Code No.: **MDS 504**

Course Title:**Mathematics for Data Science**

Unit 2: Introduction to Matrices and Vectors **[15Hrs.]**

Vectors, Basic properties of R^n; Norms and balls; Vector addition and scalar multiplication, Vector spaces and subspaces; Matrices, Operations on matrices, including matrix multiplication; Functions, linear functions, and linear transformations; Matrices as transformations, Dot products, angles, and perpendicularity; Linear combinations, span, and linear independence; Bases, orthonormal bases, and projections, Applications in the theory of probability and data science

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Exercises on Vectors

1. For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in R^n and a real number c, *then* prove that the norm satisfy the properties:

   a. $\|x\| \geq 0$ **and** $\|x\| = 0$ **iff** $x = 0$. **b.** $\| cx \| = | c | \, \| x \|$

   c. $\| x - y \| = \| y - x \|$ **(*Symmetry*)**

2. Let a) $\boldsymbol{X} = (1, 2)$, $\boldsymbol{Y} = (2, 1)$ b) $X = (1, 2, 1)$, $Y = (2, -1, 2)$

   Compute the norms $\| \boldsymbol{X} \|$ and $\| \boldsymbol{Y} \|$ and verify the norm properties

   i. $\| \boldsymbol{X} \| = \| -\boldsymbol{X} \|$ ii. $\| \boldsymbol{X} + \boldsymbol{Y} \| \leq \| \boldsymbol{X} \| + \| \boldsymbol{Y} \|$

   iii. $| \boldsymbol{X}.\boldsymbol{Y} | \leq \| \boldsymbol{X} \| \, \| \boldsymbol{Y} \|$ iv. $\| \boldsymbol{X} + \boldsymbol{Y} \|^2 + \| \boldsymbol{X} - \boldsymbol{Y} \|^2 = 2\,(\| \boldsymbol{X} \|^2 + \| \boldsymbol{Y} \|^2)$.

3. Prove the formula $x.y = \| x \| \, \| y \| \cos \theta$, where θ is the angle between x and y. Give the Geometrical Interpretation of dot product $x.y = \| x \| \, \| y \| \cos \theta$.

4. Find the distance and angle between each pair of vectors (or points):

   a) $a = (1, 2)$, $b = (2, 1)$ b) $x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

   Ans: a) $\sqrt{2}$, $\cos^{-1}\frac{4}{5}$ b) $\sqrt{11}$, $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$

5. Verify Schwarz inequality and triangle inequality for:

   a) $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ b) $\boldsymbol{v}_1 = (1, -1, 2)$, $\boldsymbol{v}_2 = (3, -3, 0)$

6. a) Find scalar and vector projection of $\boldsymbol{Y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto $\boldsymbol{X} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$.

   Ans: $\frac{11}{\sqrt{26}}$ and $\frac{(44, -11, 33)}{26}$.

   b) Let $X = (1, 0)$ and $Y = (1, 2)$, be two vectors in R^2. Construct a vector orthogonal to X.

   Hint: By Gram-Schmidt Orthogonalization Process, the non-zero vector

   $$R = Y - \left(Y.\frac{X}{\| X \|}\right)\frac{X}{\| X \|} = (0, 2) \neq O \text{ is orthogonal to } X.$$

7. Given two vectors a) $X = (1, 0, 0)$ and $Y = (0, 1, 0)$ in R^3, b) $X = (1, 2, 3)$ and

$Y = (-1, -2, -3)$. Determine the scalar and vector projections of Y onto X.
Find a vector orthogonal to X and normalize it.

Hint:

a) S.P. of Y onto $X =$ 0 and V.P. of Y onto $X =$ (0,0,0).

By orthogonal property, $R = Y - \left(Y.\frac{X}{\|X\|}\right)\frac{X}{\|X\|}$ = (0, 1, 0) ≠ O is orthogonal to X.

To normalize R, the unit vector in the direction of $R = \frac{R}{\|R\|} =$ (0, 1, 0).

Exercises on Vector Space and Linear Independence

1. The set of real numbers R , under the usual rules of addition and multiplication form a real vector space.

2. ***The*** set of all n–dimensional Euclidean vectors $\boldsymbol{R^n}$ or ordered ***n–tuple*** $(x_1, x_2, \ldots, x_n)$ of real numbers, under addition and multiplication by scalars defined by
$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $c(x_1, x_2, \ldots, x_n) = (c\,x_1, c\,x_2, \ldots, c\,x_n)$, c being a real number, is a real vector space .

3. Show that the set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in 3–space with $x = 0$ (i.e., first coordinate zero) is a vector space.

4. The set of all vectors of the form $c(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$, where c is a scalar, is a vector space. Prove.

Hint: The set of all vectors of the form $c(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$ may be represented by

$\boldsymbol{S} = \{c(1, 2, 3) = (c, 2c, 3c), c \in \boldsymbol{R}\}$. Obviously, $\boldsymbol{S} \subset \boldsymbol{R}^3$. Proceed to prove A1-A5 and M1 –M5.

5. Let S be the set of all ordered pairs (x, y) of real numbers, with addition defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and multiplication by scalar defined by $c(x, y) = (cx, y)$, c being a real number. Show that S is not a vector space.

Hint:

Here $\boldsymbol{S} = \{(cx, y): x, y \in \boldsymbol{R}\}$, $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$, $c(x, y) = (cx, y)$.

We give an example. Let $\boldsymbol{v} = (1, 2)$, $c = 1$, $d = 2$, then

$(c + d)\,\boldsymbol{v} = 3(1, 2) = (3, 2)$ and $c\boldsymbol{v} + d\boldsymbol{v} = 1(1, 2) + 2(1, 2) = (1, 2) + (2, 2) = (3, 4)$.

This violates one of the vector space axioms $(c + d)\,\boldsymbol{v} \neq c\boldsymbol{v} + d\boldsymbol{v}$.

Hence, $\boldsymbol{S}$ is not a vector space.

6. Let $\boldsymbol{V}$ be the set of all ordered pairs of real numbers, with addition defined by $(x, y) + (x', y') = (x + x', 0)$ and multiplication by scalar defined by $c(x, y) = (cx, 0)$. Is $\boldsymbol{V}$, with these operations, a vector space?

Hint: V is not a vector space, since $c\boldsymbol{v} = c(x, y) = (cx, 0)$ does not imply $1\boldsymbol{v} = \boldsymbol{v}$.
Actually, $1(x, y) = (x, 0) \neq (x, y)$.

Vector Sub-Space:

7. For any $n \times n$ matrix A, the set W of all solutions of homogeneous equation $AX = 0$ forms a subspace of R^n.

Hint:
Let X and Y be arbitrary vectors in W and c be any scalar.
Then , $AX = 0$ and $AY = 0$. So $A(X + Y) = AX + AY = 0 + 0 = 0$; and $A(cX) = cAX = c0 = 0$.

