CS271: DATA STRUCTURES

Name: Walter Valera, William Wang, Ramatu Kamara

Instructor: Dr. Stacey Truex

Project 7: Graph Analysis

Proofs

Complete a formal proof of the following: An undirected graph is bipartite if and only if it contains no cycles of odd length.

Proof To prove this statement true, we must first prove both sides of the biconditional. That is, we must prove that:

1. If an undirected graph G is bipartite, then it contains no cycles of odd length.
2. If an undirected graph G contains on cycles of odd length, then it is bipartite.

Thus, we will continue by proving the **first statement** to be true: Let $G = (V, E)$ be an undirected bipartite graph. By the definition of bipartite, $G.V$ can be partitioned into two sets, V_1 and V_2, where every edge in E connects a vertex in V_1 to a vertex in V_2. That means there are no edges between the vertices of the same set. Now suppose for the sake of contradiction that G is contains a cycle of odd length, we will call it C. We can define C as the following sequence of vertices:

$$C =< v_1, v_2, v_3, ..., v_n, v_1 >$$

We will define the length of C to be n, where n is odd. Using the two sets V_1 and V_2 we defined earlier, we can assign a "color" to each vertex. For any $v \in V_1$, $v.color = c_1$, and for any $v \in V_2, v.color = c_2$. Since C is a cycle, each vertex in the sequence must alternate in color, because by definition of bipartite, every edge connects a vertex from V_1 to a vertex in V_2. Thus, the colors of the vertices in C will go as follows:

$$1.\ v_1.color = c_1$$
$$2.\ v_2.color = c_2 \neq c_1 = v_1.color$$
$$3.\ v_3.color = c_1$$

If we continue doing this, we will notice a pattern. The pattern is that a vertex v_i will have the same color as v_1 if the path from v_1 to v_i has an even length ($i - 1$ is even). It will have a different color if the path has an odd length ($i - 1$ is odd). The last edge in C is (v_n, v_1). Since C has length n, the path from v_1 to v_n has length $n - 1$. Since we originally assumed n to be odd, $n-1$ must be even. Therefore, v_n will have the same color as v_1. This implies that the edge (v_n, v_1) connects two vertices of the same color, or in other words, it connects two vertices of the same set. This is a contradiction, as it violates the definition of a bipartite graph we set up earlier ("there are no edges between vertices of the same set"). Our initial assumption that a bipartite graph G contains a cycle of odd length must then be false. So it must be true that if G is bipartite, then it has no cycles of odd length.

Now we must prove the **second statement** to be true: Let $G = (V, E)$ be an undirected graph with cycles of even length (no cycles of odd length).

Let us also assume that, without loss of generality, a graph is bipartite if and only if all of its connected components are bipartite. Thus, suppose for the sake of contradiction, G is not bipartite, meaning at least one of its connected components is non-bipartite. We will call this component J. Since J is a connected component of G, J itself is a connected graph. Choose an arbitrary vertex $s \in J$. For each vertex x in J let $d(x)$ be the length of a shortest path from s to x.

We define:

$$V_1' = \{x \in J : d(x) \text{ is even}\}, \qquad V_2' = \{x \in J : d(x) \text{ is odd}\}.$$

Clearly $V_1' \cup V_2' = J$ and $V_1' \cap V_2' = \emptyset$.

If very edge of J had one endpoint in V_1', and other one in V_2', then J would be bipartite with bipartition (V_1', V_2'), which contradicts our assumption that J is non-bipartite. Therefore there must be an edge (u, v) in J with both endpoints in the same set, either $u, v \in V_1'$ or $u, v \in V_2'$. If both endpoints were in V_2', the argument below would be the same, so we only describe the case $u, v \in V_1'$. Then $d(u)$ and $d(v)$ are both even.

Let P_u be a shortest path from s to u and P_v be a shortest path from s to v. Consider the closed walk obtained by following P_u from s to u, then the edge (u, v), and then P_v backwards from v to s. This walk contains a cycle, and its length is

$$d(u) + 1 + d(v),$$

which is odd because $d(u)$ and $d(v)$ are the same parity (either both odd or both even, but in this case, both even), so their sum will always be even. Hence J, and therefore G, contains a cycle of odd length, contradicting our assumption that G has no cycles of odd length.

Since we have shown that both sides of the biconditional are true, we can thus conclude that the entire statement is true. Q.E.D.