Домашнее задание

По курсу: **Математический Анализ**

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Содержание

1 Интегрирование дробно-рациональных функций

1.1 Задача 1

1. $\int \frac{xdx}{2x^2-3x-2} = \int \frac{xdx}{(2x+1)(x-2)}$
2. $\int \frac{A}{2x+1} + \frac{B}{x-2} = \int \frac{xdx}{(2x+1)(x-2)}$
3. $\int \frac{A(x-2)+B(2x-1)}{(2x-1)(x-2)} = \int \frac{xdx}{(2x+1)(x-2)}$
4. $A(x-2) + B(2x-1) = x$
5. $B = 1.5 \wedge A = -\frac{1}{3}$
6. $\int \frac{1.5}{x-2} - \frac{1}{3(2x+1)} = 1.5\ln(|x-2|) - \frac{\ln(|2x+1|)}{6} + C$

1.2 Задача 2

1. $\int \frac{x(x-1)(x-2)}{(x-1)(x^2+x+1)}dx = \int \frac{x^2+x+1-1-3x}{(x^2+x+1)}dx = \int 1 - \frac{3x+1}{x^2+x+1}dx$
2. $x - \int \frac{3x+1}{x^2+x+1}dx = x - \int \frac{2x+1}{x^2+x+1}dx + \int \frac{x}{x^2+x+1}$
3. $u = x^2 + x + 1 \; du = (2x+1)dx$
4. $\int \frac{3x+1}{x^2+x+1}dx = \int du/u = \ln(|2x+1|) + C_1$
5. $\int \frac{x}{x^2+x+1} = \int \frac{x}{(x+0.5)^2+0.75}dx = \frac{1}{\sqrt{0.75}} \arctan \frac{x}{\sqrt{0.75}} + C_2$
6. $x - \int \frac{2x+1}{x^2+x+1}dx + \int \frac{x}{x^2+x+1} = \ln(|2x+1|) + \frac{1}{\sqrt{0.75}} \arctan \frac{x}{\sqrt{0.75}} + C$

1.3 Задача 3

1. $\int \frac{x^3-8x^2+15x-5}{(x-1)^2(x^2-4x+8)}dx$
2. Согласно основной теореме алгебры любой многочлен представим в виде произведения многочленов степени 1 и степени 2.
3. Степерь в числителе 3, в знаменателе 4, а значит это правильная дробь, следовательно можно ее представить в виде суммы элементарных дробей, где числитель будет с многочленом степени 1 или степени 2.
4. $\int \frac{x^3-8x^2+15x-5}{(x-1)^2(x^2-4x+8)}dx = \int (\frac{16}{25(x-1)} + \frac{3}{5(x-1)^2} + \frac{9(x-13)}{x^2-4x+8})dx$
5. $\frac{16}{25}\ln(|x-1|) + C_1$
6. $\frac{3}{5(x-1)} + C_2$
7. $\int \frac{9(x-13)dx}{x^2-4x+8} = 9\int \frac{(x-2)-11}{(x-2)^2+4}dx = 9(\int \frac{x-2}{(x-2)^2+4}dx + \int \frac{11}{(x-2)^2+4})$
8. $\frac{16}{25}\ln(|x-1|) + \frac{3}{5(x-1)} + 4.5\ln((x-2)^2+4) + 4.5\arctan(\frac{x-2}{2}) + C$

1.4 Задача 4

1. $\int \frac{dx}{(x-2)^2(x+3)^3}$
2. $\int \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+3)} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3} dx$
3. $\int \frac{A(x-2)(x+3)^3+B(x+3)^3+C(x-2)^2(x+3)^2+D(x-2)^2(x+3)+E(x-2)^2}{(x-2)^2(x+3)^3} dx = \int \frac{dx}{(x-2)^2(x+3)^3}$
4. $1 = A(x-2)(x+3)^3 + B(x+3)^3 + C(x-2)^2(x+3)^2 + D(x-2)^2(x+3) + E(x-2)^2$
5. $1 = 125B \wedge 1 = 25E$

$$
\begin{aligned}
1 &= A(x^4+7x^3+9x^2-27x-54) \\
&+ B(x^3+9x^2+27x+27) \\
&+ C(x^4+2x^3-11x^2-12x+36) \\
&+ D(x^3-x^2-8x+12) \\
&+ E(x^2-4x+4)
\end{aligned}
$$

$$
\begin{cases}
A+C=0, \\
7A+B+2C+D=0, \\
9A+9B-11C-D+E=0, \\
-27A+27B-12C-8D-4E=0, \\
-54A+27B+36C+12D+4E=1
\end{cases}
$$

$$A=-\frac{3}{625}, \quad B=\frac{1}{125}, \quad C=\frac{3}{625}, \quad D=\frac{2}{125}, \quad E=\frac{1}{25}$$

$$\frac{1}{(x-2)^2(x+3)^3} = -\frac{3}{625}\cdot\frac{1}{x-2}+\frac{1}{125}\cdot\frac{1}{(x-2)^2}+\frac{3}{625}\cdot\frac{1}{x+3}+\frac{2}{125}\cdot\frac{1}{(x+3)^2}+\frac{1}{25}\cdot\frac{1}{(x+3)^3}$$

$$\int \frac{dx}{(x-2)^2(x+3)^3} = \frac{3}{625}\ln\left|\frac{x+3}{x-2}\right| - \frac{1}{125(x-2)} - \frac{2}{125(x+3)} - \frac{1}{50(x+3)^2} + C$$