8. Let W_1 and W_2 be two subspaces of a vector space V over a field F. Then the intersection $W_1 \cap W_2$ is also a subspace of V.

Hint:
Let W_1 and W_2 be two subspaces of a vector space V over a field F.
Suppose $W = W_1 \cap W_2$ and $w_1, w_2 \in W$. Obviously, $W \subset V$ and for any scalars $c_1, c_2 \in F$,
Then, $c_1w_1 + c_2w_2 \in W_1 \cap W_2$ (How?)

9. Let $V= M$ be the vector space of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.Then,

(i) the subset $\boldsymbol{W}$ of all symmetric 2×2 matrices of the form $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ forms a subspace of the vector space V.,

(ii) the subset of all skew–symmetric 2×2 matrices S of the form $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ forms a subspace of the vector space V.

10. Show that the set of all vectors (x, y) in $\boldsymbol{R}^2$ such that $x + 2y = 0$ is a subspace of $\boldsymbol{R}^2$.
Hint: Solved in Class

11. Show that a subset $\boldsymbol{W}$ of $\boldsymbol{V} = \boldsymbol{R}^3 = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \boldsymbol{R}\}$ is subspace if

a) $\boldsymbol{W} = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y = 0\}$ b) $\boldsymbol{W} = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + 2z = 0\}$.

Hint: Solved in Class

Linearly Dependent and Independent

12. Show that the following sets of vectors are linearly independent:
a) $\{(1, 2), (1, 3)\}$ **b)** $\{(1, 1, 1), (1, -1, 0), (0, 1, 1)\}$
13. Show that the following sets of vectors in $\boldsymbol{R}^3$ are linearly dependent:
a) $\{(1, 0, 1), (1, 1, 0), (-1, 0, -1)\}$ **b)**$\{(1, 2, 3), (2, 4, 6), (3, 6, 9)\}$.
14. In the vector space $\boldsymbol{R}^2$, express the given vector $\boldsymbol{v}$ as a linear combination of the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Also find the coordinates of $\boldsymbol{v}$ with respect to $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.
a) $\boldsymbol{v} = (1, 2)$, $\boldsymbol{v}_1 = (1, -1)$, $\boldsymbol{v}_2 = (0, 1)$ **Ans:** $\boldsymbol{v}_1 + 3\boldsymbol{v}_2$; $(1, 3)$

b) $\boldsymbol{v} = (-5, 4)$, $\boldsymbol{v}_1 = (1, 2)$, $\boldsymbol{v}_2 = (3, 1)$. **Ans:** $\frac{17}{5}\boldsymbol{v}_1 - \frac{14}{5}\boldsymbol{v}_2; \left(\frac{17}{5}, \frac{-14}{5}\right)$

15. In the vector space $\boldsymbol{R}^3$, express the vector (1, 2, 3) as a linear combination of the vectors (1, 0, 1), (0, 1, 1) and (1, −1, 1).

Ans: (1, 2, 3) = (1, 0, 1) + 2(0, 1, 1) + 0(1, –1, 1)

16. Show that the following sets span $\boldsymbol{R}^2$:
a) (1, 1), (0, 1) b) (0, 1), (1, 0), (1, 1) .
17. Show that the set {(1, 1), (−1, 0)} form a basis of $\boldsymbol{R}^2$.
18. Show that the vectors {(1, 1, 1), (1, −1, 1), (2, 0, 3)} forms a basis of $\boldsymbol{R}^3$. Find the coordinates of (1, 3, 2) with respect to this basis. **Ans:** (1, –2, 1)
19. Find the coordinates of the vector (x, y, z) with respect to the basis {(1, 0, 1), (0, 0, 1), (0, 1, −1)}. **Ans:** $(x, z - x + y, y)$
20. Prove that the space of all 3 × 2 matrices over the field of real numbers $\boldsymbol{R}$, has dimension 3 × 2.

Theorem : ***A set of non–zero orthogonal vectors is linearly independent.***
Proof: Proved in class ■

Remarks: The converse of above theorem may not be true. In other words, a set of non–zero linearly independent vectors may not be orthogonal. For examples, the sets A= {(1, 2), (1, 3)};B= {(1,1,1), (1, −1, 0), (0, 1, 1)} and C= {(1, 1, 2), (3,1, 2), (0,1,4)} are linearly independent although they are not orthogonal.(Verify)

Orthogonal and Orthonormal Basis

Application of Theorem **: (Orthogonal Basis):**

Every finite dimensional non–zero vector space has an orthogonal basis.

Proof: Its Applications Solved in class

Note 1: A set of orthonormal vectors is linearly independent.

2: Every finite dimensional non–zero space has an orthonormal basis.

Remark: To obtain an orthonormal set from a given set of orthogonal vectors, we should divide each vector by its norm. Thus, if { $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$} be a **orthogonal basis** vectors of $\boldsymbol{V}$, then the **orthonormal basis** vectors of $\boldsymbol{V}$ are { $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$}, where $\boldsymbol{w}_1 = \frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|}$, $\boldsymbol{w}_2 = \frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|}$, ... , and $\boldsymbol{w}_n = \frac{\boldsymbol{u}_n}{\|\boldsymbol{u}_n\|}$.

Example 1:

The set of vectors $\boldsymbol{v}_1 = (1, 1)$ and $\boldsymbol{v}_2 = (-1, 1)$ forms an orthogonal set.

Example 2: Show that the standard basis $\boldsymbol{e_1}$ **= (1, 0, 0),** $\boldsymbol{e_2}$ **= (0, 1, 0),** $\boldsymbol{e_3}$ **= (0, 0, 1)** forms an orthonormal basis for $\boldsymbol{R}^3$**.**

Solution: Solved in class

Example 3: ***Let v_1 = (2, 3) and v_2 = (-1, 2) be a basis for R^2. Using scalar product, construct an orthogonal basis*** **and hence** ***an orthonormal basis for R^2 from the basis {v_1,v_2 }.***

Hint: Use Gram Schimdt Orthogonalization process to find orthogonal set of vectors = {$\boldsymbol{u}_1$,$\boldsymbol{u}_2$ },

where $\boldsymbol{u}_1$ = (2, 3) and $\boldsymbol{u}_2$ = $\left(\frac{-21}{13}, \frac{14}{13}\right)$. Solved in class

Example 4: ***Let v_1 = (1, 0, 0), v_2 = (1, 1, 0) and v_3 = (1, 1, 1) be a basis for R^3.Using Gram Schimdt orthogonalization process, construct an orthogonal basis*** **and** ***an orthonormal basis for R^3 from the basis {v_1,v_2,v_3}.***

Hint: Solved in class.

