Eigenvalues and Eigenvectors

Exercise

1. Write down the (i) characteristic polynomial, (ii) the characteristic equation, (iii) the characteristic values, (iv) multiplicity of the characteristic value and (v) the characteristic vector of the matrix:

a) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Ans: i) $\lambda^2 - 1$ ii) $\lambda^2 - 1 = 0$ iii) $\lambda = 1, -1$ iv) 1

v) If $\lambda_1 = 1$, $\boldsymbol{v_1} = a_1(1, 0)$, $a_1 \neq 0$ and if $\lambda_2 = -1$, $\boldsymbol{v_2} = a_2(0, 1)$, $a_2 \neq 0$.

b) $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

Ans: i) $\lambda^2 - 7\lambda + 6$ ii) $\lambda^2 - 7\lambda + 6 = 0$ iii) $\lambda = 6, 1$ iv) 1

v) If $\lambda_1 = 6$, $\boldsymbol{v_1} = a_1(4, 1)$, $a_1 \neq 0$ and if $\lambda_2 = 1$, $\boldsymbol{v_2} = a_2(1, 1)$, $a_2 \neq 0$.

Example 1: *Find the eigenvalues of the matrix* $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Hint:

The characteristic equation is $|A - \lambda I| = 0$.

or, $\begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$ i.e, $\lambda = 1, \lambda = 1 \pm i$.

Example 2: *Show that the eigenvalues of a triangular matrix are just the diagonal elements of the matrix.*

Solution:

We may take the following upper triangular matrix of order n

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

or, $$\begin{vmatrix} a_{11-\lambda} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22-\lambda} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33-\lambda} & \dots & a_{3n} \\ & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn-\lambda} \end{vmatrix} = 0$$

or, $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)\dots(a_{nn} - \lambda) = 0 \quad \therefore \quad \lambda = a_{11}, a_{22}, \dots, a_{nn}$.

This shows that the eigenvalues of A are $a_{11}, a_{22}, \dots, a_{nn}$, which are just the diagonal elements of A.

Example 3: Find the characteristic values and the characteristic vector of the matrix: $A = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$

Solution: We need to find a scalar λ and a non–zero vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$ associated with the eigenvalue λ such

That $AX = \lambda X$ or $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

Now, the characteristic polynomial of A is $A - \lambda I$ and its characteristic equation is $|A - \lambda I| = 0$ i.e.

$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$ or, $(1 - \lambda)(1 + \lambda) = 0$ or, $\lambda = 1, -1$.

Thus, $\lambda = \lambda_1 = 1$ and $\lambda = \lambda_2 = -1$ are the eigen–values of A.
Clearly, multiplicity of characteristic value is 1.
To find the eigenvectors associated with these eigen–values, we need to solve, $\{(A - \lambda I)\}\, X = O$

or, $\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

i) For $\lambda = \lambda_1 = 1$, we have $\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or, $2y = 0$.

So, $y = 0$ and x is arbitrary. Taking $x = 1$ ($\neq 0$, why?), we have $X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

as an eigen–vector of A associated with the eigen–value $\lambda = \lambda_1 = 1$, and any other eigen–vector for $\lambda_1 = 1$ is $\boldsymbol{v_1} = a_1(1, 0)$, $a_1 \neq 0$.

ii) For $\lambda = \lambda_2 = -1$, we have $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So, $2x = 0$ or $x = 0$, and y is arbitrary. Taking $y = 1$ ($\neq 0$ why?), we have

$X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as an eigen–vector of A associated with the eigen–value $\lambda = \lambda_2 = -1$, and any other eigen–vector of $\lambda_2 = -1$ is $\mathbf{v_2} = a_2\,(0, 1)$, $a_2 \neq 0$.

Exercise

Write down the characteristic values, and the characteristic vector of the matrix:

a) $A = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$ **Ans:** $\lambda = -3, 2$. If $\lambda_1 = -3$, $\boldsymbol{v_1} = a_1(1, -1)$, $a_1 \neq 0$ and if $\lambda_2 = 2$, $\boldsymbol{v_2} = a_2(3, 2)$, $a_2 \neq 0$.

b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ **Ans:** $\lambda = 1, 3$; If $\lambda_1 = 1$, $\boldsymbol{v_1} = a_1(1, -1)$, $a_1 \neq 0$ and if $\lambda_2 = 3$, $\boldsymbol{v_2} = a_2(1, 1)$, $a_2 \neq 0$.

Eigenvalues and eigenvectors of the 3×3 matrices

Example 1: Find the characteristic values and the characteristic vectors of the matrix: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

The eigen–values of A are $\lambda = \lambda_1 = 1$, $\lambda = \lambda_2 = 1$ and $\lambda = \lambda_3 = -1$.

To find the eigen–vectors, we need to solve $\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i) For $\lambda = \lambda_1 = 1$, we have

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or,} \quad \begin{bmatrix} -x+y \\ x-y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or,} \quad x - y = 0 \quad \text{or, } y = x \text{ for any } z.$$

Taking $x = 1$, $y = 1$ and $z = 1$, (Why?) we have $X_1 = (1, 1, 1)$ as an eigen–vector of A associated with the eigen–value $\lambda = \lambda_1 = 1$.

ii) $\lambda = \lambda_2 = -1$, we have $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or, $x + y = 0$, $2z = 0$, or $z = 0$.

Taking $x = 1$, we have $y = -1$. So, the eigen–vector associated with the eigen–value $\lambda = -1$, is

$$X_2 = (1, -1, 0).$$

Example 2:

Write down the characteristic values and the characteristic vectors of the matrix: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Solution:

If λ is an eigen–value of A and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the corresponding eigen–vector of A, then

$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$ which gives $\lambda = 2, 3, 4$, as the eigen–values of A.