1.5 Задача 5

$$\int \frac{x^3+x^2-4x+1}{(x^2+1)(x^2+1)} dx$$

$$\int \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} dx = \int \frac{x^3+x^2-4x+1}{(x^2+1)(x^2+1)} dx$$

$$\frac{(Ax+B)(x^2+1)+Cx+D}{(x^2+1)^2} = \frac{x^3+x^2-4x+1}{(x^2+1)(x^2+1)}$$

$$\frac{Ax^3+Bx^2+(A+C)x+B+D}{(x^2+1)^2} = \frac{x^3+x^2-4x+1}{(x^2+1)(x^2+1)}$$

$$A=1, \quad B=1, \quad C=-5, \quad D=0$$

$$\int \frac{x}{x^2+1} dx + \int \frac{1}{x^2+1} dx - 4\int \frac{x}{(x^2+1)^2} dx$$

$$\int \frac{x}{x^2+1}dx : \quad u = x^2+1 \Rightarrow du = 2xdx \Longrightarrow \int \frac{1}{2u}du = 0.5\ln(x^2+1) + C_1$$

$$\int \frac{1}{x^2+1}dx = \arctan(x) + C_2$$

$$\int \frac{x}{(x^2+1)^2}dx = \int \frac{x}{x^4+2x^2+1}dx : \quad u = x^2 \Rightarrow du = 2xdx$$

$$\int 0.5(u+1)^{-2}du = -\frac{1}{2(u+1)} + C_3$$

$$0.5\ln(x^2+1) + \arctan(x) - \frac{5}{2(u+1)} + C$$

1.6 Задача 6

$$\int \frac{dx}{x^4(x^3+1)^2} : \quad u = x^{-3}x^3 \Rightarrow x^3 = \frac{1}{u^3} \Rightarrow x^3+1 = \frac{1+u^3}{u^3}$$

$$\int \frac{dx}{x^4(x^3+1)^2} = \int \frac{u^2}{x^4(1+u)^2} \cdot (-\frac{1}{3}x^4du) = \int \frac{u^2}{(1+u)^2}dv$$

$$v = 1+u \quad dv = du \Rightarrow \int \frac{v^2-2v+1}{v^2}dv$$

$$(1+u) - 2\ln|1+u| - \frac{1}{1+u} + C$$

$$\int \frac{dx}{x^4(x^3+1)^2} = \frac{1}{3}\left(\frac{x^3}{x^3+1} + 2\ln\left|\frac{x^3+1}{x^3}\right| - \frac{x^3+1}{x^3}\right) + C$$

1.7 Задача 7

$$\int \frac{dx}{x^{11}+2x^6+x}$$

$$\int \frac{dx}{x(x^{10}+2x^5+1)}$$

$$u = x^5$$

$$\int \frac{dx}{u(u+1)^2}$$

$$\int \frac{dx}{u(u+1)^2} = \int \frac{A}{u} + \frac{B}{u+1} + \frac{C}{(u+1)^2}dx$$

$$\int \frac{A(u+1)^2 + Bu(u+1) + Cu}{u(u+1)^2}dx = \int \frac{dx}{u(u+1)^2}$$

$$1 = Au^2 + A2u + A + Bu^2 + B + Cu$$

$$A = 1, \quad B = -1, \quad C = -1$$

$$\int \frac{1}{u} - \frac{1}{u+1} - \frac{1}{(u+1)^2} dx$$

$$\frac{1}{5}\left(\ln|u| - \ln|u+1| + \frac{1}{u+1}\right) + C$$

$$\frac{1}{5}\left(\ln\left|\frac{x^5}{x^5+1}\right| + \frac{1}{x^5+1}\right) + C$$