Use Gram Schimdt Orthogonalization process the vectors $\boldsymbol{u}_1 = (1, 0, 0)$, $\boldsymbol{u}_2 = (0, 1, 0)$ and $\boldsymbol{u}_3 = (0, 0, 1)$ are non-zero and mutually perpendicular. The required orthonormal basis for $\boldsymbol{R}^3$ is

$$\boldsymbol{w}_1 = \frac{\boldsymbol{u}_1}{||\boldsymbol{u}_1||} = (1, 0, 0); \quad \boldsymbol{w}_2 = \frac{\boldsymbol{u}_2}{||\boldsymbol{u}_2||} = (0, 1, 0); \text{ and } \quad \boldsymbol{w}_3 = \frac{\boldsymbol{u}_3}{||\boldsymbol{u}_3||} = (0, 0, 1).$$

Hence the required orthonormal basis for $\boldsymbol{R}^3$ is $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Exercises

2. Prove that the vector $\boldsymbol{v}_3 = (0, 1, 0, 0)$ is orthogonal to each of the vectors $\boldsymbol{v}_1 = (1, 0, 1, 0)$ and $\boldsymbol{v}_2 = (1, 0, 0, 1)$.

4. Show that the set of vectors $\boldsymbol{V} = \{(1, 0, 1), (0, 1, 0), (-1, 0, 1)\}$ forms an orthogonal basis of $\boldsymbol{R}^3$ and hence obtain the corresponding orthonormal basis of $\boldsymbol{R}^3$.

5. Find an orthonormal basis from the basis $\{(1, 0, 1), (1, 1, 0), (1, 1, 1)\}$ of $\boldsymbol{R}^3$ using Gram Schimdt Orthogonalization Process. **Ans:** $\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$.

Matrices

Theorem A: If A, B and C are matrices of order $m \times n$, $n \times p$, $p \times q$ respectively, then

$(AB)C = A(BC)$ (*Associative law*)

Complete Proof:

We shall prove the associative law only leaving the other as exercise for the students.

To prove $(AB)C = A(BC)$, let us take $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be three matrices of order $m \times n$, $n \times p$, $p \times q$ respectively.

Then $AB = D = (d_{ij})$ is an $m \times p$ matrix, where $d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ satisfying $1 \le i \le m$ and $1 \le j \le p$

and $BC = E = (e_{ij})$ is an $n \times q$ matrix, where $e_{ij} = \sum_{k=1}^{p} b_{ir} c_{rj}$ satisfying $1 \le i \le n$ and $1 \le j \le q$.

Then clearly, $(AB)C$ and $A(BC)$ are both $m \times q$ matrices.

Thus, order of $(AB)C$ = order of $A(BC)$.

Now, $(i, j)^{\text{th}}$ element of $(AB)C = (i, j)^{\text{th}}$ element of DC

$$= \sum_{s=1}^{p} d_{is} c_{sj} = \sum_{s=1}^{p} \left(\sum_{k=1}^{n} a_{ik} b_{ks}\right) c_{sj}$$

$$= \sum_{k=1}^{n} \left(a_{ik} \sum_{s=1}^{p} b_{ks} c_{sj}\right) = \sum_{k=1}^{n} a_{ik} e_{kj}$$

= the $(i, j)^{\text{th}}$ element of AE

= the $(i, j)^{\text{th}}$ element of $A(BC)$

$\therefore A(BC) = (AB)C$.

■

Theorem B: ***If A, B and C are matrices of appropriate size suitable for addition and multiplication, then***

$A(B + C) = AB + AC$ (*Left distributive law*)

Proof:

Let $A = (a_{ij})$, $B = (b_{ij})$ and $C= (c_{ij})$ be three matrices of order $m \times n, n \times p$ and $n \times p$ respectively. Then $A(B + C)$ is an $m \times p$ matrix. Also, since both AB and AC are $m \times p$ matrices, $AB + AC$ is also an $m \times p$ matrix.

Thus, order of $A(B + C)$ = order of $(AB + AC)$

Again, we have $B + C = (b_{ij} + c_{ij})$.

Now, $(i, j)^{\text{th}}$ element of $A(B + C) = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$

$$= \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{jk} c_{kj}$$

$= (i, j)^{\text{th}}$ element of AB + $(i, j)^{\text{th}}$ element of AC

$= (i, j)^{\text{th}}$ element of $AB + AC$.

$\therefore A(B + C) = AB + AC$. ■

Exercises

1. Prove the right distributive law: $(B + C)A = BA + CA$.

2. If A and B are matrices of appropriate size suitable for multiplication, then prove that $AB \neq BA$, in general .

Hint: When A and B are square matrices of the same order, both AB and BA exist, but they are not necessarily equal.

For example, let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Then $AB \neq BA$ (Verify).

3. If A and B are matrices of appropriate size suitable for multiplication, then $AB = O$ may or may not imply

***A = O* or *B = O*. (*Non–cancellation law*)**

Hint: The cancellation law is not necessarily true.

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$. (Verify).

Thus AB is a zero matrix though neither A nor B is a zero matrix.

4. If $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, then prove that $A_\theta A_\phi = A_{\theta+\phi}$.

Hint: $A_\theta A_\phi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = A_{\theta+\phi}$.

5. A square matrix A is said to be **idempotent** if $A^2 = A$. Show that the following matrices are idempotent:

a) $\begin{pmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$.

6. A square matrix A is called a **nilpotent** matrix of order r, if $A^r = 0$ but $A^{r-1} \neq 0$. Show that the following matrices are ***nilopotent*** of index 3:

a) $M = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$ b) $N = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}$

7. A square matrix A is called a **involutory** $A^2 = I$. Show that the following matrices are involutory:

a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ b) $\begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$.

8. For any positive integer n, show that $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

9. If $a_1, a_2, \ldots, a_n$ are numbers, and $A_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}$, show that $A_1A_2\ldots A_n = \begin{pmatrix} 1 & \sum_{i=1}^{n} a_i \\ 0 & 1 \end{pmatrix}$.

10. If $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \end{pmatrix}$, show that A^TA and AA^T are symmetric.

11. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 4 \end{pmatrix}$, verify that $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is skew–symmetric and

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

Simple Properties of Transpose:

Theorem A: If A be any matrx and k is a scalar, then ***a)*** $(A^T)^T = A$ ***b)*** $(kA)^T = kA^T$.

Proof:

We prove b). Let $A = (a_{ij})_{m\times n}$ be a matrix of order $m\times n$, then we have $kA = (ka_{ij})_{m\times n}$

Now, $(kA)^T = (ka_{ij})^T{}_{m\times n} = (ka_{ji})_{n\times m} = k\,(a_{ji})_{n\times m} = kA^T$. $\therefore\ (kA)^T = kA^T$.

Hence, the proof is complete. Part a) is obvious.

Theorem B: If A and B are two matrices , then $(A + B)^T = A^T + B^T$, where A and B are of the same order.

Proof:

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. Then $A + B$ will also be an $m \times n$ matrix so that $(A + B)^T$ will be an $n \times m$ matrix.Again, A^T and B^T are both $n\times m$ matrices. Hence $A^T + B^T$ exists and will also be a matrix of order $n \times m$.

