Proof Writing Problem Set I

Linear Algebra ⋄ MATH 228-02 ⋄ Fall 2015

September 26, 2015

SOLUTIONS

1.] Let $A = [a_{ij}]$ be an $m \times n$ matrix. Let $c, d \in \mathbb{R}$ be scalars. Use the definition of matrix addition and scalar multiplication to show that the following distributive law holds for scalar multiplication:

$$(c + d)A = cA + dA.$$

Proof. Let $A = [a_{ij}]$ be an $m \times n$ matrix, such that $a_{ij} \in \mathbb{R}$ is the element of the matrix A in the i^{th} row and j^{th} column where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Also, let $c, d \in \mathbb{R}$ be scalars, then $c + d \in \mathbb{R}$ is also a scalar since addition is closed in the real number system. By the definition of scalar multiplication, it follows

$$(c + d)A = (c + d)[a_{ij}] = \big[(c + d)a_{ij}\big].$$

From here, we know $(c+d)a_{ij} = ca_{ij} + da_{ij}$ since real numbers obey the distributive property. Hence, from the definition of matrix addition and scalar multiplication, we have

$$\begin{aligned}(c + d)A &= [ca_{ij} + da_{ij}] \\ &= [ca_{ij}] + [da_{ij}] \\ &= c[a_{ij}] + d[a_{ij}] \\ &= cA + dA.\end{aligned}$$

We have shown $(c + d)A = cA + dA.$

□

2.] Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$ be $m \times n$ matrices. Use the definition of matrix addition to show that matrix addition is associative:

$$(A + B) + C = A + (B + C).$$

Proof. Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$ be $m \times n$ matrices, such that a_{ij}, b_{ij}, $c_{ij} \in \mathbb{R}$ are the elements of the matrices A, B, and C, respectively, in the i^{th} row and j^{th} column where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. By the definition of matrix multiplication, it follows that

$$\begin{aligned}(A + B) + C &= \big([a_{ij}] + [b_{ij}]\big) + [c_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] \\ &= [(a_{ij} + b_{ij}) + c_{ij}].\end{aligned}$$

Now, we have a single matrix whose i^{th}, j^{th} element is $(a_{ij} + b_{ij}) + c_{ij}$. Since a_{ij}, b_{ij}, and c_{ij} are real numbers and are associative, it follows that

$$(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + b_{ij} + c_{ij} = a_{ij} + (b_{ij} + c_{ij}).$$

Therefore, by the definition of matrix addition, we obtain

$$\begin{aligned}(A + B) + C &= [a_{ij} + (b_{ij} + c_{ij})] \\ &= [a_{ij}] + [b_{ij} + c_{ij}] \\ &= [a_{ij}] + \big([b_{ij}] + [c_{ij}]\big) \\ &= A + (B + C).\end{aligned}$$

We have shown that $(A + B) + C = A + (B + C).$

□

3.] Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix. Let $c \in \mathbb{R}$ be a scalar. Using the definition of matrix and scalar multiplication, prove the following equation:

$$c(AB) = A(cB).$$

Proof. Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix, such that $a_{ij}, b_{jk} \in \mathbb{R}$ are the elements in the i^{th} row and j^{th} column of A and the j^{th} row and k^{th} column of B, respectively, where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, p$. Also, let $c \in \mathbb{R}$ be a scalar. Then by the definition of matrix multiplication, we obtain

$$c(AB) = c[a_{ij}][b_{jk}] = c\left[\sum_{j=1}^{n} a_{ij}b_{jk}\right].$$

By the definition of scalar multiplication, we have

$$c(AB) = \left[c\sum_{j=1}^{n} a_{ij}b_{jk}\right].$$

From here, we can bring the constant c into the summation since c, a_{ij} and b_{jk} are real numbers that obey the distributive property:

$$c(AB) = \left[\sum_{j=1}^{n} ca_{ij}b_{jk}\right].$$

Further, since real number multiplication is associative, it follows that $ca_{ij}b_{jk} = a_{ij}(cb_{jk})$. We have

$$c(AB) = \left[\sum_{j=1}^{n} a_{ij}(cb_{jk})\right].$$

Finally, by applying the definitions of matrix and scalar multiplication, we arrive at the desired result:

$$\begin{aligned} c(AB) &= \left[\sum_{j=1}^{n} a_{ij}(cb_{jk})\right] \\ &= [a_{ij}][cb_{jk}] \\ &= [a_{ij}](c[b_{jk}]) \\ &= A(cB). \end{aligned}$$