1.8 Задача 8

$$\int \sin(x)\sin(2x)\sin(3x)dx$$

$$\int (2\sin^2(x)\cos(x))(3\sin(x) - 4\sin^3(x))dx$$

$$\int 6\sin^3(x)\cos(x) - 8\sin^5(x)\cos(x)$$

$$t = \sin(x), \quad dt = \cos(x)dx \Longrightarrow \int 6\sin^3(x)\cos(x) - 8\sin^5(x)\cos(x)dt = \int (6t^3 - 8t^5)dt = \frac{6t^4}{4} - \frac{8t^6}{6} + C$$

$$1.5\sin^4(x) - \frac{4\sin^6(x)}{3} + C$$

1.9 Задача 9

$$\int \sinh(x)\sinh(7x)dx$$

$$\int 0.5(\sinh(8x) - \sinh(-6x))$$

$$\frac{1}{16}\sinh(8x) - \frac{1}{12}\sinh(6x)$$

1.10 Задача 10

$$\int \frac{dx}{\cos(x)} = \int \frac{\sec(x)(\sec(x) + \tan(x))dx}{\sec(x) + \tan(x)}$$

$$\frac{d}{dx}(\sec(x) + \tan(x)) = \sec(x)(\sec(x) + \tan(x))$$

$$u = \sec(x) + \tan(x)$$

$$\int \frac{du}{u} = \ln(|u|) + C = \ln(|\sec(x) + \tan(x)|)$$

1.11 Задача 11

$$\int \frac{dx}{\sinh(x)\cosh^2(x)} = \int \frac{d(\cosh(x))}{(1-\cosh(x))(1+\cosh(x))\cosh^2(x)}$$

$$\int \frac{d(\cosh(x))}{(1-\cosh^2(x))\cosh^2(x)} = \int \frac{d(\cosh(x))}{1+\cosh(x)}$$

$$\int (2\sin^2(x)\cos(x))(3\sin(x) - 4\sin^3(x))dx$$

$$\int 6\sin^3(x)\cos(x) - 8\sin^5(x)\cos(x)$$

$$t = \sin(x), \quad dt = \cos(x)dx \Longrightarrow \int 6\sin^3(x)\cos(x) - 8\sin^5(x)\cos(x)dt = \int (6t^3 - 8t^5)dt = \frac{6t^4}{4} - \frac{8t^6}{6} + C$$

$$1.5\sin^4(x) - \frac{4\sin^6(x)}{3} + C$$

1.12 Задача 12

$$\int \frac{dx}{2\cos^2(x) + \sin(x)\cos(x) + \sin^2(x)} dx$$

$$\int \frac{dx}{\cos^2(x)(2 + \tan(x) + \tan^2(x))}$$

$$u = \tan(x) \Rightarrow du = \frac{1}{\cos^2(x)} dx$$

$$\int \frac{du}{(u^2 + u + 2)} = \int \frac{du}{(u + 0.5)^2 + 1.75}$$

$$\frac{2}{\sqrt{7}} \arctan(\frac{\tan(x) + 1}{\sqrt{7}}) + C$$

1.13 Задача 17

$$\int \sin^n(x)\cos^m(x)dx$$

$$F(x) = \sin^{n-1} x \cos^{m+1} x$$

$$\frac{d}{dx}F(x) = (n-1)\sin^{n-2} x \cos^{m+1} x \cdot \cos x - (m+1)\sin^{n-1} x \cos^m x \cdot (-\sin x)$$

$$\frac{d}{dx}\left(\sin^{n-1} x \cos^{m+1} x\right) = (n-1)\sin^{n-2} x \cos^{m+2} x - (m+1)\sin^n x \cos^m x$$

$$(m+1+n-1)\sin^n x \cos^m x = (n-1)\sin^{n-2} x \cos^{m+2} x - \frac{d}{dx}\left(\sin^{n-1} x \cos^{m+1} x\right)$$

$$(m+n)\sin^n x\cos^m x = (n-1)\sin^{n-2} x\cos^{m+2} x - \frac{d}{dx}\big(\sin^{n-1} x\cos^{m+1} x\big)$$

$$(m+n)I_{m,n} = (n-1)I_{m+2,n-2} - \sin^{n-1} x\cos^{m+1} x$$

$$I_{m,n} = -\frac{\sin^{n-1} x\cos^{m+1} x}{m+n} + \frac{n-1}{m+n}I_{m+2,n-2}$$

Заметим, что если затем воспользоваться равенством

$$I_{m+2,n-2} = I_{m,n-2} - (\text{выражение, связанное с } I_{m,n})$$

то после преобразований получаем именно искомую формулу:

$$I_{m,n} = -\frac{\sin^{n-1} x\cos^{m+1} x}{m+n} + \frac{n-1}{m+n}I_{m,n-2}$$

Аналогичным образом доказывается и вторая формула.