Now, the $(i, j)^{th}$ element of $(A + B)^T$ = the $(j, i)^{th}$ element of $(A + B)$

$= a_{ji} + b_{ji}$

= the $(i, j)^{th}$ element of A^T + the $(i, j)^{th}$ element of B^T

= the $(i, j)^{th}$ element of $(A^T + B^T)$.

$\therefore\ (A + B)^T = A^T + B^T$. ■

Theorem C : ***If A and B are two matrices, then*** $(AB)^T = B^TA^T$, ***where A is of size*** $m \times n$ ***and B is of size*** $n \times p$.

Proof:

Let $A = (a_{ij})$ be a matrix of order $m \times n$ and $B = (b_{ij})$ be a matrix of order $n\times p$. Then AB is defined and is of order $m \times p$ matrix and hence $(AB)^T$ will be a $p\times m$ matrix.

Again, $A^T = (c_{ij})$ will be an $n \times m$ matrix where $c_{ij} = a_{ji}$ and $B^T = (d_{ij})$ will be $p \times n$ matrix where $d_{ij} = b_{ji}$ and hence $B^T A^T$ exists and will be of order $p \times m$. Thus, $(AB)^T$ and $B^T A^T$ are of the same order. Further,

the $(i, j)^{th}$ element of $(AB)^T$ = the $(j, i)^{th}$ element of AB

$= a_{j1}b_{1i} + a_{j2}b_{2i} + \ldots + a_{jn}b_{ni}$

$= \sum_{k=1}^{n} a_{jk}\, b_{ki} = \sum_{k=1}^{n} c_{kj}\, d_{ik} = \sum_{k=1}^{n} d_{ik}\, c_{kj}$

$= \sum_{k=1}^{n}$ [$(i,k)^{th}$ element of B^T] × [$(k,j)^{th}$ element of A^T]

= the $(i, j)^{th}$ element of B^TA^T.

$\therefore\ (AB)^T = B^T A^T$. ■

Definition: A real square matrix A is said to be **orthogonal** if $AA^T = I = A^TA$.

Example: **Show that the** matrices $X = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $Y = \frac{1}{3}\begin{pmatrix} -1 & 2 & 3 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ **and** $A = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}$ **are orthogonal,** since $XX^T = I = Y^TY = A^TA$.(Verify)

Theorem: D ***If A is orthogonal matrix, then A^T is also orthogonal.***

Proof:

By definition, if A is an orthogonal matrix, then $AA^T = A^TA = I.$

By transposing, we get $(AA^T)^T = (A^TA)^T = I$

or, $(A^T)^TA^T = A^T.(A^T)^T = I.$ Hence A^T is orthogonal.

■

Adjoint of a Square Matrix:

Example: Find the adjoint of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$, Ans: $Adj.\ A = \begin{pmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{pmatrix}$.

And verify that $A \times (Adj.A) = (Adj.A) \times A = |A|\, I_3$, *where I_3 is the 3×3 unit matrix.*

Inverse of a Square Matrix: $A^{-1} = \frac{1}{|A|}.(Adj.A)$.

Example: Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Solution: The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is non–singular, since $|A| = -2 \neq 0$.

Also, $B = A^{-1} = \frac{1}{|A|}.(Adj.A) = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, since

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = BA.$$

■

Formula: Below, we have shown how the inverses of 2 × 2 and 3 × 3 matrices look like:

1. For 2 × 2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is $A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

2. For 3 × 3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the inverse is

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

There are various properties of adjoint and inverse of a square matrix.

Theorem A : The inverse of a matrix, if it exists, is unique.

Proof: Proved in class

Theorem B: If two matrices A and B of the same order are invertible (or non–singular), then their product AB is invertible; and the inverse of the product is equal to the product of the inverses taken in the reverse order, i.e., $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: Proved in class ■

Theorem C: The inverse of the transpose of a non–singular matrix A is the transpose of its inverse, that is

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof:

Let A be a non–singular matrix, i.e., $|A| \neq 0$. So A is invertible. Since $|A| = |A^T|$, $|A^T| \neq 0$; that is, $|A^T|$ is non–singular. Since $AA^{-1} = A^{-1}A = I$, taking their transpose, we have

$$(AA^{-1})^T = (A^{-1}A)^T = I.$$

or, $(A^{-1})^T A^T = A^T(A^{-1})^T = I.$ $[\because (AB)^T = B^T A^T]$

Hence, $(A^{-1})^T$ is the inverse of A^T . That is, $(A^T)^{-1} = (A^{-1})^T$. ■

Workedout Examples

Example 1: ***Show that the matrix*** $A = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ ***is non–singular. Find adjoint and inverse A .***

Solution:

Here, $|A| = 11 \neq 0$. So, A is non–singular, and has the inverse A^{-1}. The cofactors of the elements of A are $A_{11} = 3$, $A_{12} = 2$, $A_{13} = -6$, $A_{21} = -4$, $A_{22} = 1$, $A_{23} = 8$, $A_{31} = 4$, $A_{32} = -1$, $A_{33} = 3$.

Now, $Adj.A$ = Transpose of $\begin{pmatrix} 3 & 2 & -6 \\ -4 & 1 & 8 \\ 4 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 4 \\ 2 & 1 & -1 \\ -6 & 8 & 3 \end{pmatrix}$.

$$\therefore A^{-1} = \frac{1}{|A|}.(Adj.A) = \frac{1}{11}\begin{pmatrix} 3 & -4 & 4 \\ 2 & 1 & -1 \\ -6 & 8 & 3 \end{pmatrix}.$$

Alternatively,

A^{-1} can be found as, $A^{-1} = \dfrac{1}{|A|}.(Adj.A) = \dfrac{1}{11}\begin{pmatrix} \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} \end{pmatrix} = \dfrac{1}{11}\begin{pmatrix} 3 & -4 & 4 \\ 2 & 1 & -1 \\ -6 & 8 & 3 \end{pmatrix}$.

Example 3: Prove that if A is non–singular, then the relation $AB=AC$ implies $B=C$.

Solution:

If A is non–singular, then A^{-1} exists and $AB = AC$. Pre multiplying by A^{-1} gives

$A^{-1}(AB) = A^{-1}(AC)$

or, $(A^{-1}A)B = (A^{-1}A)C$,

i.e. $IB = IC$. Therefore $B = C$.

Exercises

1. Verify the property $A.(Adj.A) = (Adj.A).A = |A|I$ and find the inverse for each of the following non–singular matrices:

a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ b) $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ c) $C = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Ans: a) $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ **b)** $B^{-1} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ **c)** $C^{-1} = \dfrac{1}{\cos 2\theta}\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

2. Which of the following matrices have reciprocal? If they exist, find their reciprocals:

a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$ b) $B = \begin{pmatrix} 5 & -2 & 4 \\ -2 & 1 & 1 \\ 4 & 1 & 0 \end{pmatrix}$.