□

4.] Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, and $C = [c_{jk}]$ be an $n \times p$ matrix. Use the definition of matrix addition and multiplication to prove the following distributive law for matrices:

$$(A + B)C = AC + BC.$$

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, and $C = [c_{jk}]$ be an $n \times p$ matrix. Let a_{ij}, $b_{ij} \in \mathbb{R}$ be the elements in the i^{th} row and j^{th} column of the matrices A and B, respectively, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Also, let $c_{jk} \in \mathbb{R}$ be the element in the j^{th} row and k^{th} column of the matrix C for $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p$. Then by the definition of matrix addition, we have

$$(A + B)C = \big([a_{ij}] + [b_{ij}]\big)[c_{jk}] = \big[(a_{ij} + b_{ij})\big][c_{jk}].$$

By the definition of matrix multiplication, we obtain

$$(A + B)C = \left[\sum_{j=1}^{n}(a_{ij} + b_{ij})c_{jk}\right].$$

Since a_{ij}, b_{ij}, and c_{jk} are real numbers, they obey the distributive law so that $(a_{ij} + b_{ij})c_{jk} = a_{ij}c_{jk} + b_{ij}c_{jk}$. Furthermore, addition of the real numbers is associative, which means the summation can be written as follows:

$$\sum_{j=1}^{n}(a_{ij} + b_{ij})c_{jk} = \sum_{j=1}^{n} a_{ij}c_{jk} + b_{ij}c_{jk} = \sum_{j=1}^{n} a_{ij}c_{jk} + \sum_{j=1}^{n} b_{ij}c_{jk}.$$

From this, we have

$$\begin{aligned}
(A + B)C &= \left[\sum_{j=1}^{n}(a_{ij} + b_{ij})c_{jk}\right] \\
&= \left[\sum_{j=1}^{n} a_{ij}c_{jk} + \sum_{j=1}^{n} b_{ij}c_{jk}\right] \\
&= \left[\sum_{j=1}^{n} a_{ij}c_{jk}\right] + \left[\sum_{j=1}^{n} b_{ij}c_{jk}\right] \\
&= [a_{ij}][c_{jk}] + [b_{ij}][c_{jk}] \\
&= AC + BC,
\end{aligned}$$

which follows from the definition of matrix addition and multiplication. We have successfully shown that $(A + B)C = AC + BC$.

□

5.] An $n \times n$ matrix is said to be *upper triangular* if all the entries above the diagonal are zero. More precisely, if $A = [a_{ij}]$, then A is upper triangular precisely when $a_{ij} = 0$ for all $i > j$. Using the definition of matrix multiplication, show that if A and B are two $n \times n$ upper triangular matrices, then the product $C = AB$ is also upper triangular.

Proof. Let $A = [a_{ij}]$ and $B = [b_{jk}]$ be a $n \times n$ matrices, such that $a_{ij}, b_{jk} \in \mathbb{R}$ are the elements in the i^{th} row and j^{th} column of A and the j^{th} row and k^{th} column of B, respectively, where $i, j, k = 1, 2, \ldots, n$. Suppose A and B are upper triangular so that if $i > j$, then $a_{ij} = 0$ and if $j > k$, then $b_{jk} = 0$. Then by the definition of matrix multiplication, the matrix $C = AB$ is an $n \times n$ matrix denoted by $C = [c_{ik}]$ for $i, k = 1, 2, \ldots, n$. We wish to show $c_{ik} = 0$ if $i > k$. Consider the following matrix product:

$$[c_{ik}] = [a_{ij}][b_{jk}] = \left[\sum_{j=1}^{n} a_{ij}b_{jk}\right].$$

We note that for $i \leq k$, our summation can be written as

$$[c_{ik}] = \left[\sum_{j=1}^{i-1} a_{ij}b_{jk} + \sum_{j=i}^{k} a_{ij}b_{jk} + \sum_{j=k+1}^{n} a_{ij}b_{jk}\right].$$

The first summation from $j = 1$ to $j = i - 1$ is zero as $a_{ij} = 0$ since $i > j$. Also, the summation on the right from $j = k + 1$ to $j = n$ is zero as $b_{jk} = 0$ since $j > k$. Hence, we are left with

$$[c_{ik}] = \left[\sum_{j=i}^{k} a_{ij} b_{jk} \right].$$

We note that this sum has terms in it if $i \leq k$. As i approaches k, the number of terms decreases, which means if $i > k$, no more terms are within the sum. Therefore, for $i > k$ we have $c_{ik} = 0$, as desired. This shows that the matrix $C = AB$ is upper triangular.

$\square$