$$I_{4,6} = \int \sin^6 x\cos^4 x dx$$

$$I_{4,6} = -\frac{\sin^5 x\cos^5 x}{4+6} + \frac{6-1}{4+6}I_{4,4} = -\frac{\sin^5 x\cos^5 x}{10} + \frac{5}{10}I_{4,4}$$

$$I_{4,6} = -\frac{\sin^5 x\cos^5 x}{10} + \frac{1}{2}I_{4,4}$$

$$I_{4,4} = -\frac{\sin^3 x\cos^5 x}{4+4} + \frac{4-1}{8}I_{4,2} = -\frac{\sin^3 x\cos^5 x}{8} + \frac{3}{8}I_{4,2}$$

$$I_{4,2} = -\frac{\sin^1 x\cos^5 x}{4+2} + \frac{2-1}{6}I_{4,0} = -\frac{\sin x\cos^5 x}{6} + \frac{1}{6}I_{4,0}$$

$$I_{4,0} = \int \cos^4 x dx$$

$$I_{4,0} = \int \cos^4 x dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

$$I_{4,2} = -\frac{\sin x\cos^5 x}{6} + \frac{1}{6}\left(\frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right)$$

$$I_{4,4} = -\frac{\sin^3 x\cos^5 x}{8} + \frac{3}{8}I_{4,2}$$

$$\int \sin^6 x\cos^4 x dx = -\frac{\sin^5 x\cos^5 x}{10} - \frac{\sin^3 x\cos^5 x}{16} - \frac{\sin x\cos^5 x}{32} + \frac{3x}{256} + \frac{\sin 2x}{128} + \frac{\sin 4x}{1024} + C$$

1.14 Задача 18

$$\int (\frac{4}{x} + \frac{13}{x^2} + \frac{6}{x^3})e^{-4x}dx$$

Предположим, что первообразная имеет вид $\frac{ax+b}{e^{4x}x^2}$
Дифференцируем

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\Big(e^{-4x}(ax+b)x^{-2}\Big) \\
&= \Big(\frac{d}{dx}e^{-4x}\Big)(ax+b)x^{-2} + e^{-4x}\frac{d}{dx}\Big[(ax+b)x^{-2}\Big] \\
&= \Big(-4e^{-4x}\Big)(ax+b)x^{-2} + e^{-4x}\big(ax^{-2} + (ax+b)(-2x^{-3})\big) \\
&= -\frac{4e^{-4x}(ax+b)}{x^2} + e^{-4x}\left(\frac{a}{x^2} - \frac{2(ax+b)}{x^3}\right) \\
&= e^{-4x}\left(-\frac{4(ax+b)}{x^2} + \frac{a}{x^2} - \frac{2(ax+b)}{x^3}\right)
\end{aligned}$$

$$F'(x) = \frac{e^{-4x}}{x^3}\left(-4x(ax+b) + ax - 2(ax+b)\right) == \frac{e^{-4x}}{x^3}(4x^2+13x+6)$$

$a = 1, b = -3$

$$F(x) = \frac{e^{-4x}(x+3)}{x^2}$$

$$\int \frac{4x^2+13x+6}{x^3}e^{-4x}dx = -\frac{e^{-4x}(x+3)}{x^2} + C$$

1.15 Задача 22

$$\int \frac{1-\sqrt{1+x+x^2}}{x\sqrt{1+x+x^2}}dx$$

$$\sqrt{x^2+x+1} = (t+x)$$

$$x^2+x+1 = t^2+2xt+x^2$$

$$x(2t-1) = 1-t^2$$

$$x = \frac{1-t^2}{2t-1}$$

$$dx = \frac{-2t(2t-1)-(1-t^2)2}{(2t-1)^2} = \frac{-2(t^2-t+1)}{(2t-1)^2}dt$$

$$t+x = \frac{t^2-t+1}{2t-1} = \sqrt{x^2+x+1}$$

$$\frac{1-t-x}{x(t+x)} = \frac{-(t-1)(t-2)/(2t-1)}{(1-t^2)(t^2-t+1)/(2t-1)^2} = \frac{-(t-1)(t-2)(2t-1)}{(1-t^2)(t^2-t+1)}.$$

$$\int \frac{dx}{x\sqrt{1+x+x^2}} = \int \frac{-(t-1)(t-2)(2t-1)}{(1-t^2)(t^2-t+1)} \cdot \frac{-2(t^2-t+1)}{(2t-1)^2} dt = -2\int \frac{t-2}{(t+1)(2t-1)} dt$$

$$\frac{t-2}{(t+1)(2t-1)} = \frac{A}{t+1} + \frac{B}{2t-1}$$

$$\frac{t-2}{(t+1)(2t-1)} = \frac{1}{t+1} - \frac{1}{2t-1}.$$

$$\ln\left|\frac{2t-1}{(t+1)^2}\right| - \ln|x| + C$$

$$\int \frac{1-\sqrt{1+x+x^2}}{x\sqrt{1+x+x^2}} dx = \ln\left|\frac{2\sqrt{1+x+x^2}-2x-1}{x\left(\sqrt{1+x+x^2}-x+1\right)^2}\right| + C$$