Ans: a) $A^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ -5 & 2 & 1 \\ 7/2 & -3/2 & -1/2 \end{pmatrix}$ **b)** $B^{-1} = -\frac{1}{37}\begin{pmatrix} -1 & 4 & -6 \\ 4 & -16 & -13 \\ -6 & -13 & 1 \end{pmatrix}$.

3. a) Find the reciprocal of the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ and check your result by verifying $AA^{-1} = I$.

b) Show that the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$ is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$.

c) Show that the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is its own inverse.

4. Verify the property $(A^T)^{-1} = (A^{-1})^T$ for the matrix: $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

5. Verify the property $(AB)^{-1} = B^{-1}A^{-1}$ for the following matrices:

a) $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Linear Transformations

Example 1: Let V be any vector space. Then the identity transformation $I : V \to V$ defined by $I(v) = v$, for any $v \in V$, is a linear transformation from V to V.

Hint:

Let $\mathbf{v_1}, \mathbf{v_2} \in \mathbf{V}$ and $k \in \mathbf{R}$, we have

i) $I(\mathbf{v_1} + \mathbf{v_2}) = \mathbf{v_1} + \mathbf{v_2} = I(\mathbf{v_1}) + I(\mathbf{v_2})$

and ii) $I(k\mathbf{v}) = k\mathbf{v} = kI(\mathbf{v})$, for any $\mathbf{v} \in \mathbf{V}$.

Hence, I is the linear.

Example 2: The transformation $T : R^2 \to R^2$ defined by $T(v) = T(x, y) = (x + y, y)$ is linear.

Hint: Solved in class

Example 3: ***The projection mapping into the x–axis*** $\pi_1 : \mathbf{R}^2 \to \mathbf{R}$ ***defined by*** $\pi_1(\mathbf{v}) = \pi_1(x, y) = (x, 0)$ ***is linear.***

Hint:

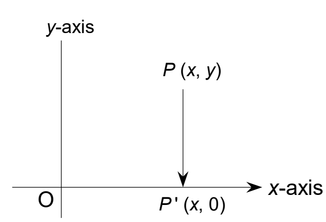

Fig. 1

Suppose $\mathbf{v_1} = (x_1, y_1)$ and $\mathbf{v_2} = (x_2, y_2)$.Then,

$\pi_1(\mathbf{v_1} + \mathbf{v_2}) = \pi_1(x_1 + x_2, y_1 + y_2) = (x_1 + x_2, 0) = (x_1, 0) + (x_2, 0) = \pi_1(\mathbf{v_1}) + \pi_1(\mathbf{v_2})$

Also, for any scalar $k \in R$, and $\mathbf{v} = (x, y) \in \mathbf{R}^2$, $\pi_1(k\mathbf{v}) = \pi_1(kx, ky) = (kx, 0) = k(x, 0) = k\pi_1(\mathbf{v})$.

Hence, π_1 is linear.

Example 4: **The *translation mapping T* : *R* → *R given by T(x) = x +1* is *not linear*.**

Hint: This is obvious, since $T(0) = 1 \neq 0$.

Exercises

1. Are the following mappings linear?
   i) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x + y, x - y)$ **Ans:** Linear
   ii) $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x, y) = (x, y, xy)$ **Ans:** Not Linear
   iii) $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (x + y + z, 0)$ **Ans:** Linear
   iv) $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x + 1, y)$ **Ans:** Not Linear
   v) $T : \mathbf{R} \to \mathbf{R}$ defined by $T(x) = x + 2$. **Ans:** Not Linear

Criteria for determining whether a given transformation is linear or not.

Theorem: ***Let V and W be two vector spaces over the same field F. A transformation $T : V \to W$ is linear then $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$, for any $c_1, c_2 \in F$ and $v_1, v_2 \in V$.***

Proof:

Suppose $T : \boldsymbol{V} \to \boldsymbol{W}$ is a linear transformation and $\boldsymbol{v_1}, \boldsymbol{v_2} \in \boldsymbol{V}$ and $c_1, c_2 \in F$. Then $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} \in \boldsymbol{V}$.

Hence, $T(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}) = T(c_1\boldsymbol{v_1}) + T(c_2\boldsymbol{v_2}) = c_1T(\boldsymbol{v_1}) + c_2T(\boldsymbol{v_2})$.■

Remarks:

For any scalars $c_i \in F$ and any vectors $\boldsymbol{v}_i \in \boldsymbol{V}$, we can obtain the following basic property of linear mappings: $T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_m\boldsymbol{v}_m) = c_1 T(\boldsymbol{v}_1) + c_2 T(\boldsymbol{v}_2) + \ldots + c_m T(\boldsymbol{v}_m)$.

Example 1: ***Find $T(a, b)$ where $T : R^2 \to R^3$ is linear and is defined by $T(1, 2) = (3, -1, 5)$ and $T(0, 1) = (2, 1, -1)$.***

Hint:

Let $(a, b) = c_1 (1, 2) + c_2 (0, 1)$. Then $(a, b) = (c_1, 2c_1 + c_2)$ which gives $c_1 = a, c_2 = b - 2a$

Also, $T(a, b) = c_1T(1, 2) + c_2T(0, 1)$

$= a(3, -1, 5) + (b - 2a)(2, 1, -1)$

$= (2b - a, b - 3a, 7a - b)$.

Example 2: **Let $T : R^2 \to R$ be the linear transformation for which $T(1, 2) = 3$ and $T(0, 3) = 2$. Find $T(5, 3)$.**

Hint:

here (5, 3) can be written as a linear combination of (1, 2) and (0, 3) in the form

$$(5, 3) = x(1, 2) + y(0, 3). \qquad \ldots\ldots..(1)$$

Then $(5, 3) = (x, 2x + 3y)$ which gives $x = 5$ and $2x + 3y = 3$ i.e., $x = 5$ and $y = -\frac{7}{3}$

Since T is linear. So, from (1), $T(5, 3) = x\,T(1, 2) + y\,T(0, 3) = 5\times 3 - \frac{7}{3} \times 2 = 15 - \frac{14}{3} = \frac{31}{3}$.

Exercises

1. Let $T : \mathbf{R}^2 \to \mathbf{R}$ be the linear transformation for which $T(1, 1) = 3$ and $T(0, 1) = -2$. Prove that
   a) $T(1, 0) = 5$ b) $T(1, ½) = 4$ c) $T(2, 5) = 0$.
2. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $T(0, 1) = (2, 1)$, $T(1, 4) = (0, -2)$. If T is linear, prove that $T(a, b) = (2b - 8a, b - 6a)$.
3. Let $T : \mathbf{R}^3 \to \mathbf{R}$ be the linear transformation for which $T(1, 1, 0) = 1$, $T(1, 1, 1) = 2$, $T(0, 1, 0) = -1$. Prove that
   a) $T(a, b, c) = 2a - b + c$ b) $T(2, 1, 0) = 5$.
4. Show that $T(a, b) = (2b - a, b - 3a, 7a - b)$, where $T : \mathbf{R}^2 \to \mathbf{R}^3$ is linear and is defined by $T(1, 2) = (3, -1, 5)$ and $T(0, 1) = (2, 1, -1)$.

