 2x_3 = 1, -x_1 - 2x_2 + 4x_3 = 13.$ Ans: (1, 3, 5).

2. Examine whether the following system of equations are consistent . If so, solve the following system of equation using Gaussian elimination method if possible:

   a) $x_1 - x_2 + x_3 = 1, 3x_1 + x_2 + 5x_3 = 11, 4x_1 + 2x_2 + 7x_3 = 16.$

   Ans: Consistent and Infinite solutions ,and $(x_1, x_2, x_3) = \left(\frac{6-3k}{2}, \frac{4-k}{2}, k\right), k \in R.$

   b) $x + 3y + 4z = 8, 2x + y + 2z = 5, 5x + 2z = 7.$

   Ans: Consistent and Infinite solutions, and $(x, y, z) = \left(\frac{7-2k}{5}, \frac{11-6k}{5}, k\right), k \in R.$

   c) $x_1 + x_2 + x_3 = -3, 3x_1 + x_2 - 2x_3 = -2, 2x_1 + 4x_2 + 7x_3 = 7.$

   Ans: Inconsistent i.e. No solutions.

   d) $x_1 + 2x_2 + 3x_3 = 4, 4x_1 + 5x_2 + 6x_3 = -7, 7x_1 + 8x_2 + 9x_3 = 10.$

   Ans: Inconsistent i.e. No solutions

Solution of System of Equation by Gauss Jordan Matrix Inversion Method

Consider the following system of linear equation in 3 variables as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

Then the above system of equations can be written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad \text{or, } AX = C$$

where, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

If $|A| \neq 0$ then $X = A^{-1}C$ be the required solution.

To find A^{-1}, the augmented of matrix A with an identity matrix I is

$$[A : I] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & : & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & : & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & : & 0 & 0 & 1 \end{pmatrix}.$$

If we reduce the above augmented matrix by using Gauss-Jordan method in the form

$$\begin{pmatrix} 1 & 0 & 0 & : & a_{11}' & a_{12}' & a_{13}' \\ 0 & 1 & 0 & : & a_{21}' & a_{22}' & a_{23}' \\ 0 & 0 & 1 & : & a_{31}' & a_{32}' & a_{33}' \end{pmatrix}.$$

Then the right hand side of the augmented matrix will represent the inverse matrix of A i.e.

$$A^{-1} = \begin{pmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{pmatrix}.$$

Consistency of a System of Linear Equations

A system of linear equations is said to be consistent if it admits a solution and inconsistent if it has no solution.

Two cases may arise:

If $|A| \neq 0$, then the linear system is consistent.

If $|A| = 0$, then the linear system is non-consistent or inconsistent.

EXERCISE B

1. Find the inverse of each of the following matrices using Gauss-Jordan method.

a) $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. The augment of the given matrix A with the unit matrix I is given by

$$[A : I] = \begin{pmatrix} 1 & 2 & : & 1 & 0 \\ 2 & 3 & : & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & : & 1 & 0 \\ 0 & -1 & : & -2 & 1 \end{pmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{pmatrix} 1 & 2 & : & 1 & 0 \\ 0 & 1 & : & 2 & -1 \end{pmatrix} R_2 \to (-1) . R_2$$

$$\sim \begin{pmatrix} 1 & 0 & : & -3 & 2 \\ 0 & 1 & : & 2 & -1 \end{pmatrix} R_1 \to R_1 - R_2 \qquad .$$

Hence $A^{-1} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$.

b) $M = \begin{pmatrix} 2 & -3 \\ -2 & 5 \end{pmatrix}$ **Ans:** $M^{-1} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

c) $B = \begin{pmatrix} 2 & -3 \\ -2 & 5 \end{pmatrix}$ **Ans:** $B^{-1} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

2. Find the inverse of the matrix using Gauss-Jordan method:

a) $P = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$

Solution:

Let $P = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$

Now, the augment of the matri*x* P with unit matri*x* I is given by

$$[P:I] = \begin{pmatrix} 1 & -2 & 3 & : & 1 & 0 & 0 \\ 0 & -1 & 4 & : & 0 & 1 & 0 \\ -2 & 2 & 1 & : & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & : & 1 & 0 & 0 \\ 0 & -1 & 4 & : & 0 & 1 & 0 \\ 0 & -2 & 7 & : & 2 & 0 & 1 \end{pmatrix} R_3 \to R_3 + 2R_1$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & -4 & : & 0 & -1 & 0 \\ 0 & -2 & 7 & : & 2 & 0 & 1 \end{pmatrix} R_2 \to (-1) . R_2$$

$$\sim \begin{pmatrix} 1 & 0 & -5 & : & 1 & -2 & 0 \\ 0 & -1 & 4 & : & 0 & 1 & 0 \\ 0 & 0 & -1 & : & 2 & -2 & 1 \end{pmatrix} \begin{matrix} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 2R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -5 & : & 1 & -2 & 0 \\ 0 & -1 & 4 & : & 0 & 1 & 0 \\ 0 & 0 & 1 & : & -2 & 2 & -1 \end{pmatrix} R_3 \to (-1) . R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & : & -9 & 8 & -5 \\ 0 & 1 & 0 & : & -8 & 7 & -4 \\ 0 & 0 & 1 & : & -2 & 2 & -1 \end{pmatrix} \begin{matrix} R_1 \to R_1 + 5R_3 \\ R_2 \to R_2 + 4R_3 \end{matrix}$$

Thus, $P^{-1} = \begin{pmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{pmatrix}$.

b) $R = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ **Ans:** $R^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & 1 & -\frac{1}{5} \end{pmatrix}$.

3. Solve the following system of equations using Gauss-Jordan inversion method:

(a) $2x + 3y = 12,\ 4x + 5y = 22$

Solution:

Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 12 \\ 22 \end{pmatrix}$.

First we find the inverse of the matrix using Gauss-Jordan method and use . Now, the augment of the coefficient matri*x* A with the unit matri*x* I is given by

$$[A : I] = \begin{pmatrix} 2 & 3 & : & 1 & 0 \\ 4 & 5 & : & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{3}{2} & : & \frac{1}{2} & 0 \\ 4 & 5 & : & 0 & 1 \end{pmatrix} R_1 \to \frac{1}{2} R_1$$

$$\sim \begin{pmatrix} 1 & \frac{3}{2} & : & \frac{1}{2} & 0 \\ 0 & -1 & : & -2 & 1 \end{pmatrix} R_2 \to R_2 - 4R_1$$

$$\sim \begin{pmatrix} 1 & \frac{3}{2} & : & \frac{1}{2} & 0 \\ 0 & 1 & : & 2 & -1 \end{pmatrix} R_2 \to (-1) R_2$$

$$\sim \begin{pmatrix} 1 & 0 & : & -\frac{5}{2} & \frac{3}{2} \\ 0 & 1 & : & 2 & -1 \end{pmatrix} R_1 \to R_1 - \frac{3}{2} R_2$$

Therefore, $A^{-1} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix}$

Hence, $X = A^{-1}B$ becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 12 \\ 22 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}. \quad \therefore x = 3 \text{ and } y = 2. \textbf{ Ans.}$$

b) $3x_1 - x_2 = 5$, $8x_1 - 3x_2 = 14$. Ans: **$x_1 = 1$ and $x_2 = -2$.**

(c) $4x - 3y = -17$, $-6x + 5y = 27$ Ans: **$x = -2$ and $y = 3$.**

4. Solve the following system of equations using Matrix Inversion method :

(a) $x_1 - 2x_2 - x_3 = 1$, $x_1 - x_2 + 2x_3 = 9$, $2x_1 - 3x_2 - x_3 = 4$.

Hint: Use $X = A^{-1}C$ and $A^{-1} = \frac{\text{Adj. } A}{|A|}$ Ans: $x_1 = 2$, $x_2 = -1$ and $x_3 = 3$.