We know $\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

i) When $\lambda = 2$, $2y = 0$, $z = 0$ or $y = 0 = z$ and x is arbitrary.
Taking $x = \text{k}$, we have $X_1 = (\text{k}_1, 0, 0)$, $k_1 \neq 0$ as an eigen–vector of A associated with $\lambda = 2$.

ii) When $\lambda = 4$, $-2x = 0$, $-z = 0$, or $x = 0 = z$ and y is arbitrary
Taking $\text{y} = \text{k}_2$, we have $X_2 = (0, \text{k}_2, 0)$, $k_2 \neq 0$ as an eigen–vector of A associated with $\lambda = 4$.

iii) When $\lambda = 3$, $-x = 0$, $y = 0$, or, $x = 0 = y$ and z is arbitrary
Taking $z = \text{k}_3$, we have $X_3 = (0, 0, \text{k}_3)$, $k_3 \neq 0$ as an eigen–vector of A associated with $\lambda = 3$.

Example 3: *Determine the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

Solution:

The characteristic equation is $|A - \lambda I| = 0$ gives the eigenvalues 7, 1, 1.

If $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the non–zero eigenvector.

If $\lambda = 7$, then $(A - \lambda I)\, X = O$ gives $(A - 7I)\, X = O$.

or, $\begin{pmatrix} -2 & 2 & 2 \\ 2 & -5 & 1 \\ 2 & 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_2$, then

$$\begin{pmatrix} -2 & 2 & 2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \to R_3 + R_2$, then

$$\begin{pmatrix} -2 & 2 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Writing in equation form, we get

$-2x + 2y + 2z = 0, \quad -3y + 3z = 0$

Taking $z = k$, we have $y = k$, and $x = 2k$.

$\therefore$ Required eigenvector corresponding to $\lambda = 7$ is $k(2, 1, 1), k \neq 0$.

Again, corresponding to the repeated eigenvalue $\lambda = 1$, the eigenvectors are given by non–zero solutions of the equation $(A - I)X = O$.

$$\text{or,} \quad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{1}{2} R_1$ and $R_3 \to R_3 - R_2$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Writing in equation form, we get

$2x + y + z = 0.$

To get two linearly independent eigenvectors, we will put successively arbitrary values of any two variables. Here, we put $x = 0$ and $y = 0$ successively.

Taking $x = 0$ and $y = k_1$, we get $z = -k_1$ and taking $y = 0$ and $z = -2k_2$, we get $x = k_2$.

$\therefore$ The eigenvectors corresponding to $\lambda = 1$ are $k_1(0, 1, -1)$ and $k_2(1, 0, -2)$ where $k_1 \neq 0$ and $k_2 \neq 0$.

Exercise

1. Find the eigenvalues and eigenvectors of the following matrices:

a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **Ans:** $\lambda_1 = 1$, $\boldsymbol{v_1} = (1, 1, 1)$; $\lambda_2 = -1$, $\boldsymbol{v_2} = (1, -1, 0)$

b) $\begin{pmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ **Ans:** $\lambda_1 = 2$, $\boldsymbol{v_1} = (1, 0, 0)$; $\lambda_2 = -2$, $\boldsymbol{v_2} = (1, -12, 4)$

c) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ d) $\begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Ans: c) $\lambda_1 = 2$, $\boldsymbol{v_1} = (1, 0, 0)$; $\lambda_2 = 4$, $\boldsymbol{v_2} = (0, 1, 0)$; $\lambda_3 = 3$, $\boldsymbol{v_3} = (0, 0, 1)$

d) $\lambda_1 = 2$, $\boldsymbol{v_1} = (1, -1, 0)$; $\lambda_2 = 3$, $\boldsymbol{v_2} = (1, 0, 0)$; $\lambda_3 = 5$, $\boldsymbol{v_3} = (1, 2, 1)$

e) $\lambda = 1$, $\boldsymbol{v} = (1, 0, 0)$.

Cayley–Hamilton Theorem for the following matrix:

Example 1: *Verify Cayley–Hamilton Theorem for the following matrix:* $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Hint:

The characteristic polynomial of A is $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -(\lambda^3 - 3\lambda^2 + 4\lambda - 2)$.

Here, $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $A^2 = A\times A = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ and $A^3 = A^2\times A = \begin{pmatrix} -2 & 2 & -2 \\ 0 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix}$.

Replacing λ by A and 1 by I, we get $A^3 - 3A^2 + 4A - 2I = O$.

Thus, $A^3 - 3A^2 + 4A - 2I = O$. Hence, Cayley – Hamilton theorem is verified.

Example 2: *Using Cayley–Hamilton Theorem, find the inverse of the matrix* $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Hint:

You find $|A| = 1 \neq 0$. So, A is non singular and hence A^{-1} exists.

The characteristic equation of A is $|A - \lambda I| = 0$ gives, $\lambda^3 - 4\lambda^2 + 4\lambda - 1 = 0$.

By Cayley–Hamilton Theorem, we get

$A^3 - 4A^2 + 4A - I = O$

or, $A(A^2 - 4A + 4I) = I = (A^2 - 4A + 4I)A$.

Hence, $A^{-1} = A^2 - 4A + 4I = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}^2 - 4\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$.

Exercise

Verify the Cayley–Hamilton Theorem for the following matrices. Also find the inverse of each :

a) $C = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ **Ans:** $A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$

c) $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ **Ans:** $B^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

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