Kernel and Image of Linear Transformation

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be two vector spaces over the same field F and let $T : \boldsymbol{V} \to \boldsymbol{W}$ be a linear transformation. Then,

a) The **kernel** of T is the set of elements $\boldsymbol{v}$ in $\boldsymbol{V}$ such that $T(\boldsymbol{v}) = \mathbf{0}$. It is denoted by $Ker\,T = \{\boldsymbol{v} \in \boldsymbol{V} : T(\boldsymbol{v}) = \mathbf{0}\}$.

b) The **image** of T, is the set of image points in $\boldsymbol{W}$. It is denoted by $Im\,T = \{\boldsymbol{w} \in \boldsymbol{W} : T(\boldsymbol{v}) = \boldsymbol{w}$ for some $\boldsymbol{v} \in \boldsymbol{V}\}$.

Theorem (A): ***Let V and W be two vector spaces over the field F and T: V→W be a linear transformation. Then i) the kernel of T is a subspace of V ii) the image of T is a subspace of W.***

Proof.

(i) Suppose $\boldsymbol{v_1}, \boldsymbol{v_2} \in Ker\ T$ of the linear transformation $T: \boldsymbol{V} \to \boldsymbol{W}$. Then, by definition, $T(\boldsymbol{v_1}) = \mathbf{0}$ and $T(\boldsymbol{v_2}) = \mathbf{0}$.

Given any two scalars $c_1, c_2 \in F$, then $T(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}) = c_1T(\boldsymbol{v_1}) + c_2T(\boldsymbol{v_2}) = c_1(\mathbf{0}) + c_2(\mathbf{0}) = \mathbf{0}$.

This implies $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} \in Ker\ T$, and hence $Ker\ T$ is a subspace of $\boldsymbol{V}$.

(ii) Suppose that $\boldsymbol{w_1}, \boldsymbol{w_2} \in Im\ T$. Then, by definition, there exist $\boldsymbol{v_1}, \boldsymbol{v_2} \in \boldsymbol{V}$ such that

$T(\boldsymbol{v_1}) = \boldsymbol{w_1}$ and $T(\boldsymbol{v_2}) = \boldsymbol{w_2}$. We also have, for any scalars $c_1, c_2 \in F$,

$$T(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}) = c_1T(\boldsymbol{v_1}) + c_2T(\boldsymbol{v_2}) = c_1\boldsymbol{w_1} + c_2\boldsymbol{w_2} \in Im\ T.$$

Hence the image of T is a subspace of $\boldsymbol{W}$.

Theorem (B): ***Let V and W be two vector spaces over the field F. and T: V→W be a linear transformation. Then if $v_1, v_2, ..., v_n$ are linearly independent vectors in V and $KerT = \{0_V\}$.Then $T(v_1), T(v_2), ..., T(v_n)$ are also linearly independent vectors in W.***

Proof.

Suppose that $\text{Ker}T = \{\mathbf{0}_V\}$ and $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are linearly independent in $\boldsymbol{V}$. We show that $T(\boldsymbol{v}_1), ..., T(\boldsymbol{v}_n)$ are linearly independent in $\boldsymbol{W}$. Let $c_1, c_2, ..., c_n \in F$ such that

$c_1T(\boldsymbol{v}_1) + ... + c_nT(\boldsymbol{v}_n) = \mathbf{0}_W$.

Then by linearity of T, we have

$T(c_1\boldsymbol{v}_1) + ... + T(c_n\boldsymbol{v}_n) = \mathbf{0}_W$

$\Rightarrow T(c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n) = \mathbf{0}_W$

$\Rightarrow c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n \in \text{Ker}\ T = \{\mathbf{0}_V\}$

$\Rightarrow c_1\boldsymbol{v}_1 + ... + c_n\boldsymbol{v}_n = \mathbf{0}_V$

$\Rightarrow c_1 = ... = c_n = 0.$

This shows that $T(\boldsymbol{v}_1), ..., T(\boldsymbol{v}_n)$ are linearly independent vectors in $\boldsymbol{W}$.

■

Algebra of Linear Transformations

Linear transformations can be combined in various ways. Some of the operations are addition, multiplication by a scalar, multiplication and inversion.

a) Addition and scalar multiplication

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be vector spaces over the same field F. Let T_1 and T_2 be linear transformations from $\boldsymbol{V}$ into $\boldsymbol{W}$. For any $\boldsymbol{v} \in \boldsymbol{V}$ and scalar $k \in F$,

i) The **sum**, $T_1 + T_2$ is defined by $(T_1 + T_2)(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v})$

and ii) The **product,** kT is defined by $(kT)(\boldsymbol{v}) = kT(\boldsymbol{v})$.

Example 1:

Let $T: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be defined by $T(\boldsymbol{v}) = T(x, y) = (x + y, y)$ and $I: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the identity mapping $I(x, y) = (x, y)$. Then, $T + I$ is given by

$(T + I)(x, y) = T(x, y) + (x, y) = (x + y, y) + (x, y) = (2x + y, 2y)$

and for any scalar k, kT is given by

$(kT)(x, y) = kT(x, y) = k(x + y, y) = (kx + ky, ky)$.

Theorem (D): ***The sum of two linear transformations is a linear transformation.***

Hint:

Let T_1 and T_2 be two linear transformations from a vector space $\boldsymbol{V}$ into another vector space $\boldsymbol{W}$ over a field F. For any $\boldsymbol{v_1}, \boldsymbol{v_2} \in \boldsymbol{V}$ and $c_1, c_2 \in F$, one can show that

$(T_1 + T_2)(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}) = c_1(T_1 + T_2)(\boldsymbol{v_1}) + c_2(T_1 + T_2)(\boldsymbol{v_2})$. Hence, $T_1 + T_2$ is linear.

Theorem (E): ***The product of a scalar and a linear transformation is a linear transformation.***

Hint:

Let $T : \boldsymbol{V} \to \boldsymbol{W}$ be a linear transformation and k be a scalar. If c_1 and c_2 are any two scalars, and $\mathbf{v_1}, \mathbf{v_2} \in \boldsymbol{V}$, then , you can show that $(kT)(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = c_1(kT)(\mathbf{v_1}) + c_2(kT)(\mathbf{v_2})$.Hence kT is linear.