(b) $3x + y + z = 15$, $x + y + z = 3$, $y - z = -1$.

Ans: $x = 6$, $y = -2$ and $z = -1$.

5. Solve the following system of equations using Gauss Jordan Matrix Inversion method :

(a) $9y - 5x = 3$, $x + y = 1$, $z + 2y = 2$.

Solution:

Let $A = \begin{pmatrix} -5 & 9 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

So that $AX = C$ and so $X = A^{-1}C$. ………….(i)

Now, the augment of the coefficient matrix A with the identity matrix I is

$$[A : I] = \begin{pmatrix} -5 & 9 & 0 & : & 1 & 0 & 0 \\ 1 & 0 & 1 & : & 0 & 1 & 0 \\ 0 & 2 & 1 & : & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{9}{5} & 0 & : & -\frac{1}{5} & 0 & 0 \\ 1 & 0 & 1 & : & 0 & 1 & 0 \\ 0 & 2 & 1 & : & 0 & 0 & 1 \end{pmatrix} R_1 \to \left(\frac{-1}{5}\right) R_1$$

$$\sim\begin{pmatrix} 1 & -\frac{9}{5} & 0 & : & -\frac{1}{5} & 0 & 0 \\ 0 & \frac{9}{5} & 1 & : & \frac{1}{5} & 1 & 0 \\ 0 & 2 & 1 & : & 0 & 0 & 1 \end{pmatrix} R_2 \to R_2 - R_1$$

$$\sim\begin{pmatrix} 1 & -\frac{9}{5} & 0 & : & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{5}{9} & : & \frac{1}{9} & \frac{5}{9} & 0 \\ 0 & 2 & 1 & : & 0 & 0 & 1 \end{pmatrix} R_2 \to \frac{5}{9} R_2$$

$$\sim\begin{pmatrix} 1 & 0 & 1 & : & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{9} & : & \frac{1}{9} & \frac{5}{9} & 0 \\ 0 & 0 & \frac{-1}{9} & : & -\frac{2}{9} & -\frac{10}{9} & 1 \end{pmatrix} \begin{matrix} R_1 \to R_1 + \frac{9}{5} R_2 \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$$\sim\begin{pmatrix} 1 & 0 & 1 & : & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{9} & : & \frac{1}{9} & \frac{5}{9} & 0 \\ 0 & 0 & 1 & : & 2 & 10 & -9 \end{pmatrix} R_3 \to (-9)\,.\,R_3$$

$$\sim\begin{pmatrix} 1 & 0 & 0 & : & -2 & -9 & 9 \\ 0 & 1 & 0 & : & -1 & -5 & 5 \\ 0 & 0 & 1 & : & 2 & 10 & -9 \end{pmatrix} \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - \frac{5}{9} R_3 \end{matrix}$$

Hence, $A^{-1} = \begin{pmatrix} -2 & -9 & 9 \\ -1 & -5 & 5 \\ 2 & 10 & -9 \end{pmatrix}$

Now, $X = A^{-1} C$ gives

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & -9 & 9 \\ -1 & -5 & 5 \\ 2 & 10 & -9 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -6-9+18 \\ -3-5+10 \\ 6+10-18 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

$\therefore$ $x = 3$, $y = 2$ and $z = -2$.

b) $4x_1+3x_2 - 5x_3 = -1,\ 3x_1+7x_2 - x_3 = 8,\ x_1 + x_2 - x_3 = 0$

Ans: $x_1 = -5$, $x_2 = 3$ and $x_3 = -2$.

c) $x + y + z = 0,\ x - 2y + z = -3,\ x + 2y - z = 5$

Ans: $x = 1$, $y = 1$ and $z = -2$.

d) $x + 2y + z = 8, 2x + 3y + 2z = 14,\ 3x + 2y + 2z = 13$

Ans: $x = 1$, $y = 2$ and $z = 3$.

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