Exercises

1. Let $T_1 : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ and $T_2 : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be defined by $T_1(x, y) = (x + y, y)$ and $T_2(x, y) = (x, 0)$.
   Find formulae defining the transformations
   a) $T_1 - T_2$ b) $2T_1 + 3T_2$ c) $T_1 o T_2$ d) $T_2 o T_2$.

   Ans: a) $\boldsymbol{(T_1 - T_2)(x, y) = (y, y)}$ b) $\boldsymbol{(2T_1 + 3T_2)(x, y) = (5x + 2y, 2y)}$
   c) $(T_1 \circ T_2)(x, y) = (x, 0)$ **d)** $(T_2 \circ T_1)(x, y) = (x, 0)$

2. Let T_1, T_2 and T_3 be the linear transformations on $\boldsymbol{R}^3$ defined by
   $T_1(x, y, z) = (0, y, z)$, $T_2(x, y, z) = (x, 0, z)$, $T_3(x, y, z) = (x, y, 0)$, then, show that
   $T_1(T_2 + T_3) = T_1T_2 + T_1T_3$.

Inverse of a Linear Transformation:

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be two vector spaces over a field F. Let $T : \boldsymbol{V} \to \boldsymbol{W}$ be a linear transformation (function). Then, T is said to be **invertible** if there exists a transformation $T' : \boldsymbol{W} \to \boldsymbol{V}$ such that

i) ToT' is an identity transformation on $\boldsymbol{W}$

and **ii)** $T'oT$ is an identity transformation on $\boldsymbol{V}$.

If T is invertible, T' is unique; and is denoted by T^{-1}. Here, T^{-1} is known as the **inverse** of T.

Theorem (A): ***The inverse of a linear transformation is linear.***

Proof:

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be two vector spaces over a field F. Suppose that the linear transformation $T : \boldsymbol{V} \to \boldsymbol{W}$ is invertible with T^{-1} as its inverse. To prove that T^{-1} is linear.

Let $\mathbf{w_1}, \mathbf{w_2} \in \boldsymbol{W}$ and let $\mathbf{v_1}, \mathbf{v_2} \in \boldsymbol{V}$ such that $T(\mathbf{v_1}) = \mathbf{w_1}$ and $T(\mathbf{v_2}) = \mathbf{w_2}$. Then, for any scalars $c_1, c_2 \in F$, we have

$$T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) = c_1\mathbf{w_1} + c_2\mathbf{w_2}.$$

By definition, $T^{-1}(c_1\mathbf{w_1} + c_2\mathbf{w_2}) = c_1\mathbf{v_1} + c_2\mathbf{v_2} = c_1T^{-1}(\mathbf{w_1}) + c_2T^{-1}(\mathbf{w_2})$.

thereby proving our theorem. ■

d) Singular and non–singular transformation:

A linear transformation T is said to be **non–singular** if $T(\mathbf{v}) = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$, i.e., *Ker* $T = \{\mathbf{0}\}$ and singular if $T(\mathbf{v}) = \mathbf{0}$ for $\mathbf{v} \neq \mathbf{0}$.

Example:

Let $T : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be a linear transformation defined by $T(\mathbf{v}) = T(x, y) = (x + y, y)$, $\mathbf{v} = (x, y) \in \boldsymbol{R}^2$.
Show that T is non–singular transformation.

Solution:

By definition, Ker $T = \{\mathbf{v} : T(\mathbf{v}) = \mathbf{0}\}$.

But, $T(\mathbf{v}) = \mathbf{0}$ implies $T(x, y) = (x + y, y) = \mathbf{0} = (0, 0)$ only when $x + y = 0$ and $y = 0$.

This is possible, if $x = 0$, $y = 0$ only, and so Ker $T = \{(0, 0)\} = \{\mathbf{0}\}$.

Hence T is non–singular transformation.

Some Examples

Example 1: ***Let $T : R^2 \to R^2$ be a linear transformation defined by $T(x, y) = (x + y, y)$ for $(x, y) \in R^2$.***
Find the formula for T^{-1}.

Solution:

Suppose $T(x, y) = (s, t)$ ...(i) and $T^{-1}(s, t) = (x, y)$. We know $T(x, y) = (x + y, y)$ ...(ii) and so $x + y = s$ and $y = t$.

Solving for x and y in terms of s and t, we get $y = t$ and $x = s - t$.

Thus, T^{-1} is given by (i) by the formula $T^{-1}(s, t) = (s - t, t)$.

Example 2: ***Let $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ defined by $T(x, y, z) = (x - 3y - 2z, y - 4z, z)$.Find a formula for T^{-1}.***

Hint:

Let us put $T(x, y, z) = (s, t, u)$ ...(i) so that $T^{-1}(s, t, u) = (x, y, z)$ ...(ii)

Also, $T(x, y, z) = (x - 3y - 2z, y - 4z, z)$ ...(iii)

From (i) and (ii), $s = x - 3y - 2z$, $t = y - 4z$, $u = z$.

Solving, we get $x = s + 3t + 14u$, $y = t + 4u$, $z = u$.

Therefore, from (i), T^{-1} is defined by $T^{-1}(s, t, u) = (s + 3t + 14u, t + 4u, u)$.

Hence, $T^{-1}(x, y, z) = (x + 3y + 14z, y + 4z, z)$.

Exercises

1. a) Let T be a linear transformation on $\boldsymbol{R}^2$ defined by $T(x, y) = (x, y + x)$. Find a formula for T^{-1}.

Ans: $T^{-1}(x, y) = (x, y - x)$

b) Let T be the linear transformation defined on $\boldsymbol{R}^2$ such that $T(x, y) = (x + 2y, 3y)$. Find T^{-1}.

Ans: $T^{-1}(x, y) = \left(x - \frac{2y}{3}, \frac{y}{3}\right)$

2. a) Let $T : R^3 \to R^3$ defined by $T(x, y, z) = (x - y, y - z, z)$,Find a formula for T^{-1}.

Ans: $T^{-1}(x, y, z) = (x + y + z, y + z, z)$.

b) Show that the following operator on $\boldsymbol{R}^3$ is invertible, and find a formula for T^{-1}.

Where $T(x, y, z) = (x + z, x - z, y)$. **Ans:** $T^{-1}(x, y, z) = \left(\frac{x+y}{2}, z, \frac{x-y}{2}\right)$.

Linear Map Associated with a Matrix

Let R^n and R^m are two vector spaces over the same field F with ordered basis say $B_1 = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ and $B_2 = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$.

Let us consider a $m \times n$ matrix $A = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11}\, a_{12} \ldots\ldots\ldots\; a_{1n} \\ a_{21}\, a_{22} \ldots\ldots\ldots\; a_{2n} \\ a_{m1}\, a_{m2} \ldots\ldots\ldots\; a_{mn} \end{pmatrix}$

Define a map $T_A : R^n \to R^m$ by $T_A(X) = AX$, for all $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in R^n$.

Then T_A is linear. This map $T_A : K^n \to K^m$ is called **linear map associated with the** $m \times n$ **matrix *A*** defined by $T_A(X) = AX$, for all $X \in K^n$.

Example:

Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ be a 2×2 matrix and $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \boldsymbol{R}^2$.Then the linear map corresponding to given matrix A with respect to the standard basis is $T_A : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$T_A(X) = AX = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{pmatrix}$$

i.e. $T_A(x_1, x_2) = (2x_1 + 3x_2, 4x_1 + 5x_2)$.

Example 1: ***Find the matrix representations of the linear transformation $T : R^2 \to R^2$ defined by $T(x, y) = (x, 2y)$ relative to the standard basis.***

Solution:

We compute the matrix of T relative to the basis $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$ for $\boldsymbol{R}^2$.

We have $T(\boldsymbol{e}_1) = T(1, 0) = (1, 0) = 1.(1, 0) + 0.(0, 1) = 1\,\boldsymbol{e}_1 + 0\,\boldsymbol{e}_2$

$T(\boldsymbol{e}_2) = T(0, 1) = (0, 2) = 0(1, 0) + 2.(0, 1) = 0\,\boldsymbol{e}_1 + 2\,\boldsymbol{e}_2$.

Now, taking the matrix of coefficients A= $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Hence, the matrix associated with T is the matrix = $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Example 2: ***Let T: $R^3 \to R^3$ is a linear transformation defined by $T(x, y, z) = (x + z, y, 0)$. Find the matrix of T with respect to the standard basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.***

Solution:

Here, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0, 0, 1)$ are the standard basis of domain R^3.

Now, $T(x, y, z) = (x + z, y, 0)$.

$\therefore \quad T(e_1) = T(1, 0, 0) = (1 + 0, 0, 0) = (1, 0, 0) = 1e_1 + 0e_2 + 0e_3$

$T(e_2) = T(0, 1, 0) = (0 + 0, 1, 0) = (0, 1,0) = 0e_1 + 1e_2 + 0e_3$

$T(e_3) = T(0, 0, 1) = (0 + 1, 0, 0) = (1, 0, 0) = 1e_1 + 0e_2 + 0e_3.$

Now, taking the matrix of coefficients $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Hence the required matrix of the linear transformation T with respect to the given basis

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 3:

Find the matrix representation of the linear transformation $T : R^2 \to R^2$ defined by $T(x, y) = (x, x + 2y)$ relative to the basis $(1, 0), (1, -1)$.

Hint: Choose $B_1 = \{v_1 = (1, 0), v_2 = (1, -1)\}$ and $B_2 = \{w_1 = (1, 0), w_2 = (0, 1)\}$.

Let $v_1 = (1, 0)$ and $v_2 = (1, -1)$ be the given basis of domain R^2. Then,

$T(v_1) = T(1, 0) = (1, 1) = 1.(1, 0) + 1.(0, 1)$

$T(v_2) = T(1, -1) = (1, -1) = 1.(1, 0) + (-1).(0, 1).$

Now, taking the matrix of coefficients $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Hence the required matrix of the linear transformation T with respect to the given basis

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Example 4:

If the linear transformation T: $R^2 \to R^2$ defined by $T(x, y) = (x, -y)$. Find the matrix of T relative to the basis $B_1 = \{ v_1 = (1, 1), v_2 = (1, 0)\}$ and $B_2 = \{ w_1 = (2, 3), w_2 = (4, 5)\}$.

Hint:

Step:1 Find the co–ordinate vector of $T(v_1)$ relative to the basis B_2

Now, $T(v_1) = T(1, 1) = (1, -1) = c_1 w_1 + c_2 w_2 = c_1(2, 3) + c_2(4, 5)$

$\therefore \quad (1, -1) = (2c_1 + 4c_2, 3c_1 + 5c_2)$.

Which gives $2c_1 + 4c_2 = 1$ and $3c_1 + 5c_2 = -1$. Solving, we get $c_1 = -9/2$ and $c_2 = 5/2$.

Hence, the co–ordinate vector of $T(v_1)$ relative to the basis B_1 is (-9/2, 5/2).

$\therefore \; T(v_1) = -\frac{9}{2} w_1 + \frac{5}{2} w_2 \qquad \ldots(1)$

Step:2 Find the co–ordinate vector of $T(v_2)$ relative to the basis B_2

$T(v_2) = T(1, 0) = (1, 0) = a_1 w_1 + a_2 w_2 = a_1 (2, 3) + a_2 (4, 5)$

Then, $(1, 0) = (2a_1 + 4a_2, 3a_1 + 5a_2)$

Which gives $2a_1 + 4a_2 = 1$ and $3a_1 + 5a_2 = 0$

Solving, we get $a_1 = -5/2$ and $a_2 = 3/2$.

Hence the co–ordinate vector of $T(v_2)$ relative to the basis B_2 is (-5/2, 3/2).

$\therefore \quad T(\boldsymbol{v_2}) = -\frac{5}{2}\ \boldsymbol{w_1} + \frac{3}{2}\ \boldsymbol{w_2} \qquad \ldots(2)$

From (1) and (2), the required matrix is the transpose of the matrix $\begin{bmatrix} -9/2 & 5/2 \\ -5/2 & 3/2 \end{bmatrix}$

Hence the matrix of T relative of the bases B_1 and B_2 is $\begin{bmatrix} -9/2 & -5/2 \\ 5/2 & 3/2 \end{bmatrix}$.

Exercises

1. Find the matrix representations of the following linear transformations relative to the usual basis (standard basis) for $\boldsymbol{R}^n$, n = 1, 2, 3.
   - i) $T : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ defined by $T(x, y) = (x, 2y)$
   - ii) $T : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ defined by $T(x, y, z) = (x, 0, z)$
   - iii) $T : \boldsymbol{R}^3 \to \boldsymbol{R}^2$ defined by $T(x, y, z) = (x, z)$
   - iv) $T : \boldsymbol{R}^1 \to \boldsymbol{R}^2$ defined by $T(x) = (x, 0)$
   - v) $T : \boldsymbol{R}^2 \to \boldsymbol{R}^3$ defined by $T(x, y) = (x, y, x + y)$

Answer:

1. **i)** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **ii)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **iii)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **iv)** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **v)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

■

Presentation:Application of Matrices and Vectors in Data Science

Q. *You have studied the matrices and vectors in MDS First semester Mathematics for Data Science. How do you apply your mathematical knowledge and skills in your Data Science career? Make a slide presentation of 10 minutes by mentioning the following area as the applications of Matrices and vectors for data science fields:*

a. *Machine learning techniques* **such as loss functions, regularization, support vector classification, and many others .**

b.***Computer Vision techniques*** image recognition, some image processing methods like image convolution etc

c. ***Natural Language Processing***: Numerous NLP algorithms, including those for text summarization, page ranking, and information retrieval, etc**.**

d….. e. ….. and other areas.

□□